Olympiad problem solving: Sophister sessions
Problem set 1, October 7, 2010

Rules: none. Attempt any problems you like, work individually or in groups (in the latter case, be quiet to not disturb others). At every problem session there will be several members of staff to discuss your solutions, give hints etc. Good luck!

1. A cube in 3d space is positioned in such a way that all its vertices have integer coordinates. Prove that the side of the cube is of integral length.

2. (**a**) Is it possible to place an uncountable family of circles in the plane, if no circle is allowed to be placed inside another one?

(**b**) Let us call a T-shaped figure in the plane the union of two perpendicular segments, with an endpoint of one of them being the midpoint of the other one. Does there exist an uncountable family of pairwise non-intersecting T-shaped figures?

3. Show that a monotonously increasing function is continuous outside some at most countable (that is, finite or countable) set.

4. Show that $n! > \frac{n^n}{e^n}$.

5. Show that the sequences $a_n = \sin(n^2)$ and $b_n = \sin(4^n)$ are not convergent.

6. Given that $|a_{i+1} - a_i| < \frac{1}{2^i}$ for all i, show that the sequence $a_1, a_2, a_3, \ldots$ is convergent.

7. Compute the limit $\lim_{n \to \infty} \sin((2 + \sqrt{3})^n \pi)$.

8. Let $a_1 = 1/2$, $a_{n+1} = a_n - a_n^2$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite nonzero limit, and compute that limit.

9. Let $a_1 = 1$, $a_{n+1} = \sin(a_n)$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite nonzero limit, and compute that limit.

10. Let $a_1 = c$, $a_{n+1} = a_n + \frac{a_n^2}{n^2}$. Show that if $0 < c < 1$, then the sequence a_n is bounded from above.

11. Assume that for a sequence of real numbers a_n we have

$$\lim_{n \to \infty} (a_n + a_{n^2}) = 0.$$

Does that imply that $\lim_{n \to \infty} a_n = 0$?

12. Show that the equation $x^{\lfloor x \rfloor} = 9/2$ has no positive rational solutions.

13. Find all $a > 0$ for which the inequality $a^x \geq ax$ holds for all positive x.

14. How many of complex roots of the equation $z^8 + z^3 + 1 = 0$ have both their imaginary part and their real part positive?

15. For a set of 2011 coins, it is known that if we remove an arbitrary coin from it, the remaining ones can be divided into two set of 1005 coins of the same total weight. Show that all coins have the same weight.

16. Let $S = x_1 + x_2 + \ldots + x_n$. Show that

$$(1 + x_1)(1 + x_2) \cdots (1 + x_n) \leq 1 + S + \frac{S^2}{2} + \ldots + \frac{S^n}{n!}.$$

17. Show that if $\cos a = b$ and $\cos b = a$ then $a = b$.

18. Show that for every a and b there exists a point $x \in [-\pi/2, \pi/2]$ for which

$$|\sin x - ax - b| \geqslant 1/8.$$

19. Show that it is impossible to place two triangles, each of area greater than 1, in a circle of radius 1 without overlaps.

20. Compute the integral $\int_0^{\pi/2}(\cos^2(\cos x) + \sin^2(\sin x))\,dx$.

21. The equation $b + ax = \sin x$ has n solutions. How many solutions can the equation $b - ax = \sin x$ have?

22. Find all integer solutions to the equation $x^2 + y^2 + z^2 = 2xyz$.

23. Let $P(x) = x(x-1)(x-2)\cdots(x-100)$. Show that the solution set to the inequality $P'(x)/P(x) > 1$ is a finite union of intervals, and compute the sum of lengths of those intervals.

24. Let a_k be the number of decimal digits of 7^k. Compute the limit of the sequence $b_k = a_k/k$.

25. Find the distance between the graphs $y = e^{2010x}$ and $y = \frac{1}{2010}\ln x$.

26. In a group of 1981 people, every person knows at least 45 other people. Show that it is possible to pick four people from that group and sit them at the four sides of a square table so that every one of them sits next to people he or she knows.

27. In a group of 10 friends, 14 quarrels (between pairs of friends) happened. Prove that there still exist 3 persons from that group who remained friends with each other.

28. Mr and Mrs Brown invited four other couples over for a party. During the party, some of them shook hands with the other ones; no two people shook their hands twice, and no person shook their partner's hand. When it was time to leave, Mr Brown asked everyone how many times they shook someone's hand, and all the numbers turned out to be different. How many hands did Mrs Brown shake?

29. For two square matrices A and B of the same size, it is known that whenever $Ax = 0$ for some vector x, we also have $Bx = 0$. Show that $B = CA$ for some matrix C.

30. Vectors $e_1, \ldots, e_k$ of an n-dimensional Euclidean space V satisfy $(e_i, e_j) < 0$ for all $i \neq j$. What is the maximal possible value of k?

31. Show that for any fixed choice of off-diagonal entries in a square matrix it is possible to assign to each place on the diagonal 0 or 1 in such a way that the resulting matrix has a non-zero determinant.

32. Show that for any choice of five vectors of a Euclidean space V it is possible to pick two of them so that the length of their sum is less than or equal to the length of the sum of three remaining vectors.