PROBLEM SOLVING: SOLUTION TO Q5 FROM THE MATHS BATTLE

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The goal of this note is to explain a solution to a (very difficult) problem 5 from the last maths battle. Recall its statement.

> Suppose that for two positive integers $a, b \geq 2$, it is known that $a^n - 1$ is divisible by $b^n - 1$ for all $n \in \mathbb{N}$. Show that $a = b^k$ for some $k \in \mathbb{N}$.

The proof of this statement relies on the following

Lemma. *Let $x_1, \ldots, x_k$ be pairwise distinct rational numbers in $(1, +\infty)$, let $c_1, \ldots, c_k$ be nonzero real numbers, and let $\{m_n\}_{n \geq 1}$ be a sequence of integers. Assume that*

$$\lim_{n\to\infty}\left(\sum_{j=1}^{k} c_j x_j^n - m_n\right) = 0.$$

Then $x_1, \ldots, x_k$ are in fact integers.

Before proving this lemma, let us show that it implies the desired statement. Indeed, since $a^n - 1$ is divisible by $b^n - 1$, we have $a \geq b$, so $a^k \leq b < a^{k+1}$ for some positive integer k. Let $x_j = \frac{a}{b^j}$ for $j = 1, \ldots, k$, $c_1 = c_2 = \cdots = c_k = 1$, and $m_n = \frac{a^n - 1}{b^n - 1}$. Then we have

$$\sum_{j=1}^{k} c_j x_j^n - m_n = \sum_{j=1}^{k} \frac{a^n}{b^{jn}} - \frac{a^n - 1}{b^n - 1} = \frac{a^n}{b^n}\frac{1 - \frac{1}{b^{kn}}}{1 - \frac{1}{b^n}} - \frac{a^n - 1}{b^n - 1} = \frac{a^n}{b^{kn}}\frac{b^{kn} - 1}{b^n - 1} - \frac{a^n - 1}{b^n - 1} = \frac{1}{b^{kn}}\frac{b^{kn} - a^n}{b^n - 1},$$

which has zero limit as $n \to \infty$: since $a^k \leq b$, this quantity does not exceed $\frac{2}{b^n - 1}$ in absolute value. Therefore all x_i are integers, and the quantity

$$\sum_{j=1}^{k} c_j x_j^n - m_n = \frac{1}{b^{kn}}\frac{b^{kn} - a^n}{b^n - 1}$$

is an integer. A sequence of integers has limit zero only if all terms starting from some point are equal to zero. Hence, $a^n = b^{kn}$, and $a = b^k$.

Proof of Lemma. Induction on k. Let us first deal with $k = 1$. Let $x_1 = \frac{p}{q}$ in lowest terms. Let $\varepsilon = \frac{1}{p+q}$. We have $|c_1 x_1^n - m_n| < \varepsilon$ for all sufficiently large n, say $n \geq n_0$. For such n, we of course have $|pc_1 x_1^n - pm_n| < p\varepsilon$ and $|qc_1 x_1^{n+1} - qm_n| < q\varepsilon$. Note that $px_1^n = qx_1^{n+1}$, so

$$|pm_n - qm_{n+1}| \leq |pc_1 x_1^n - pm_n| + |qc_1 x_1^{n+1} - qm_n| < p\varepsilon + q\varepsilon = 1.$$

Since $pm_n - qm_{n+1}$ is an integer, this can only happen for $pm_n - qm_{n+1} = 0$, that is $m_{n+1} = \frac{p}{q} m_n$. We conclude that $m_{n_0+k} = \frac{p^k}{q^k} m_{n_0}$, so m_{n_0} is divisible by q^k for all k. It is possible for $q = 1$ or $m_{n_0} = 0$. In the former case, we are done, in the latter case $m_{n_0+k} = 0$ for all k, so $\lim\limits_{n\to\infty} c_1 x_1^n = 0$, a contradiction with our assumptions $c_1 \neq 0$ and $x_1 > 1$.

Let us justify the induction step (it will be done in a very similar fashion to handling the basis of induction). Fix $1 \leq l \leq k$, and let $x_l = \frac{p}{q}$. We have

$$\lim_{n\to\infty}\left(\sum_{j=1}^{k} pc_j x_j^n - pm_n\right) = 0 \quad \text{and} \quad \lim_{n\to\infty}\left(\sum_{j=1}^{k} qc_j x_j^{n+1} - qm_{n+1}\right) = 0,$$

so

$$\lim_{n\to\infty}\left(\sum_{j=1}^{k} c_j (x_j q - p) x_j^n - (qm_{n+1} - pm_n)\right) = 0,$$

where the coefficients $c_j(x_j q - p)$ of x_j^n are nonzero for $j \neq l$, and the coefficient of x_l^n is zero. By induction hypothesis, we conclude that all numbers x_j with $j \neq l$ are integers. Since this can be done for any l, all the numbers x_j are integers. $\square$