

Olympiad problem solving: Freshmen sessions
Problem set 1, October 5,2010

Rules: none. Attempt any problems you like, work individually or in groups (in the latter case, be quiet to not disturb others). At every problem session there will be several members of staff to discuss your solutions, give hints etc. Good luck!

1. Show that the product of every three consecutive integers is (a) divisible by 3; (b) divisible by 6.
2. Show that the product of every four consecutive integers is divisible by 24.
3. What is the last decimal digit of 133^{133} ?
4. (a) Prove that an integer is divisible by 9 if and only if sum of its decimal digits is divisible by 9.
(b) Formulate and prove a similar statement for divisibility by 11.
5. The product of three integers is equal to 10987654321. Can their sum be equal to 124816?
6. A computer program takes an integer and computes the sum of its decimal digits, then the sum of decimal digits of the result etc. until the result is a single digit. If the input of the program is 199^{199} , what is its output going to be?
7. Show that in a class of 30 students there will always be three students whose birthdays are all in the same month.
8. Show that in a row of 1000 integers it is always possible to find several consecutive ones (maybe just one) whose sum is divisible by 1000.
9. Show that from a group of seven people whose (integer) ages add up to 332 one can select three people with the total age at least 142.
10. Among the integers from 1 to 1,000,000, are there more those containing 1 in their decimal notation or those whose decimal notation does not contain any 1's? The same question for integers from 1 to 10,000,000.
11. How many ways are there to elect the secretary, the treasurer, and the librarian from the 100 members of a student society? How many ways are there to elect three executive officers from the 100 society members? Why are the answers to these two questions different?
12. How many ways are there to label the faces of a cube by integers 1, 2, ..., 6? Every integer should be used, two labellings are the same if they cannot be identified rotating the cube.
13. The numbers 1, 2, 3, 4, 5, 6 are written (in this order) at the vertices of a regular hexagon. You are allowed to add 1 to two adjacent numbers, and to repeat this operation several times. Is it possible to end up with the configuration where all numbers are the same?
14. All entries of a (a) 4×4 ; (b) 3×3 matrix are equal to -1 , with the exception of the entry in the top left corner which is equal to 1. You are allowed to multiply all elements of one of the rows by -1 , or to do the same to all elements of one of the columns, and to repeat these operations several times. Is it possible to make all the matrix entries the same?
15. Prove that there is no group of 2009 people where each person knows exactly 1981 other persons from this group.

16. Prove that in a volleyball tournament at every moment it is possible to find two teams which played the same number of games.

17. (a) Prove that in each group of 6 people there exist either 3 persons who all know each other (“mutual acquaintances”), or a group of 3 people where no one knows anyone else (“mutual strangers”).

(b) Prove that in each group of 10 people there exist either 3 mutual acquaintances, or 4 mutual strangers.

18. At some point during a volleyball tournament, the observers computed the number of teams which played an odd number of games by that moment. Prove that this number is even.

19. In a group of 10 friends, 14 quarrels (between pairs of friends) happened. Prove that there still exist 3 persons from that group who remained friends with each other.

20. Mr and Mrs Brown invited four other couples over for a party. During the party, some of them shook hands with the other ones; no two people shook their hands twice, and no person shook their partner’s hand. When it was time to leave, Mr Brown asked everyone how many times they shook someone’s hand, and all the numbers turned out to be different. How many hands did Mrs Brown shake?

21. Show that for all positive integers n the numbers 2010^n and $2010^n + 10^n$ have the same number of digits in the decimal notation.

22. Is the complex number $z = \frac{3}{5} + \frac{4}{5}i$ a root of unity? In other words, does there exist a positive integer n for which $z^n = 1$?

23. Denote by S the figure consisting of all points above the graph $y = x^2$ in the plane. Is it possible to cover the plane by a finite number of figures congruent to S ?

24. Show that for every odd $k \geq 3$ there exist positive integers $a_1 < a_2 < \dots < a_k$ with

$$\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_k} = 1.$$

25. Alice and Bob have an infinite sheet of paper with a square grid on it. They play the following game: taking turns, they paint the sides of the squares (Alice is using the red colour, and Bob is using the blue colour); it is not allowed to paint any side which has been painted already. Alice was the first to make the move. Prove that Bob can play in such a way that Alice could never make a closed red contour.

26. For real numbers $a_1 \leq a_2 \leq a_3$ and $b_1 \leq b_2 \leq b_3$, we have $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$ and $a_1 a_2 + a_1 a_3 + a_2 a_3 = b_1 b_2 + b_1 b_3 + b_2 b_3$. Prove that if $a_1 \leq b_1$, then $a_3 \leq b_3$.

27. Let us call an integer n nice if any prime number in its prime number decomposition appears more than once, and the same holds for $n + 1$ (for example, the number 288 is good, since $288 = 2^5 \cdot 3^2$, $288 + 1 = 17^2$). Is the number of nice integers finite or infinite?