AMM problems January 2013, due before 31 May 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

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11684. *Proposed by Raymond Mordini, and Rudolf Rupp.* For complex a and z, let $\phi_a(z) = (a-z)/(1-\overline{a}z))$ and $\rho(a,z) = |a-z|/|1-\overline{a}z)|$.
(a) Show that whenever $-1 < a, b < 1$,

$$\max_{|z|\le 1|} |\phi_a(z) - \phi_b(z)| = 2\rho(a,b), \quad \text{and}$$
$$\max_{|z|\le 1|} |\phi_a(z) + \phi_b(z)| = 2.$$

(b) For complex α, β with $|\alpha| = |\beta| = 1$, let

$$m(z) = m_{a,b,\alpha,\beta}(z) = |\alpha\phi_a(z) - \beta\phi_b(z)|.$$

Determine the maximum and minimum, taken over z with $|z| = 1$, of $m(z)$.

11685. *Proposed by Donald Knuth.* Prove that

$$\prod_{k=0}^{\infty}\left(1 + \frac{1}{2^{2^k}-1}\right) = \frac{1}{2} + \sum_{k=0}^{\infty} \frac{1}{\prod_{j=0}^{k-1}\left(2^{2^j}-1\right)}.$$

In other words, prove that

$$(1+1)(1+\frac{1}{3})(1+\frac{1}{15})(1+\frac{1}{255})\ldots = \frac{1}{2} + 1 + 1 + \frac{1}{3} + \frac{1}{3\cdot 15} + \frac{1}{3\cdot 15\cdot 255} + \ldots.$$

11686. *Proposed by Michel Bataille.* Let x, y, z be positive real numbers such that $x + y + z = \pi/2$. Prove that

$$\frac{\cot x + \cot y + \cot z}{\tan x + \tan y + \tan z} \ge 4(\sin^2 x + \sin^2 y + \sin^2 z).$$

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11687. *Proposed by Steven Finch.* Let T be a solid torus in $\mathbb{R}^3$ with center at the origin, tube radius 1 and spine radius r with $r \geq 1$ (so that T has volume $\pi \cdot 2\pi r$). Let P be a 'random' nearby plane. Find the conditional probability that, given that P meets T, that the intersection is simply connected. For what value of r is this probability maximal? (The plane is chosen by first picking a distance form teh origin uniformly between 0 and $1 + r$, and then picking a normal vector independently and uniformly on the unit sphere.)

11688. *Proposed by Samuel Alexander.* Consider $f : \mathbb{N}^3 \to \mathbb{N}$ such that $\lim_{a\to\infty} \inf_{b,c,d\in\mathbb{N},b<a}(f(a, c, d) - f(b, c, d)) = \infty$. Show that for $B \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that

$$f(a, c, d) = k \Rightarrow \max(c, d) > B.$$

11689. *Proposed by Yagub N. Aliyev.* Two circles w_1 and w_2 intersect at distinct points B and C and are internally tangent to a third circle w at M and N, respectively. Line BC intersects w at A and D, with A nearer B than C. Let r_1 and r_2 be the radii of w_1 and w_2, respectively, with $r_1 \leq r_2$. Let $u = \sqrt{|AC| \cdot |BD|}$ and $v = \sqrt{|AB| \cdot |CD|}$. Prove that

$$\frac{u - v}{u + v} < \sqrt{\frac{r_1}{r_2}}.$$

11690. Proposed by Pál Péter Dályay. Let M be a point in the interior of a convex polygon with vertices $A_1, \ldots, A_n$ in order. For $1 \leq i \leq n$, let r_i be the distance from M to A_i, and let R_i be the radius of the circumcircle of triangle MA_iA_{i+1}, where $A_{n+1} = A_1$. Show that

$$\sum_{i=1}^{n} \frac{R_i}{r_i + r_{i+1}} \geq \frac{n}{4\cos(\pi/n)}.$$