

Problem Solving

Set A01

2 October 2012

1. For every positive integer n , let $p(n)$ denote the number of ways to express n as a sum of positive integers. For instance, $p(4) = 5$ because $4 = 3 + 1 = 2 + 2 = 2 + 1 + 1 = 1 + 1 + 1 + 1$. Also define $p(0) = 1$. Prove that $p(n) - p(n-1)$ is the number of ways to express n as a sum of integers each of which is strictly greater than 1.
2. Consider a polynomial

$$f(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1x + a_0.$$

Albert Einstein and Homer Simpson are playing the following game. In turn, they choose one of the coefficients a_0, \dots, a_{2011} and assign a real value to it. Albert has the first move. Once a value is assigned to a coefficient, it cannot be changed any more. The game ends after all the coefficients have been assigned values. Homers goal is to make $f(x)$ divisible by a fixed polynomial $m(x)$ and Alberts goal is to prevent this.

- (a) Which of the players has a winning strategy if $m(x) = x - 2012$?
 - (b) Which of the players has a winning strategy if $m(x) = x^2 + 1$?
3. Let n be a fixed positive integer. Determine the smallest possible rank of an $n \times n$ matrix that has zeros along the

main diagonal and strictly positive real numbers off the main diagonal.

4. A number is said to be triangular if it is of the form $n(n + 1)/2$, ie 0, 1, 3, 6, 10, 15, Show that n is the sum of two triangular numbers if and only if $8n + 2$ is the sum of two squares.

5. Apart from (2, 1, 1) does there exist any triplet (x, y, z) such that

$$x! = y! + z!?$$

6. If you can use circles of any size (as many as you like), is there a limit to the density with which you can fill a circle (or other shape)?

[Obviously you can always invrease the density by putting in circles small enough for the remaining gaps, but is the limit of the density 1?]

7. Show that at any time there is a point on the earth where the wind is not blowing.
8. What is the point of intersection and angle between two straight line segments of fixed (and different) lengths when you want to maximize the perimeter of the quadrilateral formed by using them as the diagonals?