ON FRIENDS AND POLITICIANS

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1. Introduction

Here we present the original proof of the following theorem by Erdős, Rényi and Sós:

Theorem 1.1 (Friendship theorem)**.** *Suppose that, in a (finite) group of people, any two people have exactly one common friend. Then there is at least one "politician" who is friends with everybody.*

It is easy to see how one can interpret this in terms of graphs - let each of the people be represented by a vertex and draw an undirected edge between two vertices when the corresponding people are friends (assume no-one is friends with themselves). We say two vertices are "neighbors" or "adjacent" if there is an edge between them. The result can then be restated as follows:

Theorem 1.2 (Friendship theorem - reformulated)**.** *Let G be a graph with n vertices such that any two distinct vertices have exactly one common neighbor (we will refer to this as the "friendship condition"). Then there is at least one vertex (a "politician vertex") which is adjacent to every other.*

It is easy to construct a graph of this form - take, for example, the "windmill graph" shown in figure 1: In fact, we can show that any graph which satisfies our

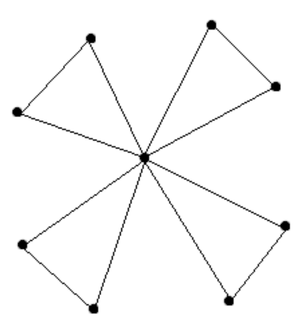

Figure 1. Example of a windmill graph

friendship condition must have this form - and thus must have an odd number of vertices.

Remark 1.3. The qualifiers "finite" and "exactly one" are important here - if we allow infinite graphs, then starting from a five-cycle, we can construct a counterexample by repeatedly add common neighbors for every pair of vertices that do not yet have one. This leads to a countably infinite graph satisfying the friendship condition with no politician.

The statement also fails if we allow more than one common neighbor - consider the four-cycle shown in figure 2

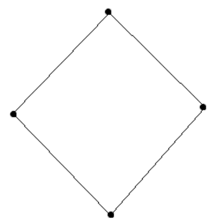

FIGURE 2. A 4-cycle

We follow Cathal Ó'Cléirigh's presentation of the proof, which is based on that of [1].

2. PROOF OF THE FRIENDSHIP THEOREM

2.1. **Outline.** Our proof is a proof by contradiction. We will assume our graph G has no politician vertex and arrive at a contradiction by computing the "degree" of each of its vertices:

Definition 2.1 (Degree of a vertex)**.** The degree of a vertex u in G is the number of vertices adjacent to it.

We proceed as follows:

(1) We show that any two non-adjacent vertices have equal degree.
(2) We extend this to show that all vertices have equal degree k for some $k \in \mathbb{N}$.
(3) We derive a formula for n (the number of vertices in the graph) in terms of k, and show that $k > 2$.
(4) We compute eigenvalues of the adjacency matrix of G
(5) We consider the trace of the adjacency matrix and apply a theorem of Dedekind to obtain a contradiction.

2.2. **Dedekind's theorem.** For our last step, we shall need the following result from number theory, which is interesting in its own right:

Theorem 2.2 (Dedekind, 1858)**.** *Let* $m \in \mathbb{N}$*. Then* $\sqrt{m} \in \mathbb{Q} \implies \sqrt{m} \in \mathbb{N}$.

Proof. Assume that $\sqrt{m} \in \mathbb{Q}$. Let n_0 be the smallest natural number with the property that $n_0\sqrt{m} \in \mathbb{N}$.

If $\sqrt{m} \notin \mathbb{N}$, then there exists some $l \in \mathbb{N}$ with $0 < \sqrt{m} - l < 1$. Let $n = n_0(\sqrt{m} - l)$. Then $n < n_0$, and

$$n\sqrt{m} = n_0\sqrt{m} - n_0 l \in \mathbb{N},$$

a contradiction. $\square$

Step 1 - Non adjacent vertices have equal degree. Take any two non-adjacent vertices u, v in our graph G. Assume u has neighbors $\{w_1, \cdots, w_k\}$ (so $\deg u = k$). Now u and v have exactly one common neighbor; re-labeling the w_i if necessary we can assume that this is w_2. Then w_2 and u must have a unique common neighbor, so w_2 is adjacent to exactly one of the other w_i; assume this is w_1 (again, re-labeling vertices if necessary).

Then v has common neighbor w_2 with w_1. It must also have a unique common neighbor z_i with each w_i for $i \geq 2$. If $z_i = z_j$ for some $i \neq j$ then z_j would have 2 common neighbors w_i and w_j with u, so all the z_i are distinct. Therefore v has at least k neighbors $\{w_2, z_2, z_3, \cdots, z_k\}$ so hence $\deg v \geq \deg u$(see figure 3).

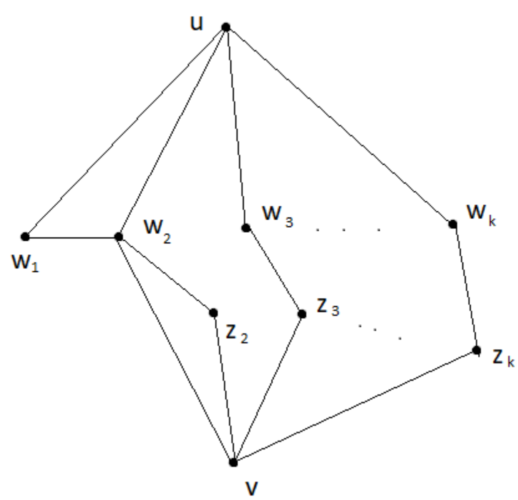

FIGURE 3.

Similarly, we can deduce that $\deg u \geq \deg v$, and hence $\deg u = \deg v = k$.

Note that so far we have not made use of the absence of a politician.

Step 2 - All vertices have equal degree. Consider all neighbors $w_1, \cdots, w_k$ of u from step one. Since all except w_2 are non-adjacent to v, we know by step 1 that all except w_2 have degree k.

Now, since w_2 is not has a politician, it has a non-neighbor x. This x must be non-adjacent to either u or v by the friendship condition, so by step 1 we have $\deg w_2 = \deg x = k$. We thus have shown that all neighbors of u have degree k, and hence k.

Step 3 - Finding a formula for n. Take any vertex u in G. Since every other vertex has a unique common neighbor with u, we can compute the number of vertices in the graph by counting "neighbors of neighbors" of u. By step 2, u and all its neighbors have degree k, so taking the sum of degrees of neighbors of u gives k^2. This counts every vertex exactly once - except for u itself, which is counted k times. Accounting for this we get

$$n = k^2 - k + 1.$$

Looking at this equation, we see that if $k = 1$ or $k = 0$ then $n = 1$ and if $k = 2$ then $n = 3$, so by the friendship condition we get a 3-cycle. Since both of these graphs have politician vertices, so we conclude that $k > 2$.

Remark 2.3. Since $k^2 \equiv k \pmod 2$, this shows that n is odd. It is also easy to prove that n must be odd if G has a politician (see below) so already we have shown that any finite graph satisfying the friendship condition has an odd number of vertices.

Step 4 - Eigenvalues of the adjacency matrix. Label all the vertices $v_1, \cdots, v_n$ of G, and consider the adjacency matrix $A = (a_{ij})$ defined by

$$a_{ij} = \begin{cases} 1 & v_i \text{ adjacent to } v_j \\ 0 & \text{otherwise.} \end{cases}$$

This matrix is obviously symmetric, with 0 on the diagonal (no vertices are adjacent to themselves), and precisely k 1s in each row and column (since the degree of every vertex is k). Also, by the unique common neighbor condition, in any two rows there is exactly one column where both have a 1. Hence

$$A^2 = \begin{pmatrix} k & 1 & \cdots & 1 \\ 1 & k & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & k \end{pmatrix} = (k-1)I_n + J$$

where I_n is the $n \times n$ identity matrix and J is the $n \times n$ matrix consisting entirely of 1s.

It is not hard to guess the eigenvectors of J. We have

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ -1 \\ \vdots \\ n-1 \end{pmatrix} = 0$$

and permuting the entries of this vector we quickly get $n-1$ linearly independent eigenvectors of A with eigenvalue 0. Also

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n \cdot \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

so J also has an eigenvalue n of multiplicity 1. It immediately follows that A has eigenvalues $k-1$ of multiplicity $n-1$ and $n+k-1$ of multiplicity 1. Recall from step 3 that $n + k - 1 = k^2$.

Now A is symmetric, so it is diagonalizable to a matrix whose (diagonal) entries are the eigenvalues of A. The same is true of A^2, and therefore the eigenvalues of A^2 are squares of the eigenvalues of A. Hence A has eigenvalues $\pm k$ of multiplicity 1 (multiplying A with a vector of 1s we see that the eigenvalue is in fact k), and eigenvalues $\sqrt{k-1}$ of multiplicity r and $-\sqrt{k-1}$ of multiplicity s, for some r, s with $r + s = n - 1$.

Step 5. Since the trace of A is 0, and the trace is the sum of the eigenvalues, we have

$$k + (r - s)\sqrt{k-1} = 0.$$

Since $k > 2$ we conclude $r \neq s$ and hence

$$\sqrt{k-1} = \frac{k}{s-r}$$

so $\sqrt{k-1} \in \mathbb{N}$ by Dedekind's theorem. Setting $h = \sqrt{k-1}$ we get

$$k = h^2 + 1$$

and re-writing our equation for the trace of A,

$$k = h(s - r).$$

so h divides $h^2 + 1$. Hence $h = 1$ and therefore $k = 2$, which we have proved to be impossible.

3. The emergence of windmills

Once we have exposed the politician, it is not hard to see what form the graph must take. Let p represent the politician vertex, and u be any other vertex in the graph. p and u must have a unique common neighbor v. If v is adjacent to any other vertex x then p is also adjacent to x, since p is a politician, and so v and p will have two common neighbors u and x, a contradiction.

Therefore the only neighbors of v are p and u, and likewise the only neighbors of u are p and v. We end up with a "windmill" similar to figure 1

Since every vertex $u \neq p$ can be paired off with another vertex u_1 in this way, the graph must have an odd number of vertices.

4. Extension of the problem

Definition 4.1 (Path)**.** We define a path in G to be a sequence of (not necessarily distinct) vertices $v_1, v_2, \cdots$ in G such that every v_i is adjacent to v_{i+1} for every i. We define the length of the path to be the length of the sequence.

It is clear that our friendship condition is equivalent to saying that there is a unique path of length $l = 2$ between any two vertices in G. We have shown that the only possible graph satisfying this condition is a windmill.

It has been conjectured by Anton Kotzig that there are no finite graphs which satisfy this condition for any $l > 2$. Kotzig's conjecture has been verified for $l \leq 33$, but the situation for arbitrary l remains something of a mystery.

References

[1] Martin Aigner and Günter M. Ziegler. *Proofs from The Book*. Springer-Verlag, Berlin, fifth edition, 2014. Including illustrations by Karl H. Hofmann.