AMM problems December 2012, due before 30 April 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

March 27, 2013

11677. *Proposed by Albert Stadler.* Evaluate

$$\prod_{n=1}^{\infty}\left(1+2e^{-m\sqrt{3}}\cosh(mn/\sqrt{3})\right).$$

11678. *Proposed by Farrukh Ataev Rakhimjanovich.* Let F_k he the kth Fibonacci number, where $F_0 = 0$ and $F_1 = 1$. For $n \geq 1$ let A_n be an $(n+1)\times(n+1)$ matrix with entries $a_{j,k}$ given by $a_{0,k} = a_{k,0} = F_k$ for $0 \leq k \leq n$ and by $a_{j,k} = a_{j-1,k} + a_{j,k-1}$ for $j, k \geq 1$. Compute the determinant of A_n.

11679. *Proposed by Tim Keller.* Let n be an integer greater than 2, and let $a_2, \ldots, a_n$ be positive real numbers with product 1. Prove that

$$\prod_{k=2}^{n}(1+a_k)^k > \frac{2}{e}\left(\frac{n}{2}\right)^{2n-1}.$$

11680. *Proposed by Benjamin Bogoşel and Cezar Lupu.* Let $x_1, \ldots, x_n$ be nonnegative real numbers. Show that

$$\left(\sum_{i=1}^{n}\frac{x_i}{i}\right)^4 \leq 2\pi^2\sum_{i,j=1}^{n}\frac{x_ix_j}{i+j}\sum_{k,\ell=1}^{n}\frac{x_kx_\ell}{(k+\ell)^3}.$$

11681. *Proposed by Des MacHale.* For any group G, let $\operatorname{Aut} G$ denote the group of automorphisms of G.
(a) Show there is no finite group G with $|\operatorname{Aut} G| = |G| + 1$.
(b) Show that there are infinitely many finite gropus G with $|\operatorname{Aut} G| = |G|$.
(c) Find all finite groups G with $|\operatorname{Aut} G| = |G| - 1$.

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11682. *Proposed by Ovidiu Furdui.* Compute

$$\sum_{n=0}^{\infty}(-1)^n\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{n+k}\right)^2.$$

11683. *Proposed by Raimond Struble.* Given a triangle ABC, let F_C be the foot of the altitude from the incenter to AB. Define F_B and F_C similarly. Let C_A be the circle with center A that passes through F_B and F_C, and define C_B and C_C similarly. The *Gergonne point* of a triangle is the point at which segments $AF_A, BF_B,$ and CF_C meet. Determine, up to similarity, all isosceles triangles such that the Gergonne point of the triangle lies on one of the circles C_A, C_B, or C_C.