AMM problems October 2012, due before 28 February 2013

January 22, 2013

116663. *Proposed by Eugen J. Ionascu.* The unit interval is broken at two randomly chosen points along its length. Show that the probability that the lengths of the resulting 3 intervals are the heights of a triangle is equal to

$$\frac{12\sqrt{5}\log((3+\sqrt{5})/2)}{25} - \frac{4}{5}.$$

11664. *Proposed by Cosmin Pohoata and Darij Grinberg.* Let a, b, and c be the side lengths of a triangle. Let s denote the semiperimeter, r the inradius, and R the circumradius of that triangle. Let $a' = s - a$, $b' = s - b$, and $c' = s - c$.
(a) Prove that $\frac{ar}{R} \leq \sqrt{b'c'}$.
(b) Prove that

$$\frac{r(a+b+c)}{R}\left(1 + \frac{R-2r}{4R+r}\right) \leq 2\left(\frac{b'c'}{a} + \frac{c'a'}{b} + \frac{a'b'}{c}\right).$$

11665. *Proposed by Raitis Ozols, student.* Let $a = (a_1, \ldots, a_n)$, where $n \geq 2$ and each a_j is a positive real number. Let $S(a) = {a_1}^{a_2} + \ldots + {a_{n-1}}^{a_n} + {a_n}^{a_1}$.
(a) Prove that $S(a) > 1$.
(b) Prove that for all $\epsilon > 0$ and $n \geq 2$ there exists a of length n with $S(a) < 1 + \epsilon$.

11666. *Proposed by Dmitry G. Fon-Der-Flaass and Max. A. Alekseyev.* Let m be a positive integer, and let A and B be nonempty subsets of $\{0, 1\}^m$. Let n be the greatest integer such that $|A| + |B| > 2^n$. Prove that $|A + B| \geq 2^n$. (Here, $|X|$ denotes the number of elements in X, and $A + B$ denotes $\{a + b : \ a \in A, b \in B\}$, where addition of vectors is componenentwise modulo 2.)

11667. *Proposed by Cezar Lupu and Dan Schwarz.* Let f, g, and h be elements of an inner product space over $\mathbb{R}$, with $\langle f, g\rangle = 0$.
(**a**) Show that

$$\langle f, f\rangle\langle g, g\rangle\langle h, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

(**b**) Show that

$$(\langle f, f\rangle\langle h, h\rangle)\langle h, f\rangle^2 + (\langle g, g\rangle\langle h, h\rangle)\langle g, h\rangle^2 \geq 4\langle g, h\rangle^2\langle h, f\rangle^2.$$

11668. *Proposed by Dimitris Stathopoulos.* For positive integer n and $i \in \{0, 1\}$, let $D_i(n)$ be the number of derangements on n elements whose number of cycles has the same parity as i. Prove that $D_1(n) - D_0(n) = n - 1$.

11669. *Proposed by Herman Roelants.* Prove that for all $n \geq 4$ there exist integers $x_1, \ldots, x_n$ such that

$$\frac{x_{n-1}^2 + 1}{x_n^2} \prod_{k=1}^{n-2} \frac{x_k^2 + 1}{x_k^2} = 1$$

satisfying the following conditions: $x_1 = 1$, $x_{k-1} < x_k < 3x_{k-1}$ for $2 \leq k \leq n-2$, $x_{n-2} < x_{n-1} < 2x_{n-2}$, and $x_{n-1} < x_n < 2x_{n-1}$.