Lattice Paths and Determinants

Daniel Mulcahy

1 Introduction

We present the proof of an interesting result relating matrix determinants and graphs, and its application both as a tool to prove other identities and to relate certain determinant problems to subject of counting paths in a lattice, following [1] and the talk by Oisín Flynn-Connolly.

2 The Lemma

Let $M = (m_{ij})_{1\le i,j\le n}$ be a real $n \times n$ matrix. We have

$$\det M = \sum_{\sigma\in S_n} m_{1\sigma(1)} \dots m_{n\sigma(n)}$$

Consider a weighted directed bipartite graph with vertex sets $\{A_1, \dots, A_n\}$ and $\{B_1, \dots, B_n\}$, and edges from each $A_i \to B_j$ weighted by m_{ij}.

Definition 2.1. The *weight* $w(P)$ of a path P in a weighted directed graph is the product $\prod_{e\in P} w(e)$ of the weights of the edges in the path.

Definition 2.2. If $\mathcal{A} = \{A_1, \dots, A_n\}$ and $\mathcal{B} = \{B_1, \dots, B_n\}$ are vertices in a directed graph, then a *path system* $\mathcal{P}$ from $\mathcal{A}$ to $\mathcal{B}$ is a collection of paths $P_i : A_i \to B_{\sigma(i)}$ for some $\sigma \in S_n$. We define $\operatorname{sgn}(\mathcal{P}) = \operatorname{sgn}(\sigma)$ and the weight of $\mathcal{P}$

$$w(\mathcal{P}) = w(P_1)w(P_2)\dots w(P_n)$$

For the graph defined above, we have

$$\det M = \sum_{\mathcal{P}:\mathcal{A}\to\mathcal{B}} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})$$

It turns out that this is a special case of a useful general result, originally proven by Ernst Lindström in 1972, and rediscovered in 1985 by Ira Gessel and Gerard Viennot to apply to various combinatiorial problems.

Let $G = (V, E)$ be a finite weighted acyclic directed graph. Let $\mathcal{A} = \{A_1, \dots, A_n\}$ and $\mathcal{B} = \{B_1, \dots, B_n\}$ be subsets of V, not necessarily disjoint.

Definition 2.3. The *path matrix* from $\mathcal{A}$ to $\mathcal{B}$ is the matrix $M = (m_{ij})_{1\le i,j\le n}$ with $m_{ij} = \sum_{P:A_i\to B_j} w(P)$ (summation over all paths).

A path system $\mathcal{P} = (P_1, \dots, P_n)$ is said to be *vertex-disjoint* if the paths P_i are pairwise vertex-disjoint, i.e. have no vertices in common.

Lemma 2.1 (Lindström,Gessel,Viennot)**.** *With $\mathcal{A}, \mathcal{B}, \mathcal{P}, M$ as before we have*

$$\det(M) = \sum_{\substack{\mathcal{P}:\mathcal{A}\to\mathcal{B} \\ \text{vertex-disjoint} \\ \text{path system}}} \operatorname{sgn}(\mathcal{P})w(\mathcal{P})$$

Proof. By grouping the path systems $\mathcal{P}_\sigma$ corresponding to different σ, we find

$$\det(M) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} \sum_{P_i : A_i \to B_{\sigma(i)}} w(P_i) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{\mathcal{P}_\sigma : \mathcal{A} \to \mathcal{B}} w(\mathcal{P}_\sigma) = \sum_{\mathcal{P} : \mathcal{A} \to \mathcal{B}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P})$$

with summation over all path systems.

We therefore must show

$$\sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{not vertex-disjoint}}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) = 0$$

Let N be the set of non-vertex-disjoint path systems. We shall define a bijective map $\pi : N \to N$ satisfying $w(\pi(\mathcal{P})) = w(\mathcal{P})$ and $\operatorname{sgn}(\pi(\mathcal{P})) = -\operatorname{sgn}(\mathcal{P})$. Let $\mathcal{P} = (P_1, \dots, P_n)$ be non-vertex-disjoint. Let i_0 be the minimal i such that P_i shares a vertex with some other P_j. Let Q be the first vertex in the path P_{i_0} that is shared with some P_j. Let $j_0 > i_0$ be the minimal $j \neq i$ such that P_j passes through Q. Thus we have paths

$$\begin{aligned} X &: A_{i_0} \to Q \\ Y &: A_{j_0} \to Q \\ Z &: Q \to B_{\sigma(i_0)} \\ W &: Q \to B_{\sigma(j_0)} \end{aligned}$$

with $P_{i_0} = Z \cdot X$ and $P_{j_0} = W \cdot Y$. Let $P'_{i_0} = W \cdot X : A_{i_0} \to B_{\sigma(j_0)}$ and $P'_j = Z \cdot Y : A_{j_0} \to B_{\sigma(i_0)}$. Then if $P'_k = P_k$ for $i_0 \neq k \neq j_0$, we may define $\pi(\mathcal{P}) = (P'_1, \dots, P'_n)$. This satisfies $w(\pi(\mathcal{P})) = w(\mathcal{P})$, as $w(P'_{i_0}) w(P'_{j_0}) = w(X) w(Y) w(Z) w(W) = w(P_{i_0}) w(P_{j_0})$ and all other factors are the same. We have $P'_k : A_k \to B_{\sigma'(k)}$ where $\sigma'(i_0) = \sigma(j_0)$, $\sigma'(j_0) = \sigma(i_0)$ and $\sigma'(k) = \sigma(k)$ for $i_0 \neq k \neq j_0$. Thus σ' differs from σ by a transposition and $\operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma)$. So

$$\operatorname{sgn}(\pi(\mathcal{P})) = \operatorname{sgn}(\sigma') = -\operatorname{sgn}(\sigma) = -\operatorname{sgn}(\mathcal{P})$$

as required. π is bijective as we see from its construction that $\pi(\pi(\mathcal{P})) = \mathcal{P}$, as i_0, Q, j_0 will be the same for $\pi(\mathcal{P})$ as $\mathcal{P}$

Thus

$$\sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{not v-d}}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) = \sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{not v-d}}} \operatorname{sgn}(\pi(\mathcal{P})) w(\pi(\mathcal{P})) = -\sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{not v-d}}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P})$$

$$\implies \sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{not v-d}}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) = 0$$

$$\therefore \sum_{\substack{\mathcal{P} : \mathcal{A} \to \mathcal{B} \\ \text{vertex-disjoint}}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P}) = \sum_{\mathcal{P} : \mathcal{A} \to \mathcal{B}} \operatorname{sgn}(\mathcal{P}) w(\mathcal{P})$$

□

3 Further Results

This result has various useful applications. Firstly, we may use it to prove the following formula for the determinant of a product of non-square matrices.

Theorem 3.1. *If $P = (p_{ij})$ is an $r \times s$ matrix and $Q = (q_{ij})$ is an $s \times r$ matrix, $r \le s$, then*

$$\det(PQ) = \sum_{\mathcal{Z}} \det(P_{\mathcal{Z}}) \det(Q_{\mathcal{Z}})$$

with summation over subsets $\mathcal{Z} \subset \{1, \ldots, s\}$ of size r, $P_{\mathcal{Z}}$ the $r \times r$ matrix with column-set $\mathcal{Z}$ and $Q_{\mathcal{Z}}$ the $r \times r$ matrix with row-set $\mathcal{Z}$

Proof. We construct a weighted directed bipartite graph between $\mathcal{A} = \{A_1, \ldots, A_r\}$ and $\mathcal{B} = \{B_1, \ldots, B_s\}$ with edges $A_i \to B_j$ having weights p_{ij}. We construct also a graph between $\mathcal{B}$ and $\mathcal{C} = \{C_1, \ldots, C_r\}$ with edges $B_i \to C_j$ having weights q_{ij}. When put together these give a graph G between $\mathcal{A}$ and $\mathcal{C}$ with path matrix $M = PQ$:

$$m_{ij} = \sum_{P:A_i \to C_j} w(P) = \sum_{k=1}^{s} p_{ik} q_{kj} = (PQ)_{ij}$$

since paths $A_i \to C_j$ pass through some B_k.

Let $N_n = \{1, \ldots, n\}$ A vertex-disjoint path system in G consists of the concatenation of a vertex-disjoint path system $\mathcal{P}_1 : \mathcal{A} \to \mathcal{Z}$ for some $\mathcal{Z} \subset \mathcal{B}$ of size r, and $\mathcal{P}_2 : \mathcal{Z} \to \mathcal{C}$. Then $w(\mathcal{P}) = w(\mathcal{P}_1) w(\mathcal{P}_2)$ and $\operatorname{sgn}(\mathcal{P}) = \operatorname{sgn}(\mathcal{P}_1) \operatorname{sgn}(\mathcal{P}_1)$ so the result follows immediately. □

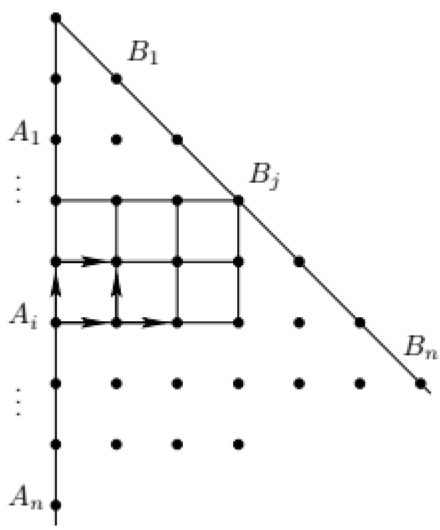
There are many other applications of the lemma to specific problems. For example, one might ask the following question:

Given some integers $a_1 < \cdots < a_n$, $b_1 < \cdots < b_n$ what is the determinant of the matrix with entries $m_{ij} = \binom{a_i}{b_j}$? (if $a < b$ then $\binom{a}{b} = 0$)

Let us consider an $a \times b$ lattice of points. How many ways are there to travel from the bottom left corner to the top right corner by a series of steps north and east between points? We may represent such a journey as a string made up of the letters N and E, and in order to travel from one corner to the other, the string must contain a Es and b Ns. The number of such strings is the number of ways to choose b positions in a string of length $a + b$, i.e. $\binom{a+b}{b}$.

Returning to the original problem, construct a graph whose vertices are integer points in the region of the xy plane between $x = 0$ and $y = -x$ with y negative (cut off at sime sufficiently low level), and whose edges point north or east between adjacent vertices all with weight one as shown below. Let $A_i = (0, -a_i)$ and $B_i = (b_i, -b_i)$. Then the graph between A_i and B_j forms an $(a_i - b_j) \times b_j$ lattice, and the path matrix M between $\mathcal{A}$ and $\mathcal{B}$ is

$$m_{ij} = \sum_{P:A_i \to B_j} 1 = \binom{(a_i - b_j) + b_j}{b_j} = \binom{a_i}{b_j}$$



Vertex-disjoint path systems must go $A_i \to B_i$ (i.e. $\sigma = 1$) otherwise the paths would cross. Thus $\det(M)$ is the number of lattice path systems $(P_i : A_i \to B_i)_{i=1}^n$, so it is always positive and in particular $\det(M) = 0 \iff \exists i : a_i < b_i$.

References

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book.* Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.