Solutions to AMM problems 11637 and 11641

TCDmath problem group
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1 AMM problem 11637

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Let $m \geq 1$ be a non-negative integer. Let $\{u\} = u - \lfloor u \rfloor$; the quantity $\{u\}$ is called the *fractional part* of u. Prove that

$$\int_0^1 \left\{ \frac{1}{x} \right\}^m x^m \, dx = 1 - \frac{1}{m+1} \sum_{k=1}^{m} \zeta(k+1).$$

Answer: *Substituting* $\{1/x\} = 1/x - \lfloor 1/x \rfloor$,

$$I = \int_0^1 dx - \int_0^1 (1/x - \lfloor 1/x \rfloor)^m x^m \, dx$$

We divide the interval $[0, 1]$ *according to the value of* $\lfloor 1/x \rfloor$. *We have*

$$\lfloor 1/x \rfloor = n \iff n \leq \frac{1}{x} < n+1 \iff \frac{1}{n+1} < x \leq \frac{1}{n}.$$

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Thus

$$
\begin{aligned}
I &= \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} (1/x - n)^m x^m dx \\
&= \sum_{n=1}^{\infty} \int_{1/(n+1)}^{1/n} (1 - nx)^m dx \\
&= \sum_{n=1}^{\infty} \left[-\frac{(1-nx)^{m+1}}{n(m+1)} \right]_{(1/(n+1)}^{1/n} \\
&= \frac{1}{m+1} \sum_{n=1}^{\infty} \frac{(1 - n/(n+1))^{m+1}}{n} \\
&= \frac{1}{m+1} \sum_{n=1}^{\infty} \frac{(1/(n+1))^{m+1}}{n} \\
&= \frac{1}{m+1} \sum_{n=1}^{\infty} \frac{(n+1)^{-(m+1)}}{n}.
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
S &= \sum_{k=1}^{m} \zeta(k+1) \\
&= \sum_{k=1}^{m} \sum_{n=1}^{\infty} n^{-(k+1)} \\
&= \sum_{n=1}^{\infty} \sum_{k=1}^{m} n^{-(k+1)} \\
&= \sum_{n=1}^{\infty} F(n),
\end{aligned}
$$

where $F(1) = m$ *while if* $n > 1$

$$
\begin{aligned}
F(n) &= \frac{1}{n^2} + \frac{1}{n^3} + \cdots + \frac{1}{n^{m+1}} \\
&= \frac{1}{n^2} \frac{1 - 1/n^m}{1 - 1/n} \\
&= \frac{1}{n(n-1)} - \frac{n^{-(m+1)}}{n-1}.
\end{aligned}
$$

Noting that

$$
F(n+1) = \frac{1}{n(n+1)} - \frac{(n+1)^{-(m+1)}}{n},
$$

we see that

$$
I = \frac{1}{m+1} \sum_{n=1}^{\infty} \left(\frac{1}{n(n+1)} - F(n+1) \right)
$$

Now

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots$$
$$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$$
$$= 1.$$

Thus

$$I = \frac{1}{m+1}\left(1 - (S - F(1)\right)$$
$$= \frac{1}{m+1}\left(1 - S + m\right)$$
$$= 1 - \frac{1}{m+1} S.$$

2 AMM problem 11641

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Let f be a convex function from $\mathbb{R}$ into $\mathbb{R}$ and suppose $f(x+y) + f(x-y) - 2f(x) \le y^2$ for all real x and y.

1. Show that f is differentiable.

2. Show that for all real x and y,

$$|f'(x) - f'(y)| \le |x - y|.$$

Answer:

1. *Let* $P = (x, f(x)),\ Q(y) = (x+y, f(x+y)),\ R(y) = (x-y, f(x-y))$.

   The slope $(f(x+y) - f(x))/y$ *of the line* $PQ(y)$ *is non-increasing as* y *decreases to 0. For suppose* $z \in (0, y)$*, and suppose the slope of* $PQ(z)$ *is greater than the slope of* $PQ(y)$*. Then the point* $Q(z)$ *lies above the line-segment* $PQ(y)$*, contradicting the definition of convexity.*

   Similarly the slope $(f(x) - f(x-y))/y$ *of the line* $R(y)P$ *is non-decreasing as* y *decreases to 0.*

   It follows that $((f(x+y) - f(x))/y,\ ((f(x) - f(x-y))/y$ *converge to limits* L, M *as* y *decreases to 0.*

   But

$$0 \le \frac{f(x+y) - f(x)}{y} - \frac{f(x) - f(x-y)}{y}$$
$$= \frac{f(x+y) + f(x-y) - 2f(x)}{y} \le y.$$

(The inequality on the left follows from the convexity of f; the inequality on the right from the condition laid down in the question.)

On letting y tend to 0, it follows that $L = M$. Hence

$$\frac{f(x+y)-f(x)}{y} \to L = M \text{ as } y \to 0,$$

ie $f(x)$ is differentiable at x with derivative $L = M$.

2. *As we saw above,*

$$0 \le \frac{f(x+y)-f(x)}{y} - \frac{f(x)-f(x-y)}{y} \le y.$$

Replacing x by $x+y$,

$$0 \le \frac{f(x+2y)-f(x+y)}{y} - \frac{f(x+y)-f(x)}{y} \le y.$$

Adding

$$0 \le \frac{f(x+2y)-f(x+y)}{y} - \frac{f(x)-f(x-y)}{y} \le 2y.$$

Continuing in this way, replacing x successively by $x+2y, x+3y, \ldots, x+ny$, and adding,

$$0 \le \frac{f(x+(n+1)y)-f(x+ny)}{y} - \frac{f(x)-f(x-y)}{y} \le ny.$$

Writing z for ny,

$$0 \le \frac{f(x+z+y)-f(x+z)}{y} - \frac{f(x)-f(x-y)}{y} \le z.$$

Letting y tend to 0,

$$0 \le f'(x+z) - f'(x) \le z,$$

which is what we have to prove.

This argument seems to assume that z is a multiple of y. However, if we are given z, we can take $y = z/n$ and then let $n \to \infty$, and the argument holds.