THE BORROMEAN RINGS DO NOT EXIST

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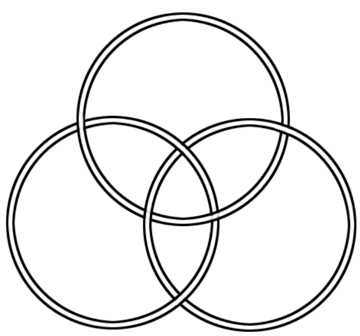
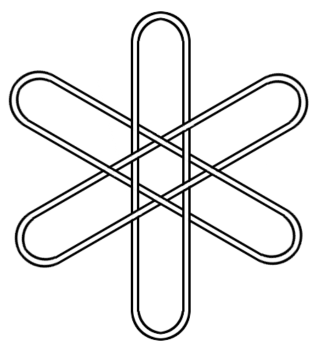
1. INTRODUCTION

The Borromean rings (pictured below) is a link comprised of three circles which the property that any two circles are pairwise disjoint, yet the link itself is not trivial. The belong to a class of links known as Brunnian links, that is, k-component links with that property that any subcollection of $k-1$ of the componets is trivial. This report will provide a proof that it is in fact impossible to construct the Borromean rings with perfect circles, as outlined in Alden Mathieu's talk. We begin with some formalitites of the theory of knots and links, then develop some results about circles in $\mathbb{R}^3$, and finally finish with the proof. In the process, we shall also prove that the Tait #18 link cannot be constructed with perfect circles and that is not equivalent to the Borromean rings, despite also being a three component Brunnian link.



(A) The Borromean Rings



(B) Tait #18

2. KNOT AND LINK PRELIMINARIES

Definition 2.1. *A knot K is a continuous embedding $K : S^1 \to \mathbb{R}^3$*

Definition 2.2. *A link L (with n components) is a collection $L = \{K_1, \ldots, K_n\}$ where $K_1, \ldots, K_n$ are knots with disjoint images.*

Remark 2.3. A knot is a link with 1 component

Example 2.4. The Borromean Rings and Tait #18 are both links with 3 components.

An important concept in knot theory is that of equivalence. We regard knots as topological objects, and so two knots are equivalent if their images are topologically equivalent. In practice, we shall use the diagrams of two knots to determine whether or not they are equivalent. Similarly, two links are equivalent if the respective (disjoint) union of the images of their components are topologically equivalent.

Definition 2.5. *The diagram of a knot K is an illustration of the projection $K(S^1) \to \mathbb{R}^2$ in which the information on whether a crossing is an under-crossing or an over-crossing is preserved*

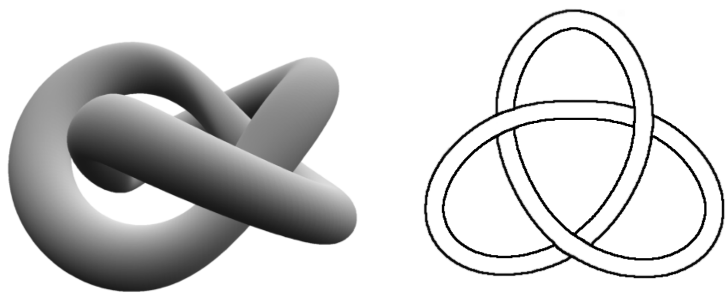
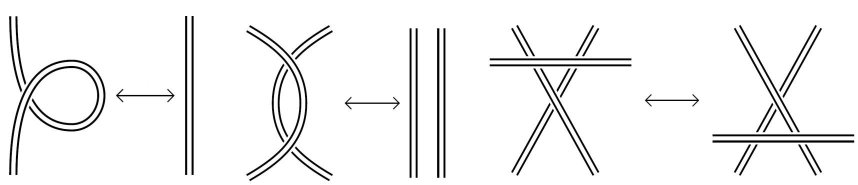


FIGURE 2. The Trefoil knot and it's diagram

This association of a knot to it's diagram may seem ambiguous as we may get a different diagram depending on what projection we take. This is taken care of by the concept of "Reidemeister moves". The following theorem will provide a method to determining if two knots are equivalent by allowing a series of Reidemeister moves for the knot diagrams.

Theorem 2.6 (**Reidemeister**). *Two knots are equivalent iff their diagrams differ by continous deformation and a finite series of Reidemeister moves. That is, moves of the form:*

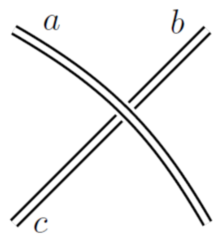


(A) Type I (B) Type II (C) Type III

FIGURE 3. The Reidemeister Moves

There are quite a number of invariants utilised in knot theory. These are certain values associated to knots that are invariant for equivalent knots. These values, while they cannot prove if two certain knots are equivalent (although they may strongly suggest that they are), are useful in determining whether two knots are not equivalent. A few examples of these are crossing number of a knot diagram, hyperbolic volume of the complement of the image of a knot, and Fox n-colourings of a knot diagram. We will be utilising the latter in this report.

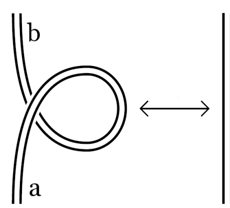
Definition 2.7. *A Fox n-colouring of a knot diagram is an association, or "colouring", of the arcs in the diagram with elements in $\mathbb{Z}/n\mathbb{Z}$ such that at any crossing, the element associated to the arc on the over-crossing a is equal to the average of the associated elements on the two arcs on the under-crossing, b and c. In other words, $2a \equiv b + c$ (mod n)*



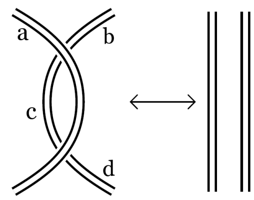
Proposition 2.8. *The number of Fox n-colourings of a knot is invariant for equivalent knots.*

Proof. It is enough to prove that the number of Fox n-colourings is invariant for the Reidemeister moves.

(I) We colour each arc on the left diagram $a, b \in \mathbb{Z}/n\mathbb{Z}$ as shown below. Due to the crossing relation, we have $2a \equiv a + b \implies a \equiv b$. Therefore for the left diagram, we have n choices for the colouring, namely the n colours, and this coincides with the right diagram as there is only one arc.



(II) We colour the four arcs on the left diagram $a, b, c, d \in \mathbb{Z}/n\mathbb{Z}$ as shown below. For the top crossing, we have that $2a \equiv b + c \implies c \equiv 2a - b$ and by the bottom crossing, we have that $2a \equiv c + d \equiv 2a - b + d \implies b \equiv d$. Therefore all Fox n-colourings for the left diagram are completely determined by the choices for a and b, which coincides with the number of colourings for the right diagram as we have two arcs with no crossings so the choice of colour for each arc is independent of each other.



(III) We colour the six arcs on the left diagram $a, b, c, d, e, f \in \mathbb{Z}/n\mathbb{Z}$ as shown below. The three crossings in the diagram give relations:

$$2c \equiv a + d \implies d \equiv 2c - a$$

$$2c \equiv b + f \implies f \equiv 2c - b$$

$$2d \equiv e + f \implies e \equiv 2d - f \equiv 2(2c - a) - (2c - b) \equiv 2c - 2a + b$$

Therefore all colourings are determined by the choices for a, b and c for the left diagram.

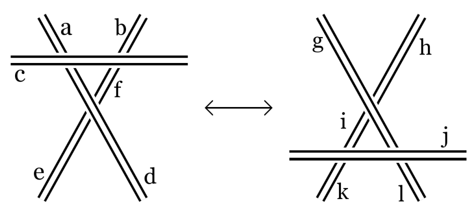
We colour the six arcs on the right diagram $g, h, i, j, k, l \in \mathbb{Z}/n\mathbb{Z}$ as shown below. The three crossings in the diagram give relations:

$$2g \equiv i + h \implies i \equiv 2g - h$$

$$2j \equiv g + l \implies l \equiv 2j - g$$

$$2j \equiv k + i \implies k \equiv 2j - i \equiv 2j - (2g - h) \equiv 2j + 2g + h$$

Therefore all colourings are determined by the choices for g, h and j for the right diagram. This coincides with the number of colourings for the left diagram as required.



□

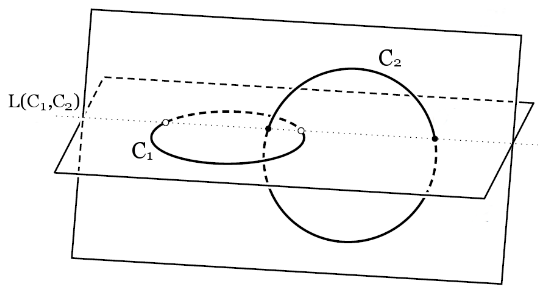
3. CIRCLE LINKS

We are concerned with constrcuting the Borromean rings using perfect circles and so we develop a few results regarding circles embedded in $\mathbb{R}^3$

Definition 3.1. *Let C be a circle in $\mathbb{R}^3$. We define $\Delta(C)$ as the disk spanned by C.*

Definition 3.2. *Let C_1, C_2 be circles in $\mathbb{R}^3$. C_1 and C_2 are linked if $C_1 \cap \Delta(C_2) = \{x_1\}$ for some $x_0 \in \mathbb{R}^3$. In other words, two circles are linked if one of the circles intersects the disk spanned by the other at exactly one point.*

Remark 3.3. Let C_1, C_2 be two disjoint circles in $\mathbb{R}^3$ that do not lie in the same plane. Let $H(C_1)$ and $H(C_2)$ denote the planes that contain C_1 and C_2 respectively and define $L(C_1, C_2) := H(C_1) \cap H(C_2)$, the line of intersection of C_1 and C_2. Now suppose C_1 and C_2 are linked. Then $C_1 \cap \Delta(C_2) = \{x_1\}$ and $C_2 \cap \Delta(C_1) = \{x_2\}$ and we have $x_1 \neq x_2$. Indeed if $x_1 = x_2$, then we would have $x_1 \in C_1 \cap \Delta(C_2)$ and $x_1 \in C_2 \cap \Delta(C_1) \implies x_1 \in C_1 \cap C_2$ which contradicts the fact that C_1 and C_2 are disjoint. So $x_1 \neq x_2$ and we find that the line segment joining x_1 and x_2 is contained in $L(C_1, C_2)$ and in particular, $L(C_1, C_2) \cap C_1 \cap C_2$ consists of exactly four points which "alternate" between points of C_1 and C_2 (see picture below). This alternating property characterises linked circles.



Definition 3.4. *Let C be a circle in $\mathbb{R}^3$ with radius r and centre c. We define the spherical dome $D(C) := \{(x, h_C(x)) : x \in \Delta(C)\} \subset \mathbb{R}^3 \times \mathbb{R}$ where $h_C : \Delta(C) \to \mathbb{R}^3$ is defined such that $h_C(x) = \sqrt{r^2 - |x - c|^2}$*

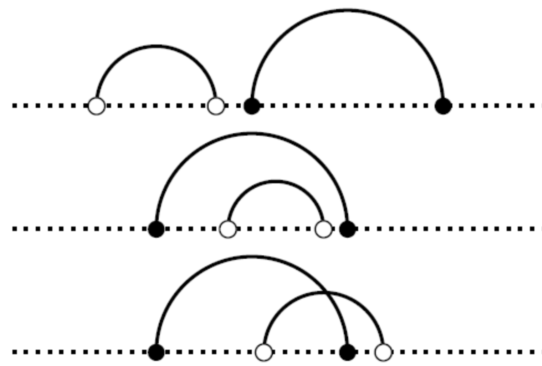
Proposition 3.5. *Let C_1, C_2 be disjoint circles in $\mathbb{R}^3$ that are not linked, then $D(C_1) \cap D(C_2) = \varnothing$*

Proof. Let C_1, C_2 be circles in $\mathbb{R}^3$ and suppose $D(C_1) \cap D(C_2) \neq \varnothing$. Then $\exists (x_0, t_0) \in D(C_1) \cap D(C_2) \subset \mathbb{R}^3 \times \mathbb{R}$. Now, $(x_0, t_0) \in D(C_1) \implies x_0 \in \Delta(C_1)$. Similarly $x_0 \in \Delta(C_2)$ and we have $x_0 \in \Delta(C_1) \cap \Delta(C_2)$ and $x_0 \in L(C_1, C_2)$. Now we consider the functions h_{C_1} and h_{C_2} restricted to $\Delta(C_1) \cap L(C_1, C_2)$ and $\Delta(C_2) \cap L(C_1, C_2)$ respectively. Under this restriction, h_{C_1} and h_{C_2} define perfect half-circles. These half-circles must intersect as $x_0 \in \Delta(C_1) \cap \Delta(C_2) \cap L(C_1, C_2)$ and therefore we have the alternating property of the end points of $D(C_1) \cap L(C_1, C_2)$ and $D(C_2) \cap L(C_1, C_2)$ (see picture below). Moreover we have this alternating property for the circles C_1 and C_2 and so they must be linked. The result follows. $\square$

We combine these concepts to prove the following result which will be used in the proof that the Borromean rings do not exist.

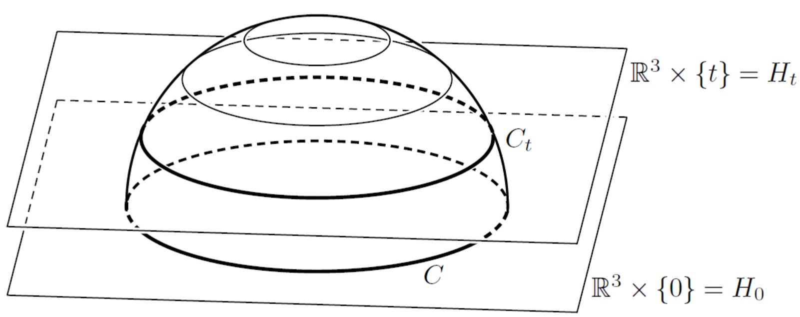
Proposition 3.6. *If a link consists of disjoint perfect circles that are pairwise not linked, then the link is trivial.*

Proof. We may assume without loss of generality that each of the circles lie in distinct planes as we may slightly move each circle to make it so. By Proposition 3.5, we have that the spherical domes of the circles are pairwise disjoint. Indentifying



(A) We have the alternating property iff the half-circles intersect

$\mathbb{R}^3$ with $\mathbb{R}^3 \times \{0\} \subset \mathbb{R}^3 \times \mathbb{R}$, which contains the sphercial domes, we construct a "movie" $\mathbb{R}^3 \times \{t\}$ with time co-ordinate t. We can see from the picture below that starting at $\mathbb{R}^3 \times \{0\}$ and increasing the time co-ordinate, we have a movie in which we see a circle that continuously shrinks to a point and then dissappears at some point in time.



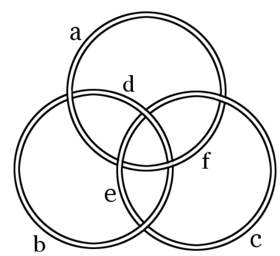
Note that as the circle shrinks in the movie, the centre of the circle and the plane that the circle lies in remains the same. Moreover, the circles remain disjoint since the spherical domes are pairwise disjoint by Proposition 3.5, hence the circles remain pairwise not linked throughout the movie. At some time t, we will have that each circle is so small that it does not intersect a plane in which another circle lies which implies the disk spanned by this circle does not intersect any plane in which another circle lies and this will remain the case for all later times in the movie. We conclude that the movie ends with all circles shrunk to the extent that they have disjoint spanning disks. This means the circles are completely seperate and so the link is trivial. $\square$

4. THE BORROMEAN RINGS DO NOT EXIST

Theorem 4.1. *Neither the Borromean Rings nor Tait #18 may be constructed from three perfect circles.*

Proof. Let us consider Fox n-colourings for the Borromean rings. Colouring the six arcs $a, b, c, d, e, f \in \mathbb{Z}/n\mathbb{Z}$ as shown in the diagram below, we have the following relations from the six crossings:

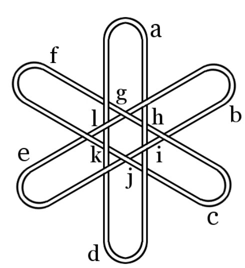
$$2a \equiv b + d \implies d \equiv 2a - b$$
$$2b \equiv c + e \implies e \equiv 2b - c$$
$$2c \equiv a + f \implies f \equiv 2c - a$$
$$2d \equiv e + c \implies 2(2a - b) \equiv (2b - c) + c \implies 4a - 2b \equiv 2b \implies 4a \equiv 4b$$
$$2e \equiv f + a \implies 2(2b - c) \equiv (2c - a) + a \implies 4b - 2c \equiv 2c \implies 4b \equiv 4c$$
$$2f \equiv d + b \implies 2(2c - a) \equiv (2a - b) + b \implies 4c - 2a \equiv 2a \implies 4c \equiv 4a$$



So we have the relation $4a \equiv 4b \equiv 4c$ but mod 5 this implies $a \equiv b \equiv c$ and so the Borromean rings only has the 3 trivial Fox 5-colourings, however, the trivial 3-component link has $5^3 = 125$ Fox 5 colourings, hence the Borromean rings are not trivial.

Now let us consider Fox n-colourings for Tait #18. Colouring the twelve arcs $a, b, c, d, e, f, g, h, i, j, k, l \in \mathbb{Z}/n\mathbb{Z}$ as shown in the diagram below, we have the following relations from the twelve crossings (I omit some tedious calculations):

$$g \equiv 2a - f, \quad h \equiv 2b - a, \quad i \equiv 2c - b, \quad j \equiv 2d - c, \quad k \equiv 2e - d, \quad l \equiv 2f - e$$
$$4(a - f) \equiv b - e, \quad 4(b - a) \equiv c - f, \quad 4(c - b) \equiv d - a$$
$$4(d - c) \equiv e - b, \quad 4(e - d) \equiv f - c, \quad 4(f - e) \equiv a - d$$



Now mod 5, the latter six equivalences simplify to:

$$f - a \equiv b - e, \quad a - b \equiv c - f, \quad b - c \equiv d - a$$
$$c - d \equiv e - b, \quad d - e \equiv f - c, \quad e - f \equiv a - d$$

Which gives $a \equiv c \equiv e$ and $b \equiv d \equiv f$ so we have that the Fox 5-colouring depends only on the choices of colour for a and b and so we have $5^2 = 25$ Fox 5-colourings for Tait #18 and therefore Tait #18 is also not trivial.

Both the Borromean rings and Tait #18 are not trivial, so by Proposition 3.6 we have that neither consists of disjoint perfect circles. □

In the proof of this theorem we have also proved that the Borromean rings and Tait #18 are not equivalent as they have a different number of Fox 5-colourings.

References

[1] Martin Aigner and Gunter M. Ziegler. *Proofs from The Book.* Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.