AMM problems May 2013, due before 30 September 2013

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

July 1, 2013

11705. *Proposed by John Loase.* Let $C(n)$ be the number of distinct multisets of two or more primes that sum to n. Prove that $C(n+1) \geq C(n)$ for all n. (For instance, $C(4) = 1$, $C(5) = 1$, and $C(6) = 2$).

11706. *Proposed by Nguyen Thanh Binh.* Let ABC and DEF be triangles in the plane.
(a) Provide a compass and straightedge construction, which may use ABC and DEF, of a triangle $A'B'C'$ that is similar to ABC and circumscribes DEF.
(b) Among all triangles $A'B'C'$ of the sort described in part (a), determine which one has the greatest area and which one has the greatest perimeter.

11707. *Proposed by José Luis Palacios.* For $N \geq 1$, consider the following random walk on the $(N+1)$-cycle with vertices $0, 1, \ldots, N$. The walk begins at vertex 0 and continues until every vertex has been visited and the walk returns to vertex 0. Prove that the expected number of visits to any vertex other than 0 is $\frac{1}{3}(2N+1)$.

11708. *Proposed by James W. Moeller.* Let $\langle E_n \rangle$ and $\langle P_n \rangle$ be two sequences of distinct orthogonal projections on an infinite-dimensional Hilbert space H, whose ranges are finite-dimensional and satisfy the *intersection property*

$$\mathrm{Ran} E_n \cap (\mathrm{Ran} P_n)^\perp = \{O\} = \mathrm{Ran} P_n \cap (\mathrm{Ran} E_n)^\perp.$$

Such sequences are *strongly uncorrelated* if $\langle E_n \rangle$ converges strongly to O while $\langle P_n \rangle$ converges strongly to I. (A sequence $\langle L_n \rangle$ of bounded linear operators on a Hilbert space H converges strongly to L if $L_n x \to Lx$ for all $x \in H$.)

Show that the set of strongly uncorrelated sequences of projections is nonempty.

*This group involves students and staff of the Department of Mathematics, Trinity College, Dublin. Please address correspondence either to Timothy Murphy (`tim@maths.tcd.ie`), or Colm Ó Dúnlaing (`odunlain@maths.tcd.ie`).

11709. *Proposed by Moubinool Omarjee.* Find

$$\int_{x=0}^{\infty} \frac{1}{x} \int_{y=0}^{x} \frac{\cos(x-y)-\cos(x)}{y} dy\, dx.$$

11710. Proposed by B. Voorhees. Let n, k, and r be positive numbers such that $n \geq k+1$ and $r \geq 1$. Show that

$$r^{n+k} - 1 \geq \frac{(kr+n)(nr+k)}{(n-k)^2} \left(1 - \left(\frac{kr+n}{nr+k} \right)^{n-k} \right).$$

11711. *Proposed by J.A. Grzesik.* Show , for integers n and k with $n \geq 2$ and $1 \leq k \leq n$, that

$$(-1)^{n-k} \binom{n}{k} k \sum_{j=1, j\neq k}^{n} \frac{1}{k-j} = -\sum_{j=1, j\neq k}^{n} (-1)^{n-j} \binom{n}{j} \frac{j}{k-j}.$$