AMM problems February 2013, due before 30 June 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

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11691. *Proposed by M.L. Glasser.* Show that the $2n$th moment $\int_0^\infty x^{2n} f(x)dx$ of the function f given by

$$f(x) = \frac{d}{dx} \arctan\left(\frac{\sinh x}{\cos x}\right)$$

is zero when n is an odd positive integer.

11692. *Proposed by Cezar Lupu and Ştefan Spătaru.* Let a_1, a_2, a_3, a_4 be real numbers in $(0, 1)$ with $a_4 = a_1$. Show that

$$\frac{3}{1 - a_1a_2a_3} + \sum_{k=1}^{3} \frac{1}{1 - a_k^3} \geq \sum_{k=1}^{3} \frac{1}{1 - a_k^2 a_{k+1}} + \frac{1}{1 - a_k a_{k+1}^2}.$$

11693. *Proposed by Eugen Ionascu and Richard Strong.* Let T be an equilateral triangle inscribed in the d-dimensional unit cube $[0, 1]^d$, with $d \geq 2$. As a function of d, what is the maximum possible side-length of T?

11694. *Proposed by Kent Holing.* Let $g(x) = x^4 + ax^3 + bx^2 + ax + 1$, where a and b are rational. Suppose g is irreducible over $\mathbb{Q}$. Let G be the Galois group of g. Let $\mathbb{Z}_4$ denote the additive group of the integers mod 4, and let D_4 be the dihedral group of order 8. Let $\alpha = (b + 2)^2 - 4a^2$ and $\beta = a^2 - 4b + 8$.
(a) Show that G is isomorphic to one of $\mathbb{Z}_4$ or D_4 if and only if neither α nor β is the square of a rational number, and G is cyclic exactly when $\alpha\beta$ is the square of a rational number.
(b) Suppose neither α nor β is square, but $\alpha\beta$ is. Let r be one of the roots of g. (Trivally, $1/r$ is also a root.) Let $s = \sqrt{\alpha\beta}$, and let

$$t = ((s + (b - 6)a)r^3 + (as + (b - 8)a^2 + 4(b + 2))r^2+ \\ ((b - 1)s + (b^2 - b + 2)a - 2a^3)r + 2(b + 2)b - 6a^2)/(2s).$$

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Show that $t \in \mathbb{Q}[r]$ is one of the other two roots of g. Comment on the special case $a = b = 1$.

11695. *Proposed by Ovidiu Furdui.* The *Stirling numbers of the first kind,* denoted $s(n,k)$, can be defined by their generating function: $z(z-1)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k)z^k$. Let m and p be nonnegative integers with $m > p-4$. Prove that

$$\int_0^1 \int_0^1 \frac{\log x \cdot \log^m(xy) \cdot \log y}{(1-xy)^p} dxdy$$

$$= \begin{cases} (-1)^m \frac{1}{6}(m+3)!\zeta(m+4), & \text{if } p = 1; \\ (-1)^{m+p-1} \frac{(m+3)!}{6(p-1)!} \sum_{k=1}^{p-1} (-1)^k s(p-1,k)\zeta(m+4-k) & \text{if } p > 1. \end{cases}$$

11696. *Proposed by Enkel Hysnelaj and Elton Bojaxhiu.*

Let T be a triangle with sides of length $a, b, c,$ inradius r, circumradius R, and seimiperimeter p. Show that

$$\frac{1}{2(r^2+4Rr)} + \frac{1}{9}\sum_{\text{cyc}} \frac{1}{c(p-c)} \geq \frac{4}{9}\sum_{\text{cyc}} \left(\frac{1}{9Rr - c(p-c)}\right).$$

11697. *Proposed by Moubinool Omarjee.* Let n and q be integers, with $2n > q \geq 1$. Let

$$f(t) = \int_{\mathbb{R}^q} \frac{e^{-t(x_1^{2n}+\ldots+x_q^{2n})}}{1+x_1^{2n}+\ldots x_q^{2n}} dx_1 \cdots dx_q.$$

Prove that $\lim_{t\to\infty} t^{q/2n} f(t) = n^{-q}(\Gamma(1/2n))^q$.