AMM problems October 2013, due before 28 February 2014

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

November 16, 2013

11726. *Proposed by Stephen Scheinberg.* Let K be Cantor's middle-third set. Let $K^* = K \times \{0\}$. Is there a function F from $\mathbb{R}^2$ to $\mathbb{R}$ such that

1. For each $x \in \mathbb{R}$, the function $t \mapsto F(x,t)$ is continuous on $\mathbb{R}$,
2. for each $y \in \mathbb{R}$, the function $s \mapsto F(s,y)$ is continuous on $\mathbb{R}$, and
3. F is continuous on the complement of K^* and discontinuous on K^*?

11727. *Proposed by Nguyen Thanh Binh.* Let R be a circle with center O. Let R_1 and R_2 be circles with centers O_1 and O_2 inside R, such that R_1 and R_2 are externally tangent and both are internally tangent to R. Give a straightedge and compass construction of the circle R_3 that is internally tangent to R and externally tangent to R_1 and R_2.

11728. *Proposed by Walter Blumberg.* Let p be a prime congruent to 7 mod 8. Prove that

$$\sum_{k=1}^{p} \left\lfloor \frac{k^2+k}{p} \right\rfloor = \frac{2p^2+3p+7}{6}.$$

11729. *Proposed by Vassilis Papanicolaou.* An integer n is called *b-normal* if all digits $0, 1, \ldots, b-1$ appear the same number of times in the base-b expansion of n. Let $\mathcal{N}_b$ be the set of all b-normal integers. Determine those b for which

$$\sum_{n \in \mathcal{N}_b} \frac{1}{n} < \infty.$$

11730. *Proposed by Mircea Merca.* Let p be the *partition function* (counting the ways to write n as a sum of positive integers), extended so that $p(0) = 1$ and $p(n) = 0$ for $n < 0$. Prove that

$$\sum_{k=0}^{\infty} \sum_{j=0}^{2k} (-1)^k p\left(n - \frac{k(3k+1)}{2} - j\right) = 1.$$

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11731. *Proposed by Meijie Ma and Douglas West.* The *integer simplex* with dimension d and side-length m is the graph T_m^d whose vertices are the nonnegative integer $(d+1)$-tuples summing to m, with two vertices adjacent when they differ by 1 in two places and are equal in all other places. Determine the connectivity, the chromatic number, and the edge-chromatic number of T_m^d (the latter when $m > d$).

11732. *Proposed by Marcel Chirita.* Let a and b be real, with $1 < a < b$, and let m and n be real with $m \neq 0$. Find all continuous functions f from $[0, \infty)$ to $\mathbb{R}$ such that for $x \geq 0$,

$$f(a^x) + f(b^x) = mx + n.$$