AMM problems November 2013, due before 31 March 2014

TCDmath problem group
Mathematics, Trinity College, Dublin 2, Ireland*

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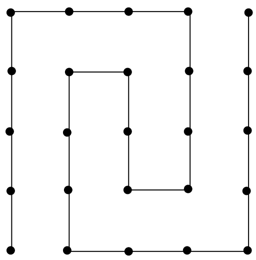

11733. *Proposed by Donald Knuth.* Let $V = \{0, 1, 2, 3, 4\}^2$. Say that nonnegative integers a and b are *adjacent* when their base-5 expansions $\ldots a_2a_1a_0$ and $\ldots b_2b_1b_0$ satisfy the condition that if $i > j \geq 0$ and $(a_i, a_j) \neq (b_i, b_j)$ then (a_i, a_j) and (b_i, b_j) are consecutive in the path through V shown at right (horizontal coordinate listed first). Thus, for example, 0 is adjacent to 1. Similarly, 48 (expansion 143_5) is adjacent to 47 (expansion 142_5) and 73 (expansion 243_5).

(a) Prove that every positive integer is adjacent to exactly two nonnegative integers.

(b) Prove that with this definition of adjacency, the nonnegative integers form a path $\langle x_0, x_1, x_2, \ldots\rangle$ starting with $x_0 = 0$.

(c) Explain how to compute efficiently from n the number x_n that comes n steps after 0, and determine $x_{1,000,000}$.

11734. *Proposed by Vahagn Aslanyan.* Find all lists (a, k, m, n) of positive integers such that

$$a^{m+n} + a^n - a^m - 1 = 15^k.$$

11735. *Proposed by Cosmin Pohoata.* Let P be a point inside triangle ABC. Let d_A, d_B, and d_C be the distances from m (sic) to A, B, and C, respectively. Let r_A, r_B, and r_C be the radii of the circumcircles of PBC, PCA, and PAB, respectively. Prove that

$$\frac{1}{d_A} + \frac{1}{d_B} + \frac{1}{d_C} \geq \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}.$$

11736. *Proposed by Mircea Merca.* For $n \geq 1$, let f be the symmetric polynomal in variables $x_1, \ldots, x_n$ given by

$$f(x_1, \ldots, x_n) = \sum_{k=0}^{n-1} (-1)^{k+1} e_k(x_1 + x_1^2, x_2 + x_2^2, \ldots, x_n + x_n^2),$$

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where e_k is the kth elementary polynomials in n variables. (For example, when $n = 6$, e_2 has 15 terms, each a product of two distinct variables.) Also, let ξ be a primitive nth root of unity. Prove that

$$f(1, \xi, \xi^2, \ldots, \xi^{n-1}) = L_n - L_0,$$

where L_k is the k-th Lucas number (that is, $L_0 = 2$, $L_1 = 1$, and $L_k = L_{k-1} + L_{k-2}$ for $k \geq 2$).

11737. *Proposed by Nguyen Thanh Binh.* Given an acute triangle ABC, let O be its circumcenter, let M be the intersection of lines AO and BC, and let D be the other intersection of AO with the circumcircle of ABC. Let E be that point on AD such that M is the midpoint of ED. Let F be the point at which the perpendicular to AD at M meets AC. Prove that EF is perpendicular to AB.

11738. *Proposed by Stefano Siboni.* Three point particles are constrained to move without friction along a unit circle. Three ideal massless springs of stiffness k_1, k_2, and k_3 connect the particles pairwise. Show that an equilibrium in which the particles occupy three distinct positions exists if and only if $1/k_1$, $1/k_2$, and $1/k_3$ can be the lengths of the sides of a triangle. Show also that if this happens, the equilibrium length L of the spring with stiffness k_1 is given by

$$L = \sqrt{k_2 k_3} \sqrt{\left(\frac{1}{k_2} + \frac{1}{k_3}\right)^2 - \frac{1}{k_1^2}}.$$

11739. *Proposed by Fred Adams, Anthony Bloch, and Jeffrey Lagarias.* Let $B(x) = \begin{pmatrix} 1 & x \\ x & 1 \end{pmatrix}$. Consider the infinite matrix product

$$M(t) = B(2^{-t})B(3^{-t})B(5^{-t}) \ldots = \prod_p B(p^{-t}),$$

where the product runs over all primes, taken in increasing order. Evaluate $M(2)$.