MA341D Answers and solutions to homework assignment 1

1. (a) The leading monomials of these are x^2 and x^5yz^4, respectively, with coefficients 1 and -3, so the S-polynomial is

$$x^3yz^4(2x+3y+z+x^2-z^2+z^3)-\frac{1}{-3}(2x^2y^8-3x^5yz^4+xyz^3-xy^4)=$$
$$\frac{2}{3}x^2y^8+\frac{1}{3}xyz^3-\frac{1}{3}xy^4+2x^4yz^4+3x^3y^2z^4+x^3yz^5-z^3yz^6+x^3yz^7.$$

(b) The leading monomials of these are z^3 and x^5yz^4, respectively, with coefficients 1 and -3, so the S-polynomial is

$$x^5yz(2x+3y+z+x^2-z^2+z^3)-\frac{1}{-3}(2x^2y^8-3x^5yz^4+xyz^3-xy^4)=$$
$$\frac{2}{3}x^2y^8+\frac{1}{3}xyz^3-\frac{1}{3}xy^4+2x^6yz+3x^5y^2z+x^5yz^2+x^7yz-x^5yz^3.$$

2. In this question, we just give answers for the reduced Gröbner bases; these should serve you for checking your computations.

(a) $xy^2z-xyz,\ x^2y^2-z,\ x^2yz-z^2,\ yz^2-z^2,\ x^2z^2-z^3.$

(b) $x^2y+xz+y^2z,\ xz^2-yz,\ xyz-y^2,\ y^3+y^2z^3+yz^2,\ xy^2+y^2z^2+yz.$

(c) $x^2y-y,\ y^2+\frac{1}{2}z,\ z^2+\frac{1}{2}z,\ xz+z.$

3. (a) Let us use the `lex` ordering with $z>x>y$. Then the leading monomials of the generators of the ideal are z and x^3, which have no common divisors, so they form a Gröbner basis. The normal form of $xy^3-z^2+y^5-z^3$ can be computed by following steps of long division:

$$xy^3-z^2+y^5-z^3\to xy^3-z^2+y^5-z^3-z^2(x^2y-z)=$$
$$=xy^3-z^2+y^5-z^2x^2y\to xy^3-z^2+y^5-z^2x^2y-x^2yz(x^2y-z)=xy^3-z^2+y^5-x^4y^2z\to$$
$$\to xy^3-z^2+y^5-x^4y^2z-x^4y^2(x^2y-z)=xy^3-z^2+y^5-x^6y^3\to xy^3-z^2+y^5-x^6y^3-z(x^2y-z)=$$
$$=xy^3+y^5-x^6y^3-x^2yz\to xy^3+y^5-x^6y^3-x^2yz-x^2y(x^2y-z)=xy^3+y^5-x^6y^3-x^4y^2\to$$
$$\to xy^3+y^5-x^6y^3-x^4y^2-x^3y^3(-x^3+y)=xy^3+y^5-x^4y^2-x^3y^4\to$$
$$\to xy^3+y^5-x^4y^2-x^3y^4-y^4(-x^3+y)=xy^3-x^4y^2\to xy^3-x^4y^2-xy^2(-x^3+y)=0.$$

We see that the normal form is zero, so the polynomial is in the ideal.

(b) Let us use the `lex` ordering with $y>x>z$. Computing a Gröbner basis of the ideal in question, we get $y-z, 2z^2+z, xz-z$. The normal form of x^3z-2y^2 can be computed by following steps of long division:

$$\begin{aligned}x^3z-2y^2 &\to x^3z-2y^2-x^2(xz-z)=x^2z-2y^2\to x^2z-2y^2-x(xz-z)=\\ &=xz-2y^2\to xz-2y^2-(xz-z)=z-2y^2\to z-2y^2+2y(y-z)=\\ &=z-2yz\to z-2yz+2z(y-z)=z-2z^2\to z-2z^2+(2z^2+z)=2z.\end{aligned}$$

We see that the normal form is nonzero, so the polynomial is not in the ideal. The normal form is the normal monomial z with the coefficient 2.

4. Let us compute the reduced Gröbner basis for the ideal generated by p^5-n, $p^{10}-d$, $p^{25}-q$ in $\mathbb{C}[p,n,d,q]$ with respect to the `glex` order. That Gröbner basis consists of the polynomials p^5-n, nd^2-q, d^3-nq, and n^2-d (we omit the calculations). Note that when computing the normal form relative to that Gröbner basis using iterated long division, every monomial is at each stage is replaced by a monomial of a smaller degree, thus the exponents of the normal form of p^n give the most economic way to pay n cents using the given coins. Performing the long division for p^{167}, we obtain the normal form p^2ndq^6, meaning we should use two pennies, a nickel, a dime and six quarters.