An introduction to rigid cohomology
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Counting points

Let $\mathbf{F}_q$ be a finite field with q elements (q a power of a prime p) and X an algebraic variety over $\mathbf{F}_q$. We want to do the following:

- Compute the number of rational points of X

We will denote it by $N(X) := |X(\mathbf{F}_q)|$.

Example (always assuming q is odd)

We may consider the affine curve X defined by

$$y^2 = x^3 + x, \quad y \neq 0$$

inside $\mathbf{A}^2_{\mathbf{F}_q}$, or its projective closure $\overline{X}$ defined by

$$y^2 z = x^3 + xz^2$$

inside $\mathbf{P}^2_{\mathbf{F}_q}$.

Example (continuing)

Of course, we have

$$N(\overline{X}) = N(X) + |\{a \in \mathbf{F}_q, a^3 + a = 0\}| + |\{a \in \mathbf{F}_q, a^3 = 0\}|$$

$$= \begin{cases} N(X) + 2 & \text{if} \quad i \notin F_q \quad (q \equiv -1 \mod 4) \\ N(X) + 4 & \text{if} \quad i \in F_q \quad (q \equiv 1 \mod 4). \end{cases}$$

Recall that, in general, we know from the exact sequence

$$1 \to \{\pm 1\} \to F_q^\times \to (F_q^\times)^2 \to 1$$

that there are exactly $\frac{q-1}{2}$ squares in $F_q^\times$.

We consider now the first case which concerns $q = 3, 7, 27, \ldots$ and can easily be done in a very general way. Since -1 is not a square in F_q, we see that, given any $a \in \mathbf{F}_q^\times$, then

either a or $-a$ is a square but not both.

Example (continuing)

For the same reason,

either $a^3 + a$ or $(-a)^3 + (-a) = -(a^3 + a)$ is a square but not both.

Thus we see that it happens exactly $\frac{q-1}{2}$ times that $a^3 + a$ has the form b^2. And when this happens, we get exactly 2 possibilities for b. It follows that $N(X) = q - 1$ and therefore $N(\overline{X}) = q + 1$.

The second case which concerns $q = 5, 9, 25, 49, \ldots$ is a lot more complicated. For example, if $q = 5$, we may draw the following table

a	-2	-1	1	2
a^2	-1	1	1	-1
a^3	2	-1	1	-2
$a^3 + a$	0	-2	2	0

It follows that no element of the form $a^3 + a$ can be a non zero square and therefore $N(X) = 0$ so that $N(\overline{X}) = 4$.

Example (continuing)

We can also work out the case of $\mathbf{F}_9 := \mathbf{F}_3[i]$. On easily computes

$$a^3 + a = \overline{a} + a = 2\mathrm{Re}(a) = -\mathrm{Re}(a) \in \mathbf{F}_3$$

and see that it is a non zero square in $\mathbf{F}_9$ if and only if $\mathrm{Re}(a) \neq 0$. Thus we obtain 6 possibilities for a and therefore $N(X) = 12$ so that $N(\overline{X}) = 16$.

The Zeta function

If $\mathbf{F}_{q^r}/\mathbf{F}_q$ is a finite extension, and X is any algebraic variety over $\mathbf{F}_q$, we will write

$$N_r(X) := |X(\mathbf{F}_{q^r})| \quad \left(= N(X \otimes_{\mathbf{F}_q} \mathbf{F}_{q^r})\right).$$

And we define the Zeta function of X as

$$Z(X,t) = \exp\left(\sum_1^\infty N_r(X)\frac{t^r}{r}\right).$$

If we can compute it, we will recover

$$N(X) = \left(\frac{\mathrm{d}\ln Z(X,t)}{\mathrm{d}t}\right)_{|0},$$

and more generally, all other $N_r(X)$ by looking at the coefficients of $\ln Z(X,t)$.

Thus, what we want to do now is the following:

- Compute the Zeta function of X

Example

Let us first verify that if X is defined over $\mathbf{F}_q$ by

$$y^2 = x^3 + x, \quad y \neq 0$$

and $\overline{X}$ denotes its projective closure as before, then we have

$$Z(\overline{X}, t) = \begin{cases} \frac{Z(X,t)}{(1-t)^2(1-t^2)} & \text{if} \quad q \equiv -1 \mod 4 \\ \\ \frac{Z(X,t)}{(1-t)^4} & \text{if} \quad q \equiv 1 \mod 4. \end{cases}$$

Since $Z(\overline{X}, t) = Z(X, t) \times Z(\overline{X} \setminus X, t)$, we simply have to identify the numerator with $Z(\overline{X} \setminus X, t)$.

Example (continuing)

First of all, an equation $x = a$ has exactly 1 solution in $\mathbf{F}_{q^r}$ for each r and therefore, the Zeta function of a rational point is

$$\exp\left(\sum_1^\infty \frac{t^r}{r}\right) = \exp\left(-\ln(1-t)\right) = \frac{1}{1-t}.$$

This gets rid of the second case where there are 4 rational points.

However, when $q \equiv -1 \mod 4$, then x^2+1 has no solution in $\mathbf{F}_{q^r}$ for r odd and exactly 2 solutions for r even. Thus the corresponding Zeta function is

$$\exp\left(\sum_1^\infty 2\frac{t^{2k}}{2k}\right) = \frac{1}{1-t^2}.$$

And the second case is settled as well.

Example (continuing)

When $q = 3$, we can deduce the first terms of the Zeta functions of our affine and projective curves above from our previous computations.

More precisely, we have $N_1(X) = 3 - 1 = 2$ and $N_3(X) = 27 - 1 = 26$ and we did directly $N_2(X) = 12$ so that

$$Z(X, t) = \exp(2t + 12\frac{t^2}{2} + 26\frac{t^3}{3}) = 1 + 2t + 8t^2 + 34t^3 \mod t^4.$$

Also, we have $N_1(\overline{X}) = 3 + 1 = 4$ and $N_3(\overline{X}) = 27 + 1 = 30$ and $N_2(\overline{X}) = 12 + 4 = 16$ so that

$$Z(\overline{X}, t) = \exp(4t + 16\frac{t^2}{2} + 28\frac{t^3}{3}) = 1 + 4t + 16t^2 + 52t^3 \mod t^4.$$

Alternatively, one can derive this by dividing out the previous one by $(1 - t)^2(1 - t^2)$ (exercise !).

Using cohomology

We can use étale ([8]) or rigid ([3]) cohomology in order to compute a Zeta function. We will do rigid cohomology here.

Theorem (Etesse-LS)

If X is a smooth algebraic variety of pure dimension d over $\mathbf{F}_q$, then

$$Z(X,t) = \prod_{i=0}^{2d} \det\left(1 - tq^d (F^*)^{-1}_{|H^i_{\mathrm{rig}}(X)}\right)^{(-1)^{i+1}}.$$

Therefore, what we want to do is the following:

- Compute the rigid cohomology of X
- Compute the action of Frobenius

Example

Assume for a while that we know the standard properties of rigid cohomology. Then, one easily sees that the Zeta function of an elliptic curve such as $\overline{X}$ above has the form

$$Z(\overline{X}, t) = \frac{1 - at + qt^2}{(1-t)(1-qt)}.$$

We will also see below that we can actually do the explicit computations. In particular, the Zeta function is completely determined once we know $N(\overline{X}) = -a + 1 + q$ (use logarithmic derivative).

When $q \equiv -1 \mod 4$, we get $a = 0$ and an easy computation shows that

$$Z(\overline{X}, t) = 1 + (q+1)t + (q^2+2q+1)t^2 + (q^3+2q^2+2q+1)t^3 \mod t^4$$

which is a generalization of the above formula (case $q = 3$).

Example (continuing)

As an application, we may take $q = 7$ and get

$$\ln Z(\overline{X}, t) = 8t + 32t^2 \mod t^3.$$

We recover $N(\overline{X}) = 8$ and discover $N_2(\overline{X}) = 64$ so that $N_2(X) = 60$. Thus, the equation $y^2 = x^3 + x$ has 60 solutions with $y \neq 0$ over $\mathbf{F}_7[i]$.

We can also do the case $q = 5$. We know that $N(\overline{X}) = 4$ so that $4 = -a + 1 + 5$ and thus $a = 2$. In other words, we have

$$Z(\overline{X}, t) = \frac{1 - 2t + 5t^2}{(1 - t)(1 - 5t)}.$$

It follows that

$$\ln Z(\overline{X}, t) = 4t + 16t^2 \mod t^3$$

form which we recover $N(\overline{X}) = 4$ but we also discover $N_2(\overline{X}) = 32$

Defining cohomology

We may define rigid cohomology whenever we are in a suitable geometric situation and show afterwards that this is well defined.

Assume for example that there exists a scheme $\mathcal{X}$ over $\mathbf{Z}_q := W(\mathbf{F}_q)$ such that

$$X = \mathcal{X} \otimes_{\mathbf{Z}_q} \mathbf{F}_q$$

and a smooth proper formal scheme $\overline{\mathcal{X}}$ over $\mathbf{Z}_q$ such that $\mathcal{X}$ is the complement of a relative normal crossing divisor with smooth components. Then, one can define

$$H^*_{\mathrm{rig}}(X) := H^*_{\mathrm{dR}}(\mathcal{X} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q).$$

Recall that de Rham cohomology is obtained by differentiating functions. We will work out an example right now.

Example

We consider again the affine curve $y^2 = x^3 + x, y \neq 0$. We will have

$$H^*_{\mathrm{rig}}(X) := H^*(A \xrightarrow{\mathrm{d}} A\mathrm{d}x)$$

(meaning $H^0_{\mathrm{rig}}(X) = \ker \mathrm{d} : A \to A\mathrm{d}x$ and $H^1_{\mathrm{rig}}(X) = A\mathrm{d}x/\mathrm{d}A$) with

$$A := \mathbf{Q}_q[x, y, \frac{1}{y}]/(y^2 - x^3 - x) \quad \text{and} \quad \mathrm{d}y = \frac{3x^2+1}{2y}\mathrm{d}x.$$

Actually, it is convenient to set $B := \mathbf{Q}_q[x, \frac{1}{x^3+x}]$, so that

$$A = B \oplus By \quad \text{and} \quad \mathrm{d}y = \frac{3x^2+1}{2(x^3+x)}y\mathrm{d}x.$$

We may then split the computation in two parts:

$$H^*_{\mathrm{rig}}(X) := H^*(B \xrightarrow{\mathrm{d}} B\mathrm{d}x) \oplus H^*(By \xrightarrow{\mathrm{d}} By\mathrm{d}x).$$

Example (continuing)

Any element of B can be written in a unique way as a finite sum

$$f(x) = \sum P_k(x)(x^3 + x)^k$$

with $\deg P_k \leq 2$. All terms can be integrated unless $k = -1$ and we obtain

$$H^1(B \xrightarrow{\mathrm{d}} B\mathrm{d}x) \simeq \mathbf{Q}_q \frac{\mathrm{d}x}{y^2} \oplus \mathbf{Q}_q x \frac{\mathrm{d}x}{y^2} \oplus \mathbf{Q}_q x^2 \frac{\mathrm{d}x}{y^2}.$$

The second part requires some more work but one finds

$$H^1(By \xrightarrow{\mathrm{d}} By\mathrm{d}x) \simeq \mathbf{Q}_q \frac{\mathrm{d}x}{y} \oplus \mathbf{Q}_q x \frac{\mathrm{d}x}{y}.$$

Using standard properties of rigid cohomology, one can show that this last vector space is actually identical to $H^1_{\mathrm{rig}}(\overline{X})$.

Frobenius action

The *Frobenius map* on an $\mathbf{F}_q$-variety X is the identity on the underlying space and raises functions to the q-th power.

Unfortunately, the map $f \mapsto f^q$ on X does not lift to $\mathcal{X}$.

Example

The endomorphism

$$F : (x, y) \mapsto (x^q, y^q)$$

of the affine plane over $\mathbf{Z}_q$ does not leave $\mathcal{X}$ stable in the example above:

$$(x^q)^3 - x^q = x^{3q} - x^q \neq (x^3 + x)^q = (y^q)^2$$

There seems to be a solution: one may replace $\mathcal{X}$ with its p-adic completion $\widehat{\mathcal{X}}$. In other words, we will replace polynomials with series that converge on the closed p-adic ball of radius one.

Example

In the case of the curve $y^2 = x^3 + x, y \neq 0$, we will replace A by

$$\widehat{A} := \mathbf{Q}_q\{x, y, 1/y\}/(y^2 - x^3 - x)$$

where

$$\mathbf{Q}_q\{x, y, 1/y\} = \{\sum_{i \in \mathbf{N}, j \in \mathbf{Z}} a_{i,j} x^i y^j, a_{i,j} \to 0\}.$$

We may then define

$$F : (x, y) \mapsto \left(x^q, y^q \sqrt{\frac{x^{3q} + x^q}{(x^3 + x)^q}} \right)$$

in order to get a lifting of Frobenius on $\widehat{A}$. We need to give a meaning to this square root.

Example (continuing)

Since

$$(x^3 + x)^q \equiv x^{3q} + x^q \mod p,$$

we can write

$$\frac{x^{3q} + x^q}{(x^3 + x)^q} = 1 + pz$$

and use

$$\sqrt{1 + pz} = \sum_{n \geq 0} \binom{n}{\frac{1}{2}} p^n z^n.$$

This series converges for $|z| < \frac{1}{|p|}$, and in particular on the closed disc of radius one.

Unfortunately, unless $\mathcal{X}$ is proper, we have

$$H^*_{\mathrm{dR}}(\widehat{\mathcal{X}} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q) \neq H^*_{\mathrm{dR}}(\mathcal{X} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q).$$

Example

$$\widehat{\mathbf{A}^1_{\mathbf{Z}_q}} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q = \mathbf{D}_{\mathbf{Q}_q}(0, 1^+)$$

and

$$H^*_{\mathrm{dR}}(\mathbf{D}_{\mathbf{Q}_p}(0, 1^+)) = H^*(\mathbf{Q}_q\{t\} \xrightarrow{\mathrm{d}} \mathbf{Q}_q\{t\}\mathrm{d}t).$$

One easily sees that the series

$$\sum_k p^k t^{p^k} \in \mathbf{Q}_q\{t\},$$

for example, is not integrable and it follows that

$$H^1_{\mathrm{dR}}(\mathbf{D}_{\mathbf{Q}_p}(0, 1^+)) \neq 0 = H^1_{\mathrm{dR}}(\mathbf{A}^1_{\mathbf{Q}_p}).$$

Actually, there exists an object $\mathcal{X}^\dagger$ that lies between $\mathcal{X}$ and $\widehat{\mathcal{X}}$ called the *weak completion* of $\mathcal{X}$ such that

$$H^*_{\mathrm{dR}}(\mathcal{X}^\dagger \otimes_{\mathbf{Z}_q} \mathbf{Q}_q) = H^*_{\mathrm{dR}}(\mathcal{X} \otimes_{\mathbf{Z}_q} \mathbf{Q}_q).$$

The idea is to consider functions that converge on a bigger ball.

Example

In the case of the curve $y^2 = x^3 + x, y \neq 0$, we will replace A with

$$A^\dagger := \mathbf{Q}_q[x, y, 1/y]^\dagger/(y^2 - x^3 - x)$$

where

$$\mathbf{Q}_q[x, y, 1/y]^\dagger = \{ \sum_{i \in \mathbf{N}, j \in \mathbf{Z}} a_{i,j} x^i y^j, \exists \lambda > 1, |a_{i,j}| \lambda^{i+|j|} \to 0 \}.$$

(overconvergent series) and the above Frobenius is actually defined on $A^\dagger$.

This technique works as well for all hyperelliptic curves ([5]) and leads to efficient algorithms.

Rigid cohomology

Now, we want to define rigid cohomology in general.

Let K be a non trivial complete ultrametric field of characteristic 0 with residue field k (and also valuation ring $\mathcal{V}$ and maximal ideal $\mathfrak{m}$). Any finite extension of $\mathbf{Q}_p$ would satisfy these assumptions.

We have to define a functor

$$X \mapsto H^*_{\mathrm{rig}}(X/K)$$

from algebraic varieties over k to vector spaces over K.

More generally, we want to define a category of coefficients $\mathrm{Isoc}^\dagger(X/K)$ and a functorial assignment

$$E \mapsto H^*_{\mathrm{rig}}(X/K, E).$$

Some geometry

We need to consider several categories:

• $\mathbf{Sch}_{/k}$ is the category of schemes over k that have a locally finite covering of the form

$$\mathrm{Spec}\ k[t_1, \ldots, t_n]/\mathfrak{a}$$

(we recall that $\mathrm{Spec}\ A$ denotes the set of prime ideals of A).

• $\mathbf{Sch}_{/K}$ is the analogous category over K.

• $\mathbf{Sch}_{/\mathcal{V}}$ is the category of schemes over $\mathcal{V}$ that have a locally finite covering of the form

$$\mathrm{Spec}\ \mathcal{V}[t_1, \ldots, t_n]/\mathfrak{a}$$

where $\mathfrak{a}$ is of finite type (condition always satisfied when the valuation is discrete).

There are functors

$$\begin{array}{ccc} \mathbf{Sch}_{/\mathcal{V}} \longrightarrow \mathbf{Sch}_{/k} & \text{and} & \mathbf{Sch}_{/\mathcal{V}} \longrightarrow \mathbf{Sch}_{/K} \\ \mathcal{X} \longmapsto \mathcal{X}_k & & \mathcal{X} \longmapsto \mathcal{X}_K \end{array}$$

called *special fiber* and *generic fiber* respectively. Locally, they are given by reduction mod $\mathfrak{m}$ (same as $k \otimes_{\mathcal{V}} -$)

$$\mathcal{V}[t_1, \ldots, t_n]/\mathfrak{a} \mapsto k[t_1, \ldots, t_n]/\overline{\mathfrak{a}},$$

and scalar extension (same as $K \otimes_{\mathcal{V}} -$)

$$\mathcal{V}[t_1, \ldots, t_n]/\mathfrak{a} \mapsto K[t_1, \ldots, t_n]/(\mathfrak{a}).$$

Note also that there is a pair of complementary closed and open immersions

$$\mathcal{X}_k \hookrightarrow \mathcal{X} \hookleftarrow \mathcal{X}_K$$

locally obtained by pulling back prime ideals.

• $\mathbf{FSch}_{/\mathcal{V}}$ is the category of formal schemes over the ring $\mathcal{V}$ of integers of K that have a locally finite covering of the form

$$\mathrm{Spf}\ \mathcal{V}\{t_1, \ldots, t_n\}/\mathfrak{a}$$

where $\mathfrak{a}$ is of finite type. Recall that $\mathrm{Spf}\ A$ denotes the set of open primes (i.e. that contain $\mathfrak{m}A$) and that

$$\mathcal{V}\{t_1, \ldots, t_n\} = \left\{ \sum_{i_1=0,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n}, \quad \mathcal{V} \ni a_{i_1,\ldots,i_n} \to 0 \right\}.$$

There exists a completion functor

$$\mathbf{Sch}_{/\mathcal{V}} \to \mathbf{FSch}_{/\mathcal{V}}, \quad \mathcal{X} \mapsto \widehat{\mathcal{X}}$$

locally given by (same as completion)

$$\mathcal{V}[t_1, \ldots, t_n]/\mathfrak{a} \mapsto \mathcal{V}\{t_1, \ldots, t_n\}/(\mathfrak{a}).$$

Moreover, there is a canonical morphism $\widehat{\mathcal{X}} \hookrightarrow \mathcal{X}$ locally obtained by pulling back open primes.

There exists also a special fiber functor

$$\begin{array}{ccc} \mathbf{FSch}_{/\mathcal{V}} & \longrightarrow & \mathbf{Sch}_{/k} \\ \mathcal{X} & \longmapsto & \mathcal{X}_k \end{array}$$

Locally, it is given by (same as $k \otimes_{\mathcal{V}} -$)

$$\mathcal{V}\{t_1, \ldots, t_n\}/\mathfrak{a} \mapsto k[t_1, \ldots, t_n]/\overline{\mathfrak{a}}.$$

Note that when $\mathcal{X}$ is a formal scheme, the canonical map (locally obtained by pulling back prime ideals) $\mathcal{X}_k \hookrightarrow \mathcal{X}$ is a homeomorphism.

In order to define the generic fibre, we need to introduce analytic varieties.

• $\mathbf{An}_{/K}$ is the category of analytic varieties over K in *Berkovich theory*. Locally (in a sense that will be made precise latter), they have the form

$$\mathcal{M}(K\{t_1, \ldots, t_n\}/\mathfrak{a})$$

where

$$K\{t_1, \ldots, t_n\} = \left\{ \sum_{i_1=0,\ldots,i_n=0}^{\infty} a_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_n^{i_n},\ K \ni a_{i_1,\ldots,i_n} \to 0 \right\}$$

and $\mathcal{M}(A)$ denotes the set of continuous multiplicative semi-norms on the K-algebra A. There exists an analytification functor

$$\mathbf{Sch}_{/K} \to \mathbf{An}_{/K}, \quad V \mapsto V^{\mathrm{an}}$$

locally given by

$$\mathrm{Spec}\ A \mapsto \mathcal{M}^{\mathrm{alg}}(A)$$

where $\mathcal{M}^{\mathrm{alg}}(A)$ denotes the set of multiplicative semi-norms on the K-algebra A.

Moreover, there is a canonical surjective map $V^{\mathrm{an}} \to V$ locally given by

$$x \mapsto \ker x := \{f \in A, \quad x(f) = 0\}.$$

There exists also a generic fibre functor

$$\mathbf{FSch}_{/\mathcal{V}} \to \mathbf{An}_{/K}, \quad \mathcal{X} \mapsto \mathcal{X}_K$$

locally given by (same as $K \otimes_{\mathcal{V}} -$)

$$\mathcal{V}\{t_1, \ldots, t_n\}/\mathfrak{a} \mapsto K\{t_1, \ldots, t_n\}/(\mathfrak{a}).$$

Also, there exists a specialization map $\mathrm{sp} : \mathcal{X}_K \to \mathcal{X}$, which is not continuous (for Berkovich topology), locally given by

$$x \mapsto \mathfrak{p} = \{f \in A, \quad x(f) < 1\}.$$

Finally, if we start with a scheme $\mathcal{X}$ over $\mathcal{V}$, there exists a natural map $\widehat{\mathcal{X}}_K \to \mathcal{X}_K^{\mathrm{an}}$ locally given by the embedding

$$\mathcal{M}(K\{t_1, \ldots, t_n\}/(\mathfrak{a})) \hookrightarrow \mathcal{M}^{\mathrm{alg}}(K[t_1, \ldots, t_n]/(\mathfrak{a})).$$

Example (K algebraically closed)

As sets, we may write

$$\mathbf{A}^1_k = k \cup \{\xi\}, \quad \mathbf{A}^1_K = K \cup \{\Xi\} \quad \text{and} \quad \mathbf{A}^1_{\mathcal{V}} = k \cup \{\xi\} \cup K \cup \{\Xi\}$$

where ξ and Ξ denote the generic points of the lines.

Of course, we may also consider $\widehat{\mathbf{A}^1_{\mathcal{V}}} = k \cup \{\xi\}$ (as sets) whose generic fibre is

$$\mathbf{D}(0, 1^+) := \mathcal{M}(K\{t\}) = \bigcup_{a \in \mathcal{V} \bmod \mathfrak{m}} \mathbf{D}(a, 1^-) \cup \{\|-\|\}$$

with the Gauss norm defined as $\|\sum a_i t^i\| = \max |a_i|$.
Specialization sends the residue disc $\mathbf{D}(a, 1^-)$ to $\overline{a}$ and the Gauss norm to ξ.

We will also describe $\mathbf{A}^{1\mathrm{an}}_K$, but we want to do the projective line before.

Example (continuing)

As sets again, we have

$$\mathbf{P}^1_k = k \cup \{\infty_k\} \cup \{\xi\}, \quad \mathbf{P}^1_K = K \cup \{\infty_K\} \cup \{\Xi\}$$

and

$$\mathbf{P}^1_{\mathcal{V}} = k \cup \{\infty_k\} \cup \{\xi\} \cup K \cup \{\infty_K\} \cup \{\Xi\}.$$

Of course, we may also write $\widehat{\mathbf{P}^1_{\mathcal{V}}} = k \cup \{\infty_k\} \cup \{\xi\}$. But now,

$$(\widehat{\mathbf{P}^1_{\mathcal{V}}})_K = \mathbf{P}^{1\mathrm{an}}_K = D(0, 1^+) \cup D(\infty, 1^-).$$

Specialization sends the rational point $(a; b) \in \mathbf{P}^{1\mathrm{an}}_K$ with $\max(|a|, |b|) = 1$ to $(\overline{a}; \overline{b}) \in \mathbf{P}^1_k$.

Finally, we have

$$\mathbf{A}^{1\mathrm{an}}_K = \mathbf{P}^{1\mathrm{an}}_K \setminus \{\infty\}$$

and the canonical map $\widehat{\mathcal{X}}_K \to \mathcal{X}^{\mathrm{an}}_K$ is the inclusion $\mathbf{D}(0, 1^+) \hookrightarrow \mathbf{A}^{1\mathrm{an}}_K$.

The smooth proper case

We can first define rigid cohomology for a liftable smooth proper variety (or more generally for the special fibre of the complement of a relative normal crossing divisor with smooth components):

Definition

Let P be a smooth proper (formal) scheme over $\mathcal{V}$ with special fibre X and generic fibre V, then

$$H^*_{\mathrm{rig}}(X) := H^*_{\mathrm{dR}}(V).$$

We must show that this definition is independent of the choice of P and functorial in X. This is not too difficult if one is ready to use crystalline cohomology ([1]) and the following result ([2]):

$$H^*_{\mathrm{cris}}(X/W(k)) \otimes_{W(k)} K \simeq H^*_{\mathrm{dR}}(P) \otimes_{\mathcal{V}} K$$

(at least when k is perfect and the valuation is discrete).

Tubes

We will generalize a little bit the previous case.

Let $X \hookrightarrow P$ be a locally closed embedding of an algebraic k-variety into a formal $\mathcal{V}$-scheme. Then, the *tube* of X in P is

$$]X[_P := \{x \in P_K, \quad \mathrm{sp}(x) \in X\}.$$

Note that

$$x \in]X[_P \quad \Leftrightarrow \quad \mathrm{sp}(x) \in X.$$

If P is affine and

$$X = \{a \in P_k, \quad \forall i, \overline{f}_i(a) = 0 \quad \text{and} \quad \exists j, \overline{g}_j(a) \neq 0\},$$

one easily checks that

$$]X[_P = \{x \in V, \quad \forall i, x(f_i) < 1 \quad \text{and} \quad \exists j, x(g_j) = 1\}.$$

Let's do it: one may assume that

$$X = \{a \in P_k, \quad \overline{f}(a) = 0\}.$$

Then, we can simply recall that, by definition,

$$f(\mathrm{sp(x)}) = 0 \Leftrightarrow x(f) < 1.$$

Example

We have

$$]0[_{\widehat{\mathbf{A}^n_{\mathcal{V}}}} = \mathbf{B}^n(0, 1^-).$$

When X is open in P_k so that $X = \mathcal{X}_k$ with $\mathcal{X}$ open in P, the tube of X in P is exactly the generic fibre of $\mathcal{X}$: we have $]X[_P = \mathcal{X}_K$. In order to verify this assertion, we may assume $P = \mathrm{Spf}\ A$ and $\mathcal{X} = \mathrm{Spf}\ A\{1/f\}$ so that

$$]X[_P = \{x \in \mathcal{M}(A_K), \quad x(f) = 1\} \simeq \mathcal{M}(A_K\{1/f\}) = \mathcal{X}_K.$$

We can now define rigid cohomology for proper (possibly singular) algebraic varieties:

Definition

Let $X \hookrightarrow P$ be a *closed* embedding of a *proper* algebraic variety into a *smooth* (around X) formal scheme, then

$$H^*_{\mathrm{rig}}(X) := H^*_{\mathrm{dR}}(]X[_P).$$

Note that we recover the previous case when $X = P_k$. Note also that this definition also makes sense when X is not proper in which case, it should be called *convergent* cohomology.

We must show that this definition is independent of P and functorial in X. In order to do that, we can use Ogus convergent cohomology and a comparison theorem. Alternatively, we can give a direct proof based on the *Poincaré lemma* and the *weak fibration theorem* that we describe now.

Theorem (Local Poincaré lemma)

Let V be an affinoid variety. If $\mathcal{F}$ is a coherent $\mathcal{O}_V$-module, we have a short exact sequence

$$0 \to \Gamma(V, \mathcal{F}) \to \Gamma(V \times \mathbf{D}(0, 1^-), p^*\mathcal{F}) \stackrel{\partial/\partial t}{\to} \Gamma(V \times \mathbf{D}(0, 1^-), p^*\mathcal{F}) \to 0.$$

Proof.

The point is to consider a series $\sum_n s_n t^n$ that converges on the open disc. One first checks that we can integrate it with respect to t which means that the series

$$\sum_n \frac{s_n}{n+1} t^{n+1}$$

also converges on the open disc. This gives exactness on the right. Next, one has to show that the derivative of the series $\sum_n s_n t^n$ with respect to t is zero if and only if the series is constant. But the condition means that $n s_n t^{n-1} = 0$ for $n > 0$. This gives exactness in the middle. Exactness on the left is clear. □

Theorem (Weak fibration theorem)

Let X be an algebraic variety, $X \hookrightarrow P$ (resp. $X \hookrightarrow P'$) be a locally closed embedding of X into a formal scheme, and $P' \to P$ a smooth (around X) morphism. Then, locally on X, we have

$$]X[_{P'} \simeq]X[_P \times \mathbf{B}^n(0, 1^-).$$

Proof.

One easily reduces to the case of an étale map of affine formal schemes (so that $n = 0$). It is therefore sufficient to show that if an étale map $A \to A'$ induces an isomorphism

$$A/(f_1, \ldots, f_n) \simeq A'/(f_1, \ldots, f_n)$$

then, it induces an isomorphism

$$A\{f_1/r, \ldots, f_n/r\} \simeq A'\{f_1/r, \ldots, f_n/r\}$$

when $r < 1$ is close to 1. This is formal. □

Towards the general case

In order to go further, we will consider the following situation:

$$X \hookrightarrow P \xleftarrow{\mathrm{sp}} P_K \xleftarrow{\lambda} V$$

where the first map is a locally closed embedding of a k-algebraic variety X into a formal $\mathcal{V}$-scheme P and λ is a morphism from an analytic variety V to the generic fibre of P. Most of the times, λ will be the inclusion of a neighborhood V of $]X[_P$ in P_K.

Definition

The *tube* of X in V is

$$]X[_V := \{x \in V, \quad \mathrm{sp}(\lambda(x)) \in X\}.$$

Note that

$$x \in]X[_V \quad \Leftrightarrow \quad \lambda(x) \in]X[_P \quad \Leftrightarrow \quad \mathrm{sp}(\lambda(x)) \in X.$$

Example (Monsky-Washnitzer)

Let A be a finitely presented $\mathcal{V}$-algebra,

$$X := \mathrm{Spec}\ A_k \quad \text{and} \quad V := (\mathrm{Spec}\ A_K)^{\mathrm{an}}.$$

The choice of a presentation

$$A := \mathcal{V}[t_1, \ldots, t_r]/\mathfrak{a}$$

defines an embedding

$$\mathrm{Spec}\ A \hookrightarrow \mathbf{A}^n_{\mathcal{V}} \hookrightarrow \mathbf{P}^n_{\mathcal{V}}.$$

We will then consider:

$$X = \mathrm{Spec}\ A_k \hookrightarrow \widehat{\mathbf{P}^n_{\mathcal{V}}} \xleftarrow{\mathrm{sp}} \mathbf{P}^{n,\mathrm{an}}_K \hookleftarrow (\mathrm{Spec}\ A_K)^{\mathrm{an}} = V.$$

Example (continuing)

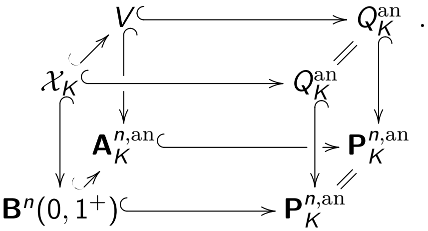
Alternatively, if we denote by Q the projective closure of $\mathrm{Spec} A$ in $\mathbf{P}^n_{\mathcal{V}}$, we may consider

$$X = \mathrm{Spec}\ A_k \hookrightarrow \widehat{Q} \xleftarrow{\mathrm{sp}} Q_K^{\mathrm{an}} \hookleftarrow (\mathrm{Spec}\ A_K)^{\mathrm{an}} = V.$$

If we write $\mathcal{X} = \mathrm{Spf}\ \widehat{A}$, we have

$$]X[_V =]X[_{\hat{Q}} = \mathcal{X}_K = V \cap \mathbf{B}^n(0, 1^+).$$

Actually, we can draw the following commutative diagram



Working with neighborhoods

Back to

$$X \hookrightarrow P \xleftarrow{\mathrm{sp}} P_K \xleftarrow{\lambda} V,$$

we denote by

$$i_X :]X[_V \hookrightarrow V$$

the inclusion map.

Definition

The sheaf $i_X^{-1}\mathcal{O}_V$ is the sheaf of *overconvergent* functions on the tube.

Note that Berkovich $i_{X*}i_X^{-1}$ "=" rigid $j_X^\dagger$. More generally, we will consider the sheaves $i_X^{-1}\Omega_V^r$ of *overconvergent* forms on the tube.

Example

If $X \hookrightarrow P$ is a *closed* embedding, then $]X[_V$ is *(Berkovich) open* in V and $i_X^{-1}\mathcal{O}_V = \mathcal{O}_{]X[_P}$.

Example (MW)

In the Monsky-Washnitzer situation, one easily sees that the subsets

$$V_\lambda := V \cap B(0, \lambda^+)$$

with $\lambda > 1$ form a fundamental system of affinoid neighborhoods of

$$]X[_V = V \cap B(0, 1^+)$$

in V (use the fact that $]X[_V$ is compact to reduce to the affine space).

Actually, if

$$A = \mathrm{Spec}\ V[t_1, \ldots, t_n]/\mathfrak{a},$$

then $V_\lambda = \mathcal{M}(A_{\lambda K})$ with

$$A_\lambda := \mathcal{V}\{t_1/\lambda, \ldots, t_n/\lambda\}/(\mathfrak{a}).$$

Example (Continuing)

It follows that

$$\Gamma(]X[_P, i_X^{-1}\mathcal{O}_V) = \varinjlim A_{\lambda K} = A_K^\dagger$$

with

$$A^\dagger := V[t_1, \ldots, t_n]^\dagger/(\mathfrak{a}).$$

and

$$\mathcal{V}[t_1, \ldots, t_n]^\dagger = \left\{ \sum_{\underline{i}=\underline{0}}^{\underline{\infty}} a_{\underline{i}} \underline{t}^{\underline{i}}, \quad \exists \lambda > 1, \mathcal{V} \ni |a_{\underline{i}}| \lambda^{|\underline{i}|} \to 0 \right\}.$$

The general case

We may now give a general definition of rigid cohomology:

Definition

Let $X \hookrightarrow P$ be an embedding of an algebraic variety into a smooth proper formal scheme (around X). Then,

$$H^*_{\mathrm{rig}}(X) := H^*(]X[_P, i_X^{-1}\Omega^\bullet_{P_K}).$$

In order to show that this definition is independent of P and functorial in X, we will need the *strong fibration theorem* (see [6]) and the *overconvergent Poincaré lemma* (see also [6]).

Example (MW)

In the Monsky-Washnitzer situation, we get

$$H^*_{\mathrm{rig}}(X) = H^*_{dR}(A^\dagger_K) =: H^*_{\mathrm{MW}}(X).$$

Diagonal embedding

For more generality, we work in a relative situation

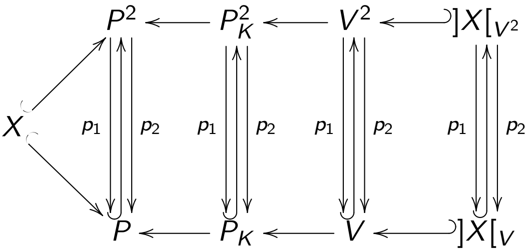
$$\begin{array}{ccccccc} X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O \end{array}$$

where the lower line is of the same type as the upper one and vertical arrows are morphisms that make the diagram commutative. The absolute case correspond to the case $C = \mathrm{Spec} k$, $S = \mathrm{Spf} \mathcal{V}$ and $O = S_K = \mathcal{M}(K)$.

If we embed diagonally X in $P^2 = P \times_S P$, we also have a map

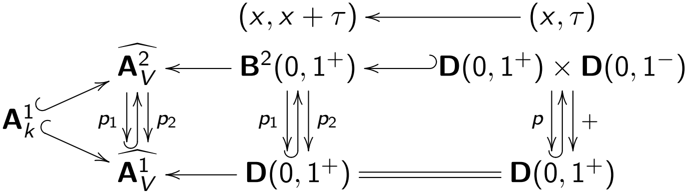
$$V^2 = V \times_O V \to P_K \times_{S_K} P_K = P_K^2$$

and we may consider the following diagram:



Example (Affine line)

In the simplest case of the affine $\mathcal{V}$-line, we get a quite interesting picture



Overconvergence

We also want to work with general coefficients and we introduce the following:

Definition

An *overconvergent stratification* on an $i_X^{-1}\mathcal{O}_V$-module E is an isomorphism of $i_X^{-1}\mathcal{O}_{V^2}$-modules

$$\epsilon : p_2^\dagger E := i_X^{-1} p_2^* i_{X*} E \simeq i_X^{-1} p_1^* i_{X*} E =: p_1^\dagger E$$

satisfying

$$p_{13}^\dagger(\epsilon) = p_{12}^\dagger(\epsilon) \circ p_{23}^\dagger(\epsilon) \quad \text{on} \quad]X[_{V^3} \quad \text{and} \quad \Delta^\dagger(\epsilon) = \mathrm{Id} \quad \text{on} \quad]X[_V.$$

These are called the *cocycle* and *normalization* conditions.

Example (Affine line)

In case of the affine $\mathcal{V}$-line, an overconvergent stratification on a free module of rank one is multiplication by a function

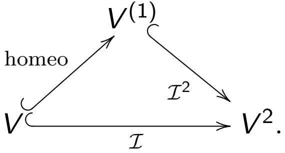
$$F(x,\tau) = 1 + \sum_{k=1}^{\infty} f_k(x)\tau^k \quad \text{on} \quad \mathbf{D}(0,1^+) \times \mathbf{D}(0,1^-)$$

satisfying

$$F(x,\tau_1+\tau_2) = F(x,\tau_1)F(x+\tau_1,\tau_2).$$

Connections

If V is defined by the ideal $\mathcal{I}$ in V^2, the *first infinitesimal neighborhood* of V is the subvariety defined by $\mathcal{I}^2$:



There exists an exact sequence

$$0 \to \Omega^1_V \to \mathcal{O}_{V^{(1)}} \to \mathcal{O}_V \to 0$$

and the derivation may be defined as $\mathrm{d}f = \overline{1 \otimes f - f \otimes 1}$. We will denote by $p_1^{(1)}, p_2^{(1)} : V^{(1)} \to V$ the maps induced by the projections.

We may always pull back an overconvergent stratification to $V^{(1)}$ and get

$$\begin{array}{ccccccc} E & \hookrightarrow & p_2^{(1)\dagger}E & \xrightarrow{\simeq} & p_1^{(1)\dagger}E & = & E \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\mathcal{O}_{V^{(1)}} \\ m & & \longrightarrow & & & & m \otimes 1 + \nabla(m) \end{array}$$

where

$$\nabla : E \to E \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega^1_V$$

happens to be an *integrable connection* on E that will be said to be *overconvergent*.

One easily checks that ∇ is well defined using the normalization condition and integrability comes from the cocycle condition. It is a connection by linearity: more precisely,

$$(1 \otimes f)(m \otimes 1 + \nabla(m)) = fm \otimes 1 + \nabla(fm)$$

gives

$$\nabla(fm) = f\nabla(m) + (1 \otimes f - f \otimes 1)(m \otimes 1) = f\nabla(m) + m \otimes \mathrm{d}f.$$

Example (Affine line)

An integrable connection on a free module on one generator e on the closed disk is determined by

$$\partial_x(e) = f(x)e$$

with some $f \in K\{x\}$. Using the above cocycle condition for F, one shows by induction that, necessarily,

$$F(x,\tau) = \sum_{k=0}^{\infty} f_k(x)\tau^k \quad \text{with} \quad f_k(x)e = \frac{1}{k!}\partial_x^k(e).$$

The (over-) convergence condition reads:

$$\forall \eta < 1, \quad \|\frac{1}{k!}\partial_x^k(e)\|\eta^k \to 0.$$

This applies to Dwork's exponential $\partial_x(e) = -\pi e$ with $\pi^{p-1} = -p$ which is overconvergent and non trivial with $F(x,\tau) = \exp(\pi\tau)$.

Main independence theorem

We come back to our diagram

$$\begin{array}{ccccccccc}
X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V & \hookleftarrow &]X[_V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow u & & \downarrow p \\
C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O & \hookleftarrow &]C[_O
\end{array}$$

that we will denote by $(X, V)/O$.

And we define relative rigid cohomology with coefficients (using derived functors for sake of clarity):

Definition

An *overconvergent isocrystal* on $(X, V)/O$ is a coherent $i_X^{-1}\mathcal{O}_V$-module E with an overconvergent integrable connection. Then,

$$\mathrm{R}p_{\mathrm{rig}}E := \mathrm{R}p_*(E \otimes_{i_X^{-1}\mathcal{O}_V} i^{-1}\Omega^\bullet_V).$$

Overconvergent isocrystals on $(X, V)/O$ form a category $\mathbf{Isoc}^\dagger(X, V/O)$ and we have:

Theorem (Berthelot)

Assume

1. *P is smooth over S in the neighborhood of X,*
2. *the Zariski closure of X in P is proper over (the Zariski closure of C in) S,*
3. *$V = P_K \times_{S_K} O$.*

Then, we have

1. *$\mathbf{Isoc}^\dagger(X, V/O)$ only depends on X and not on P or V.*
2. *$\mathrm{R}p_{\mathrm{rig}}E$ is functorial and only depends on X and not on P or V.*

We will give a proof of this theorem later on. Before, we need to have a closer look at the geometrical aspect of the question. We will also introduce the necessary cohomological formalism.

Remarks

If we remove the second assumption, we obtain that everything depends on the Zariski closure on X in P but not on P or V. There is a variant with support: If Z is a closed subset of X, the inclusion

$$\beta :]Z[_V \hookrightarrow]X[_V$$

is an *open* immersion. And we define

$$\mathrm{R}p_{\mathrm{rig},Z}(E) := \mathrm{R}p_*(]X[_V, \beta_!\beta^{-1}E \otimes_{i_X^{-1}\mathcal{O}_V} i^{-1}\Omega^\bullet_V).$$

Then the conclusions of the theorem also hold for $\mathrm{R}p_{\mathrm{rig},Z}(E)$. There is also a dual theory. One can replace V with a neighborhood of $]X[_V$ in V such that the inclusion $\alpha :]X[_V \hookrightarrow V$ is closed and $E = \alpha^{-1}\mathcal{E}$ where $\mathcal{E}$ is a coherent module with an integrable connection on V. Then

$$\mathrm{R}p_{\mathrm{rig},c}(E) := \mathrm{R}p_*(]X[_V, \alpha^!(\mathcal{E} \otimes_{\mathcal{O}_V} \Omega^\bullet_V)).$$

Again, the conclusions of the theorem also hold for $\mathrm{R}p_{\mathrm{rig},c}(E)$.

Algebraic varieties

We will quickly review the basics of algebraic geometry in a setting suitable for generalization to formal and analytic geometry.

A *k-algebra of finite type* is a k-algebra of the form

$$A := k[t_1, \ldots, t_n]/\mathfrak{a}$$

and an *affine variety* over k is a topological space X together with a fixed homeomorphism

$$X \simeq \mathrm{Spec}\ A$$

(set of prime ideals with the Zariski topology). The *ring of functions* on X is $\mathcal{O}(X) := A$.

If $x \in X$ corresponds to $\mathfrak{p}$, $k(x) := \mathrm{Frac}(A/\mathfrak{p})$ and if $f \in A$, then $f(x)$ will denote the image of f in $k(x)$. We have a bijection (Nullstellensatz):

$$X_0 := \{x \in X, \quad [k(x) : k] < \infty\} \simeq \mathrm{Spm}\ A.$$

Example (k algebraically closed)

We have

$$\begin{array}{ccccc} k \cup \{\xi\} & \xrightarrow{\simeq} & \text{Spec } k[t] & =\!=\!= & : \mathbf{A}^1_k \\ a & \longleftrightarrow & (t-a) & & \text{closed points} \\ \xi & \longleftrightarrow & 0 & & \text{generic point} \end{array}$$

A *morphism* of affine varieties is a pair made of a continuous map $\varphi : Y \to X$ and a homomorphism $\varphi^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ such that

$$f(\varphi(y)) = 0 \Leftrightarrow \varphi^*(f)(y) = 0.$$

Note that φ is determined by φ^* but the converse is false.

A subset U of X is an *affine domain* if the inclusion of U in X extends to a a morphism $U \hookrightarrow X$ which is universal for all morphisms of affine varieties $Y \to X$ that factors through U. Then, this affine structure on U is unique and U is open in X. Moreover, the affine domains of X form a basis for the topology of X.

Example (k algebraically closed)

The non empty affine domains of $\mathbf{A}_k^1$ are all the subsets

$$U := \mathbf{A}_k^1 \setminus \{a_1, \ldots, a_n\},$$

where $a_1, \ldots a_n \in k$, with affine structure given by

$$k[t, \frac{1}{t - a_1}, \ldots, \frac{1}{t - a_n}].$$

An *affine net* on a topological space X is a covering $\mathcal{U}$ of X such that

1. Any finite intersection of elements of $\mathcal{U}$ has a covering by elements of $\mathcal{U}$.
2. Any element of $\mathcal{U}$ has an affine structure and any inclusion of elements of $\mathcal{U}$ is the inclusion of an affine domain.

An *algebraic variety* is a topological space X with a maximal affine net.

A morphism of algebraic varieties $Y \to X$ is, up to refinement, a compatible family of morphisms of affine varieties $V \to U$ for all V in some open affine net $\mathcal{V}$ of Y and some U in an open affine net $\mathcal{U}$ of X.

Of course, this works over any field and we will as well consider the category Sch/K of algebraic varieties over K.

Note that this definition of algebraic varieties does not require sheaf theory that will be introduced later on.

Example (k algebraically closed)

We may consider

$$\begin{array}{ccc} k \cup \{\infty\} \cup \{\xi\} & \xrightarrow{\simeq} & \mathrm{Proj}\ k[t,s] =: \mathbf{P}^1_k \\ a & \longleftrightarrow & (t-as) \\ \infty & \longleftrightarrow & (s) \\ \xi & \longleftrightarrow & (0) \end{array}$$

where $\mathrm{Proj}\ k[t,s]$ denote the set of homogeneous prime ideals $\mathfrak{p} \subset k[t,s]$ other than (t,s).

An open affine net may be given by

$$\mathbf{A}^1_k = \mathrm{Spec} k[t], \quad \mathbf{P}^1_k \setminus \{0\} \simeq \mathrm{Spec} k[s], \quad \mathbf{A}^1_k \setminus 0 = \mathrm{Spec} k[t, \frac{1}{t}].$$

The maximal affine net is given by $\emptyset$ and all $\mathbf{P}^1_k \setminus S$ where S is any non-empty finite set of closed points.

Valuations

A *valuation* on a field K is a map $v : K \mapsto \Gamma \cup \{\infty\}$ where Γ is a totally ordered (additive) abelian group such that

1. $\forall a \in K, \quad a = 0 \Leftrightarrow v(a) = \infty$
2. $\forall a, b \in K, \quad v(a+b) \geq \min(v(a), v(b))$
3. $\forall a, b \in K, \quad v(ab) = v(a) + v(b)$

Then

$$\mathcal{V} := \{a \in K, \quad v(a) \geq 0\}$$

is a local ring with maximal ideal

$$\mathfrak{m} := \{a \in K, \quad v(a) > 0\}$$

and fraction field K. Actually, $\mathcal{V}$ is a *valuation ring* (an integral domain satisfying

$$\forall a \in \mathrm{Frac}(\mathcal{V}), \quad a \in \mathcal{V} \quad \text{or} \quad \frac{1}{a} \in \mathcal{V}).$$

Conversely, given a subring $\mathcal{V}$ of a field K, we may order $\Gamma := K^\times/\mathcal{V}^\times$ by setting $\Gamma^+ := (\mathcal{V} \setminus 0)/\mathcal{V}^\times$ and consider the canonical map

$$\begin{aligned} K &\xrightarrow{v} \{0\} \cup \Gamma \\ 0 \neq a &\longmapsto \bar{a} \\ 0 &\longmapsto 0 \end{aligned}$$

If $\mathcal{V}$ is a valuation ring, then Γ is totally ordered (multiplicative) and v is a valuation.

Assume for a while that v is surjective (or replace Γ with $v(K^\times)$). One can show that $\dim \mathcal{V} = \mathrm{rank}\,\Gamma$ (prime ideals of $\mathcal{V}$ correspond to convex subgroups of Γ). Actually, $\mathrm{rank}\,\Gamma = 1$ if and only if Γ is isomorphic to an ordered subgroup of $\mathbf{R}$. The valuation is said to be *discrete* if Γ is isomorphic to $\mathbf{Z}$.

Example

If p is a non-zero prime, the *p-adic valuation* on $\mathbf{Q}$ is the unique valuation such that $v(p) = 1$. It is discrete.

Absolute values

An *ultrametric absolute value* is a map $|-|: K \to \mathbf{R}_{\geq 0}$ such that

1. $\forall a \in K, \quad a = 0 \Leftrightarrow |a| = 0$
2. $\forall a, b \in K, \quad |a+b| \leq \max(|a|, |b|)$
3. $\forall a, b \in K, \quad |ab| = |a| + |b|$

Using an isomorphism

$$\begin{array}{ccc} (\mathbf{R}, +, \leq) \cup \{\infty\} & \xrightarrow{\simeq} & (\mathbf{R}_{>0}, \times, \geq) \cup \{0\} \\ x & \longmapsto & \epsilon^{-x} \\ -\log_\epsilon y & \longleftarrow & y \end{array}$$

with $\epsilon > 1$, we see that an ultrametric absolute value is, up to notation, a rank one valuation. For example, the *p-adic absolute value* is defined by $|a| = p^{-v(a)}$ on $\mathbf{Q}$.

An absolute value $|-|$ defines a *metric* on K by $d(a,b) = |b-a|$ and the completion $\hat{K}$ of K is naturally a field with an absolute value. If $0 < |\pi| < 1$, then the valuation ring of $\hat{K}$ is

$$\hat{\mathcal{V}} := \varprojlim \mathcal{V}/(\pi^{r+1}).$$

For example, the completion of $\mathbf{Q}$ for the p-adic absolute value is the field $\mathbf{Q}_p$ of p-adic numbers whose valuation ring is

$$\mathbf{Z}_p := \varprojlim \mathbf{Z}/(p^{r+1}).$$

The absolute value extends uniquely to the algebraic closure $\mathbf{Q}_p^{\mathrm{alg}}$ of $\mathbf{Q}_p$ whose completion $\mathbf{C}_p$ is also algebraically closed (but the valuation is not discrete anymore).

Formal schemes

Let K be a field which is complete for a non trivial ultrametric absolute value and $\mathcal{V}$ its valuation ring. Recall that

$$\mathcal{V}\{t_1, \ldots, t_n\} := \left\{ \sum_{\underline{i}=\underline{0}}^{\underline{\infty}} a_{\underline{i}} \underline{t}^{\underline{i}}, \quad \mathcal{V} \ni a_{\underline{i}} \to 0 \right\}.$$

Alternatively, if $\pi \in K$ with $0 < |\pi| < 1$, we have

$$\mathcal{V}\{t_1, \ldots, t_n\} = \varprojlim \mathcal{V}/(\pi^{r+1})[t_1, \cdots, t_n].$$

A $\mathcal{V}$-algebra is said to be *topologically of finite presentation* if it has the form

$$A := \mathcal{V}\{t_1, \ldots, t_n\}/\mathfrak{a}$$

where $\mathfrak{a}$ is finitely generated.

We denote by $\mathrm{Spf}\ A$ the set of *open primes* of A (primes ideals that contain $\mathfrak{m}A$). Note that the reduction map $A \mapsto A_k := A/\mathfrak{m}A$ induces a homeomorphism

$$\mathrm{Spec} A_k \simeq \mathrm{Spf}\ A.$$

An *affine formal scheme* over $\mathcal{V}$ is a topological space X together with a fixed homeomorphism

$$X \simeq \mathrm{Spf}\ A$$

The ring of functions on X is $\mathcal{O}(X) := A$.

Morphisms of affine formal schemes are defined as before (any homomorphism is continuous).

Affine domains are defined as before. They form a basis of open subsets as before.

A *formal scheme* over $\mathcal{V}$ is a topological space X with a maximal affine net. Morphisms are defined as before.

Semi-norms

Let K be a non-trivial complete ultrametric field.

A *semi-norm* on a K-algebra A is a map $x : A \mapsto \mathbf{R}_{\geq 0}$ such that

1. $\forall a \in K, \quad x(a) = |a|$.
2. $\forall f, g \in A, \quad x(f+g) \leq \max(x(f), x(g))$
3. $\forall f, g \in A, \quad x(fg) \leq x(f)x(g)$

It is called *multiplicative* if we have an equality in the last condition. It is a said to be a *norm* if

$$x(f) = 0 \Rightarrow f = 0.$$

Example

We may consider the multiplicative semi-norm defined on $K[t]$ by

$$a(f) := |f(a)| \quad \text{if} \quad a \in K.$$

Example (Continuing)

We may also consider the *Gauss norm*

$$\xi(f) = \max\{|a_i|\} \quad \text{for} \quad f := \sum a_i t^i \in K[t].$$

More generally (the previous cases are $r = 0$ and $r = 1$ respectively),

$$\xi_{a,r}(f) = \max\{|a_i| r^i\} \quad \text{for} \quad f := \sum a_i (t-a)^i \in K[t].$$

A (semi-) norm x defines a *(semi-) metric* on A by $d(f,g) = x(g-f)$ and the completion of A is naturally a normed ring.

Example

The completion of $K[t_1, \ldots, t_n]$ for the Gauss norm $\xi(f) = \max\{|a_{\underline{i}}|\}$ is

The kernel

$$\mathfrak{p} := \{f \in A, \quad x(f) = 0\}$$

of a multiplicative semi-norm x is a prime ideal in A. A multiplicative semi-norm x on A induces a multiplicative norm on $A/\mathfrak{p}$ that extends uniquely to the fraction field and then to its completion $\mathcal{H}(x)$ (which is therefore a complete ultrametric field):

$$\begin{array}{ccccccc} A & \twoheadrightarrow & A/\mathfrak{p} & \hookrightarrow & \mathcal{H}(x) & \xrightarrow[|-|]{} & \mathbf{R}_{\geq 0} \\ f & & \longmapsto & & f(x) & & \end{array} \qquad (A \xrightarrow{x} \mathbf{R}_{\geq 0})$$

Thus, if we denote by $f(x) \in \mathcal{H}(x)$ the image of $f \in A$, we have

$$x(f) = |f(x)|.$$

Example

In the above example, $\mathcal{H}(a) = K$ and $\mathcal{H}(\xi)$ is the completion of $K(t)$ for the Gauss norm (which is also the fraction field of $K\{t\}$).

Affinoid varieties

An *affinoid algebra* is a quotient of a *Tate algebra*

$$K\{t_1, \ldots, t_n\} := \left\{ \sum_{\underline{i}=\underline{0}}^{\underline{\infty}} a_{\underline{i}} \underline{t}^{\underline{i}}, \quad K \ni a_{\underline{i}} \to 0 \right\}.$$

The *Gelfand spectrum* of A is the set $\mathcal{M}(A)$ of all continuous (just means $|f(x)| \leq \|f\|$) multiplicative semi-norms on A. It is *compact Hausdorff* for the topology of simple convergence (induced by the inclusion

$$\mathcal{M}(A) \subset \prod_{f \in A} [0, \|f\|] \;).$$

Example

In the above example, a multiplicative semi-norm $\xi_{a,r}$ extends (uniquely) to a continuous multiplicative semi-norms on $K\{t\}$ if and only if $|a| \leq 1$ and $r \leq 1$.

An *affinoid variety* over K is a topological space V together with a fixed homeomorphism

$$V \simeq \mathcal{M}(A).$$

The ring of functions on V is $\mathcal{O}(V) := A$ and we can define $\mathcal{H}(x)$ for $x \in X$ as well as $f(x)$ if $f \in A$. We have a bijection (Nullstellensatz):

$$V_0 := \{x \in V, \quad [\mathcal{H}(x) : K] < \infty\} \simeq \mathrm{Spm}\ A.$$

Example

If $D := D(a, r)$ is a closed disc in K then $\xi_D := \xi_{a,r}$ only depends on D and we get a map

$$\begin{array}{ccc} \{\text{closed discs } \subset \mathcal{V}\} & \longrightarrow & \mathcal{M}(K\{t\}) =\!=\!=: \mathbf{D}(0, 1^+). \\ D & \longmapsto & \xi_D \end{array}$$

It is bijective when K algebraically closed and maximally complete[1].

[1] $|a_{i+1} - a_i| \leq r_i \Rightarrow \exists a, \forall i, |a - a_i| \leq r_i$

Affinoid domains

A *morphism of affinoid varieties* is a pair made of a continuous map $\varphi : V' \to V$ and a homomorphism $\varphi^* : \mathcal{O}(V) \to \mathcal{O}(V')$ such that

$$|f(\varphi(x'))| = |\varphi^*(f)(x')|.$$

Note that, here again, φ is determined by φ^*.

A subset W of V is an *affinoid domain* if the inclusion extends to a morphism $W \hookrightarrow V$ which is universal for all morphisms $V' \to V$ that factors through W. Then, the affinoid structure on W is unique and W is closed in V. Moreover, affinoid domains form a basis of compact neighborhoods for V.

Example (K algebraically closed)

The connected affinoid domains of the closed unit disk are the subsets

$$W = \{x \in \mathbf{D}(0,1^+), \quad |x-a| \leq r \quad \text{and} \quad |x-a_i| \geq r_i\}$$

with $r, r_1, \dots, r_n \in |K| \cap\,]0,1]$ and $a, a_1, \dots, a_n \in \mathcal{V}$. The affinoid structure is given by

$$K\{\frac{t}{r}, \frac{r_1}{t-a_1}, \dots, \frac{r_n}{t-a_n}\}.$$

In particular, we will consider

$$\mathbf{C}(0,1) = \{x \in \mathbf{D}(0,1^+), \quad |x| = 1\}$$

and

$$\mathbf{D}(0,r^+) = \{x \in \mathbf{D}(0,1^+), \quad |x| \leq r\}$$

for $r \in |K| \cap\,]0,1]$.

Analytic varieties

A *quasi-net* on a topological space V is a covering $\mathcal{W}$ of V such that

$$\forall x \in V, \quad \exists\, W_1, \ldots, W_n \in \mathcal{W}, \quad \begin{cases} x \in W_1 \cap \cdots \cap W_n \\ \exists U \text{ open}, x \in U \subset W_1 \cup \cdots \cup W_n \}. \end{cases}$$

A subset V' of V is said to be *admissible* with respect to $\mathcal{W}$ if the set

$$\{W \in \mathcal{W}, \quad W \subset V'\}$$

is a quasi-net on V'.

The quasi-net $\mathcal{W}$ is said to be a *net* if any finite intersection of elements of $\mathcal{W}$ is admissible. A covering of an admissible subset by admissible subsets is called *admissible* if it is a quasi-net.

Example

1. The open subsets of a topological space form a net with open subsets as admissible subsets and open coverings as admissible coverings.
2. An affine net on an algebraic variety is a net.

Example

1. The set of all affinoid domains of an affinoid variety X is a net and any *finite* affinoid covering of an affinoid domain is admissible. We will always endow X with this net.
2. The infinite affinoid covering of $\mathbf{D}(0, 1^+)$ by $\mathbf{C}(0, 1)$ and $\mathbf{D}(0, r^+)$ for $r \in |K| \cap\,]0, 1[$ is not admissible because ξ has no neighborhood of any of these types.
3. The open disc

$$\mathbf{D}(0, 1^-) = \{x \in \mathbf{D}(0, 1^+), \quad |x| < 1\}$$

An *affinoid net* on a locally Hausdorff topological space X is a net $\mathcal{W}$ such that any element of $\mathcal{W}$ has an affinoid structure and any inclusion of elements of $\mathcal{W}$ is the inclusion of an affinoid domain.

An *analytic variety (in Berkovich sense)* is a topological space X with a maximal affinoid net. An admissible subset is called an *analytic domain*. Any open subset is an analytic domain.

Example

The *analytic affine line* is made of all multiplicative (not necessarily continuous) semi-norms on $K[t]$:

$$\mathbf{A}_K^{1\mathrm{an}} = \mathcal{M}^{\mathrm{alg}}(K[t]) = \cup_{R \in |K^\times|} \mathcal{M}(K\{t/R\}).$$

As above, we have when K is algebraically closed and maximally complete,

$$\begin{array}{ccc} \{\text{closed discs } \subset K\} & \xrightarrow{\simeq} & \mathbf{A}_K^{1\mathrm{an}}. \\ D & \longmapsto & \xi_D \end{array}$$

A *morphism of analytic varieties* is, up to refinement, a compatible family of morphisms of affinoid varieties $V \to U$ for all V in some affinoid net $\mathcal{V}$ of Y and some U in an affinoid net $\mathcal{U}$ of X.

Note that if V is a Hausdorff analytic variety, there exists a unique structure of rigid analytic variety on

$$V_0 := \{x \in V, \quad [\mathcal{H}(x) : K] < \infty\}$$

such that affinoid domains and admissible affinoid coverings correspond bijectively to affinoid open subsets and admissible affinoid coverings. This is functorial and fully faithful.

Finally, an analytic variety is said to be *good* if any point has an affinoid neighborhood.

Example

An affinoid variety is good. More generally, a proper analytic variety over an affinoid variety is good. Also, if V is an algebraic variety, then V^{an} is good. But the generic fibre of $\widehat{\mathbf{A}}^2 \setminus (0,0)$ is not good.

Presheaves

We introduce the vocabulary that will be necessary to give a formal approach to rigid cohomology.

A *presheaf* (of sets) on a category $\mathcal{C}$ is a contravariant functor

$$\begin{aligned} T : \quad \mathcal{C} &\longrightarrow \mathbf{Sets}. \\ X &\longmapsto T(X) \\ \left(Y \xrightarrow{f} X\right) &\longmapsto \left(T(X) \xrightarrow{T(f)} T(Y)\right) \end{aligned}$$

In other words, we require

$$T(f \circ g) = T(g) \circ T(f) \quad \text{and} \quad T(\mathrm{Id}_X) = \mathrm{Id}_{T(X)}.$$

A *morphism of presheaves* $\alpha : T' \to T$ is a natural transformation: a family of maps $\alpha_X : T'(X) \to T(X)$ making commutative all

$$\begin{array}{ccc} T'(X) & \xrightarrow{\alpha_X} & T(X) \\ \downarrow{\scriptstyle T'(f)} & & \downarrow{\scriptstyle T(f)} \\ T'(Y) & \xrightarrow{\alpha_Y} & T(Y). \end{array}$$

We get a category $\widehat{\mathcal{C}}$ of all presheaves and an embedding (Yoneda)

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \widehat{\mathcal{C}} \\ X & \longmapsto & \left(Y \overset{\hat{X}}{\longmapsto} \mathrm{Hom}(Y, X) \right) \end{array}$$

Actually, we will have for all X in $\mathcal{C}$ and T in $\widehat{\mathcal{C}}$,

$$\mathcal{T}(X) \simeq \mathrm{Hom}(\hat{X}, T).$$

Be careful that the Yoneda functor does not preserve direct limits ($\emptyset$ or $\cup$ for example).

Example (Topological spaces)

If X is a topological space and $\mathbf{Open}(X)$ denotes the category of open subsets of X and inclusion maps, a presheaf on X (meaning on $\mathbf{Open}(X)$) is given by a set $T(U)$ for $U \subset X$ open, and compatible restriction maps

$$T(U) \to T(U'), \quad s \mapsto s_{|U'}$$

whenever $U' \subsetneq U$. Yoneda's embedding identifies an open subset U with the presheaf:

$$\hat{U} : U' \mapsto \begin{cases} \{\subset\} & \text{if } U' \subset U \\ \emptyset & \text{otherwise} \end{cases} .$$

If $X = \cup U_i$ is an open covering, we can also consider the subobject of $\hat{X}$:

$$R := \cup \hat{U}_i : U' \mapsto \begin{cases} \{\subset\} & \text{if } \exists i \in I, U' \subset U_i \\ \emptyset & \text{otherwise} \end{cases}$$

which is different from $\hat{X}$.

Example (continuing)

If $f : Y \to X$ is any continuous map, it induces a functor

$$f^{-1} : \mathbf{Open}(X) \to \mathbf{Open}(Y)$$

and by composition, a functor

$$\hat{f}_* : \widehat{\mathbf{Open}(Y)} \to \widehat{\mathbf{Open}(X)}.$$

in other words, we have

$$\hat{f}_*(T)(U) = T(f^{-1}(U)).$$

One can check that giving a presheaf on the category **Top** of all topological spaces is equivalent to giving a presheaf T_X (its *realization*) on each topological space X and a compatible family of morphisms $T_X \mapsto f_* T_Y$ for all continuous map $f : Y \to X$.

Topology

Let $\mathcal{C}$ be any category (think of **Top** for example).

A *sieve* of $X \in \mathcal{C}$ is a subobject $R \subset \hat{X}$. If $f : Y \to X$ is a morphism, then

$$f^{-1}(R)(Z) := \{s \in \hat{Y}(Z), f \circ s \in R(Z)\}$$

defines a sieve in Y.

A *topology* on $\mathcal{C}$ is a set of sieves of the objects of $\mathcal{C}$, called *covering sieves* such that

1. X is a covering sieve of X.
2. if R is a covering sieve of X and $f : Y \to X$ any map, then $f^{-1}(R)$ is a covering sieve of Y.
3. Let R be a sieve of X. Assume that there exists a covering sieve S of X such that whenever $f : Y \to X$ belongs to $S(X)$, then $f^{-1}(R)$ is a covering sieve of Y. Then R is a covering sieve of X.

A category endowed with a topology is a *site*.

The various topologies on a category $\mathcal{C}$ are ordered from *coarse* (only X covers X) to *discrete* (any R covers X).

Example (Topological spaces)

We turn $\mathbf{Open}(X)$ into a site by calling a sieve $R \subset \hat{U}$ a covering if the family

$$\{U' \in \mathbf{Open}(X), \quad R(U') \neq \emptyset\}$$

is an open covering of U.

In the same way, $R \subset \hat{X}$ will be a covering sieve in **Top** if

$$\{U \in \mathbf{Open}(X), \quad R(U) \neq \emptyset\}$$

is an open covering of X.

Pretopology

A *pretopology* on a category $\mathcal{C}$ is a set of families $\{X_i \to X\}$ called *covering families* such that

1. $\{\mathrm{Id}_X : X \to X\}$ is a covering family
2. If $\{X_i \to X\}$ is a covering family and $f : Y \to X$ any morphism, then $\{X_i \times_X Y \to Y\}$ (exists and) is a covering family
3. If $\{X_i \to X\}$ is a covering family and for each i, $\{X_{ij} \to X_i\}$ is also a covering family, then $\{X_{ij} \to X\}$ is a covering family.

A sieve $R \subset X$ will be called a *covering sieve* for this pretopology if there exists a covering family $\{f_i : X_i \to X\}_{i \in I}$ such that for all $i \in I$, $\mathrm{Im}\hat{f}_i \subset R$. This way, we get a topology on $\mathcal{C}$. Conversely, if $\mathcal{C}$ is a site with fibred products, the set of all families $\{f_i : X_i \to X\}_{i \in I}$ such that $\cup \, \mathrm{Im}\hat{f}_i$ is a covering sieve of X, is a pretopology that induces the topology of $\mathcal{C}$.

Example (Topological spaces)

Usual open coverings define a pretopology giving the above

Sheaves

A *sheaf* on a site $\mathcal{C}$ is a presheaf $\mathcal{F} : \mathcal{C} \to \mathbf{Sets}$ such that, for all covering sieve R of X, we have

$$(\mathcal{F}(X) =) \quad \mathrm{Hom}(\hat{X}, \mathcal{F}) \simeq \mathrm{Hom}(R, \mathcal{F}).$$

If the topology comes from a pretopology, this is equivalent to

$$\mathcal{F}(X) \longrightarrow \prod_i \mathcal{F}(X_i) \rightrightarrows \prod_{i,j} \mathcal{F}(X_i \times_X X_j)$$

being exact for any covering family (Exercise : show that $\mathcal{F}(\emptyset) = 0$).

The category of all sheaves on the site $\mathcal{C}$ is the *topos* $\widetilde{\mathcal{C}} \subset \widehat{\mathcal{C}}$.

Example

The constant presheaf $U \mapsto \mathbf{Z}$ on a topological space is not a sheaf in general but the presheaf $U \mapsto \mathbf{Z}^{\pi_0(U)}$ is a sheaf.

Example (Topological spaces)

1. A sheaf on a topological space is a presheaf that satisfies: given an open covering $U = \cup U_i$ and a family of $s_i \in \mathcal{F}(U_i)$ such that $(s_i)_{|U_j} = (s_j)_{|U_i}$, there exists a unique $s \in \mathcal{F}(U)$ such that $s_{|U_{U_i}} = s_i$.
2. A presheaf on **Top** is a sheaf if and only for any X, the realization $\mathcal{F}_X$ is a sheaf on X.

Example

1. For the *coarse* topology, any presheaf is a sheaf and the corresponding topos is $\widehat{\mathcal{C}}$.
2. The only sheaf for the *discrete* topology is the constant sheaf 0 and the corresponding topos is the category **0**.
3. The *canonical* topology on a category $\mathcal{C}$ is the finest topology such that any object of $\mathcal{C}$ is a sheaf (meaning that $\hat{X}$ is a sheaf).

If $\mathcal{C}$ is a site, the inclusion functor $\widetilde{C} \hookrightarrow \widehat{C}$ has a left adjoint $T \mapsto \tilde{T}$ (which is a actually section) given by applying twice

$$T'(X) = \varinjlim T(R)$$

with R running through all covering sieves of X.

The sheaf $\tilde{T}$ is called the *sheaf associated* to the presheaf T. When $\mathcal{F}$ is a sheaf, there is a canonical bijection

$$\mathrm{Hom}(\tilde{T}, \mathcal{F}) \simeq \mathrm{Hom}(T, \mathcal{F}).$$

Example

The sheaf associated to the constant presheaf $U \mapsto \mathbf{Z}$ on a topological space X is given by $U \mapsto \mathbf{Z}^{\pi_0(U)}$.

Morphisms of sites

Example (Topological spaces)

Recall that if $f : Y \to X$ is any continuous map, there is an obvious functor

$$f^{-1} : \mathbf{Open}(X) \to \mathbf{Open}(Y)$$

that gives, by composition, a functor

$$\hat{f}_* : \widehat{\mathbf{Open}(Y)} \to \widehat{\mathbf{Open}(X)}.$$

The functor $\hat{f}_*$ has left and right adjoints given by

$$\hat{f}^{-1}(T)(V) = \varinjlim_{V \subset f^{-1}(U)} T(U)$$

$$\text{and} \quad \hat{f}^{!}(T)(V) = \varprojlim_{f^{-1}(U) \subset V} T(U).$$

If $g : \mathcal{C} \to \mathcal{C}'$ is any functor (think of $g = f^{-1}$ in the previous example), composition induces a functor

$$\hat{g}^{-1} : \qquad \begin{array}{ccc} \widehat{\mathcal{C}'} & \longrightarrow & \widehat{\mathcal{C}} \\ T' & \longrightarrow & T' \circ g \end{array}$$

that has left and right adjoints: we have

$$\mathrm{Hom}(\hat{g}_! T, T') = \mathrm{Hom}(T, \hat{g}^{-1} T'))$$

$$\text{and} \quad \mathrm{Hom}(\hat{g}^{-1} T', T) = \mathrm{Hom}(T, \hat{g}_* T').$$

By analogy with topological spaces, we often write $g =: f^{-1}$, and we have a sequence of adjointness:

$$\hat{g}_! =: \hat{f}^{-1}, \quad \hat{g}^{-1} =: \hat{f}_*, \quad \hat{g}_* =: \hat{f}^!.$$

Note that $\widehat{g(X)} = \hat{g}_!(\hat{X})$ (or with the other notation $\widehat{f^{-1}(X)} = \hat{f}^{-1}(\hat{X})$). The point now is to extend this to sheaves and not merely presheaves.

Example (Topological spaces)

If $f : Y \to X$ is any continuous map, the functor $\hat{f}_*$ induces a functor on sheaves, $\tilde{f}_* : \widetilde{\mathbf{Open}(Y)} \to \widetilde{\mathbf{Open}(X)}$ that has an exact left adjoint $\tilde{f}^{-1}$: we have

$$\mathrm{Hom}(\tilde{f}^{-1}\mathcal{F}, \mathcal{G}) = \mathrm{Hom}(\mathcal{F}, \tilde{f}_*\mathcal{G}).$$

If $\mathcal{C}$ and $\mathcal{C}'$ are two sites, a functor $f^{-1} : \mathcal{C} \to \mathcal{C}'$ is said to be *continuous* if $\hat{f}_*$ preserves sheaves. By composition, the induced functor $\tilde{f}_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ has a left adjoint

$$\tilde{f}^{-1} : \quad \begin{array}{ccc} \widetilde{\mathcal{C}} & \longrightarrow & \widetilde{\mathcal{C}'} \\ \mathcal{F} & \longrightarrow & \widetilde{\hat{f}^{-1}(\mathcal{F})} \end{array}$$

If $\tilde{f}^{-1}$ is *exact*, we then say that $f : \mathcal{C}' \to \mathcal{C}$ is a *morphism of sites*.

Note that when f^{-1} is left exact (and we have fibred products in $\mathcal{C}$), then f^{-1} defines a morphism of sites if and only if it preserves covering families for some pretopologies.

Morphisms of topos

A *morphism of topos*

$$(f^{-1}, f_*) : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$$

is a couple of adjoint functors (f^{-1}, f_*) with f^{-1} exact.

Thus, we see that any morphism of site $f : \mathcal{C}' \to \mathcal{C}$ defines a morphism of topos

$$(\tilde{f}^{-1}, \tilde{f}_*) : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$$

The converse is true: a functor $f^{-1} : \widetilde{\mathcal{C}'} \longrightarrow \widetilde{\mathcal{C}}$ extends (uniquely) to a morphism of topos if and only if it defines a morphism of sites $f : \widetilde{\mathcal{C}} \longrightarrow \widetilde{\mathcal{C}'}$ for the canonical topologies. In practice, we will therefore often identify the morphism of sites f and the morphism of topos (f^{-1}, f_*).

A functor $g : \mathcal{C}' \to \mathcal{C}$ between two sites is said to be *cocontinuous* if $\hat{g}_*$ preserves sheaves. Then, the induced functor $\tilde{g}_* : \widetilde{\mathcal{C}'} \to \widetilde{\mathcal{C}}$ extends uniquely to a morphism of toposes $g := (\tilde{g}^{-1}, \tilde{g}_*)$.

Restricted (or comma) category

If X is an object in a category $\mathcal{C}$, we may consider the category $\mathcal{C}_{/X}$ of objects of $\mathcal{C}$ above X. An object is a pair made of an object $Y \in \mathcal{C}$ and a structural morphism $g : Y \to X$. If Z has structural morphism $h : Z \to X$, a morphism $Z \to Y$ over X is simply a morphism $f : Z \to Y$ such that $h = g \circ f$.

More generally, if $T \in \widehat{\mathcal{C}}$, then $\mathcal{C}_{/T}$ is the category whose objects are pairs made of an object $Y \in \mathcal{C}$ and a section $s \in T(Y)$. A morphism $Z \to Y$ over T is a morphism compatible with sections: $T(f)(s) = t$.

We consider now the forgetful functor : $j_T : \mathcal{C}_{/T} \to \mathcal{C}$. If $\mathcal{C}$ is a site, we endow $\mathcal{C}_{/T}$ with the induced topology (finest making j_T continuous). Unfortunately, $j_{T!}$ is not left exact in general and we do not get a morphism of sites. However, j_T is cocontinuous and we obtain a morphism of toposes

$$j_T : \widetilde{\mathcal{C}_{/T}} \to \widetilde{\mathcal{C}}.$$

Note also that any morphism $T' \to T$ will induce a morphism of toposes

$$\begin{array}{ccc} \widetilde{\mathcal{C}_{/T'}} & \longrightarrow & \widetilde{\mathcal{C}_{/T}} \\ \mathcal{F}_{/T'} & \longleftarrow\!\shortmid & \mathcal{F}. \end{array}$$

Example (Topological spaces)

If X is a topological space, we may consider the big site $\mathbf{Top}_{/X}$ of all topological spaces over X. The inclusion map

$$\mathbf{Open}(X) \hookrightarrow \mathbf{Top}_{/X}$$

is continuous, cocontinuous and left exact giving rise to two morphisms of toposes

$$\widetilde{\mathbf{Top}_{/X}} \underset{\psi_X}{\overset{\varphi_X}{\rightleftarrows}} \widetilde{\mathbf{Open}(X)}$$

with $\varphi_X \circ \psi_X = \mathrm{Id}$ and $\varphi_{X*} = \psi_X^{-1}$.

Example (continuing)

Let T be any presheaf on **Top** (or a topological space identified with the corresponding presheaf). If $\mathcal{F}$ is a sheaf on $\mathbf{Top}_{/T}$ and X a topological space over T, the *realization* of $\mathcal{F}$ on X is

$$\mathcal{F}_X := \varphi_{X*}\mathcal{F}_{/X}.$$

Any morphism $f : Y \to X$ over T will induce a morphism

$$\alpha_f : f^{-1}\mathcal{F}_X \to \mathcal{F}_Y$$

between the realizations.

One can show that, giving a sheaf $\mathcal{F}$ on $\mathbf{Top}_{/T}$ is equivalent to giving the collection of all $\mathcal{F}_X$ and compatible morphisms α_f. We may call $\mathcal{F}$ a *crystal* if all the maps α_f are isomorphisms.

Note that if X is a topological space, then, φ_X induces an equivalence between crystals on $\mathbf{Top}_{/X}$ and sheaves on X.

Example (from geometry)

We consider the *forgetful functor* $\mathbf{Sch}_{/k} \to \mathbf{Top}$ that sends an algebraic k-variety to its underlying topological space. We endow $\mathbf{Sch}_{/k}$ with the coarsest topology making this functor cocontinuous (it will also be continuous but not left exact).

This topology is induced by the pretopology made of coverings $\{X_i \hookrightarrow X\}_{i \in I}$ of X by open domains.

Careful: the topology *induced* by the forgetful functor (finer topology making it continuous) is too fine (an absolute Frobenius for example would be a covering).

If T is a presheaf on $\mathbf{Sch}_{/k}$, we can also consider the site $\mathbf{Sch}_{/T}$ of all algebraic varieties over T. As above, giving a (pre) sheaf on $\mathbf{Sch}_{/T}$ is equivalent to giving its realizations on X for all varieties X over T with the transition morphisms.

Example (continuing)

We can do exactly the same thing with $\mathbf{Sch}_{/K}$, $\mathbf{Sch}_{/\mathcal{V}}$, $\mathbf{FSch}_{/\mathcal{V}}$ or $\mathbf{An}_{/K}$ for example. There are morphisms of toposes

$$\widetilde{\mathbf{Sch}_{/k}} \hookrightarrow \widetilde{\mathbf{FSch}_{/\mathcal{V}}} \hookrightarrow \widetilde{\mathbf{Sch}_{/\mathcal{V}}} \hookleftarrow \widetilde{\mathbf{Sch}_{/K}} \leftarrow \widetilde{\mathbf{An}_{/K}}.$$

However, the generic fibre functor

$$\mathbf{FSch}_{/\mathcal{V}} \to \mathbf{An}_{/K}$$

is not continuous (for Berkovich topology).

Nevertheless, admissible coverings define a new *pretopology* of $\mathbf{An}_{/K}$ which is finer than the usual topology and makes specialization continuous. We call it the *admissible topology*, the *Grothendieck topology*, the *rigid topology* or the *Tate topology*.

Rings and modules

Geometrical objects usually come with a sheaf of functions. We want to formalize this now.

If $\mathcal{A}$ and $\mathcal{C}$ are two categories, a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$ is a contravariant functor $\mathcal{C} \to \mathcal{A}$. If $\mathcal{A}$ is concrete (endowed with a faithful functor $\mathcal{A} \to \textbf{Sets}$) composition gives a functor "underlying presheaf of sets". If moreover, $\mathcal{C}$ is a site, a presheaf with values in $\mathcal{A}$ is said to be a *sheaf* if the underlying presheaf of sets is a sheaf. We get the category $\mathcal{A}(\mathcal{C})$ of sheaves with values in $\mathcal{A}$.

We will apply this to the case of abelian groups **Ab** or rings **Rng** for example and call the corresponding sheaves just *abelian groups* or *rings* on $\mathcal{C}$. If $\mathcal{F}$ and $\mathcal{G}$ are two abelian groups on $\mathcal{C}$, their internal hom is defined by

$$\mathcal{H}om_{\mathrm{Ab}}(\mathcal{F},\mathcal{G})(X) := \mathrm{H}om_{\mathrm{Ab}}(\mathcal{F}_{/X},\mathcal{G}_{/X}).$$

Note that, if $\mathcal{F}$ is an abelian group on $\mathcal{C}$, then

$$\mathcal{E}nd_{\mathrm{Ab}}(\mathcal{F}) := \mathcal{H}om_{\mathrm{Ab}}(\mathcal{F}, \mathcal{F})$$

has a natural structure of sheaf of rings.

A *ringed site* is a pair $(\mathcal{C}, \mathcal{O})$ made of a site and a commutative ring on this site. We say that $(X, \mathcal{O})$ is a *ringed space* when $\mathcal{C} = \mathrm{Open}(X)$.

An *$\mathcal{O}$-module* is an abelian group on $\mathcal{C}$ endowed with a morphism of rings

$$\mathcal{O} \to \mathcal{E}nd_{\mathrm{Ab}}(\mathcal{F}).$$

A *morphism of $\mathcal{O}$-modules $u : \mathcal{F} \to \mathcal{G}$* is a morphism of abelian groups making commutative the diagram

$$\begin{array}{ccc} \mathcal{O} & \longrightarrow & \mathcal{E}nd_{\mathrm{Ab}}(\mathcal{F}) \\ \downarrow & & \downarrow{\scriptstyle u\circ -} \\ \mathcal{E}nd_{\mathrm{Ab}}(\mathcal{G}) & \xrightarrow{-\circ u} & \mathcal{H}om_{\mathrm{Ab}}(\mathcal{F}, \mathcal{G}). \end{array}$$

The category $\mathcal{O}-\mathbf{Mod}$ of all $\mathcal{O}$-modules is an abelian category with an *internal hom* (defined as above) that has a right adjoint $\otimes$ called *tensor product*.

An $\mathcal{O}$-module is said to be *(locally) finitely presented* if for any X in $\mathcal{C}$, the the set $R(Y)$ of all $f : Y \to X$ such that there exists a right exact sequence

$$\mathcal{O}_{/Y}^{M} \to \mathcal{O}_{/Y}^{N} \to \mathcal{F}_{/Y} \to 0$$

defines a covering sieve of X. They form an abelian subcategory $\mathcal{O}-\mathbf{Mod}^{\mathrm{fp}}$.

A *morphism of ringed sites*

$$(\mathcal{C}', \mathcal{O}') \to (\mathcal{C}, \mathcal{O})$$

is a pair made of a morphism of sites $f : \mathcal{C}' \to \mathcal{C}$ and a morphism of rings $f^{-1}\mathcal{O} \to \mathcal{O}'$ (or equivalently, $\mathcal{O} \to f_*\mathcal{O}'$).

If $\mathcal{F}'$ is an $\mathcal{O}'$-module, then $f_*\mathcal{F}'$ will have a natural structure of $\mathcal{O}$-module. The induced functor has a left adjoint given by

$$\mathcal{F} \mapsto f^*\mathcal{F} := \mathcal{O}' \otimes_{f^{-1}\mathcal{O}} f^{-1}\mathcal{F}.$$

Example (from geometry)

The assignment $X \mapsto \mathcal{O}(X)$ defined on affine k-varieties extends uniquely to a sheaf of rings $\mathcal{O}$ on $\mathbf{Sch}_{/k}$ called the *structural sheaf*.

For any presheaf T on $\mathbf{Sch}_{/k}$, we obtain a sheaf of rings on $\mathbf{Sch}_{/T}$ by restriction. We may also consider the realization $\mathcal{O}_X$ of $\mathcal{O}$ on a given algebraic variety X.

Any morphism of algebraic varieties $f : Y \to X$ over k will induce a morphism of ringed spaces $(Y, \mathcal{O}_Y) \to (X, \mathcal{O}_X)$. But this functor is not fully faithful.

Example (continuing)

Let T be a presheaf on $\mathbf{Sch}_{/k}$. Giving an $\mathcal{O}_{/T}$-module is equivalent to giving a family of $\mathcal{O}_X$-modules for all X over T with a compatible family of *transition* morphisms $f^*\mathcal{F}_X \to \mathcal{F}_Y$. We may call it a *crystal* if all transition maps are bijective.

If X is an algebraic variety over k, then realization induces an equivalence

$$\mathbf{Crys}(X) \simeq \mathcal{O}_X{-}\mathbf{Mod} \quad \text{and} \quad \mathcal{O}_{/X}{-}\mathbf{Mod}^{\mathrm{fp}} \simeq \mathbf{Coh}(X).$$

Again, we can do the same thing with $\mathbf{Sch}_{/K}$, $\mathbf{Sch}_{/\mathcal{V}}$, $\mathbf{FSch}_{/\mathcal{V}}$ or $\mathbf{An}_{/K}$ for example. In the last case, we first define $\mathcal{O}$ for the Tate topology and then restrict to the usual topology.

We will use later on these considerations in order to give a natural treatment to the notion of overconvergent isocrystal. Our next aim is to introduce the correct formalism in order to deal with the cohomological aspect.

Derived functors

The idea behind the notion of triangulated categories and derived functors is to put altogether the cohomology groups in a natural way.

If $\mathcal{C}$ is an additive category, we may consider the category $\mathbf{C}^+(\mathcal{C})$ of all complexes $K^\bullet$ with $K^n = 0$ for $n << 0$. Any additive functor $F : \mathcal{C} \to \mathcal{C}'$ extends trivally to a functor

$$F : \mathbf{C}^+(\mathcal{C}) \to \mathbf{C}^+(\mathcal{C}').$$

Assume $\mathcal{C}$ is an abelian category with enough injectives (some $\mathcal{O}-\mathbf{Mod}$ category for example). Then any complex $K^\bullet$ has an *injective resolution* $I^\bullet$ (a quasi-isomorphism $K^\bullet \simeq I^\bullet$ with all I^n injective).

When F is left exact, we *want to* define the right derived functor of F by the formula

$$\mathrm{R}FK^\bullet := FI^\bullet$$

in order to get

Homotopy

Two morphisms of complexes $f, g : K^\bullet \to L^\bullet$ are said to be *homotopic* if there exists a collection of maps $s^n : K^n \to L^{n-1}$ such that for all n, we have

$$g^n - f^n := s^{n+1} \circ d^n + d^{n-1} \circ s^n.$$

The category $\mathbf{K}^+(\mathcal{C})$ is the category of complexes up to homotopy: objects are the same as in $\mathbf{C}^+(\mathcal{C})$ and morphisms are morphisms in $\mathbf{C}^+(\mathcal{C})$ modulo homotopy.

Any additive functor $F : \mathcal{C} \to \mathcal{C}'$ will induce a functor

$$F : \mathbf{K}^+(\mathcal{C}) \to \mathbf{K}^+(\mathcal{C}').$$

Derived category

If $K^\bullet$ is a complex in a abelian category $\mathcal{C}$, then

$$H^n(K^\bullet) := \ker d^n / \mathrm{im}\ d^{n-1}.$$

Any morphism of complexes $f : K^\bullet \to L^\bullet$ induces maps

$$H^n(K^\bullet) \to H^n(L^\bullet).$$

If all these maps are isomorphisms, then f is called a *quasi-isomorphism*.

The derived category $\mathbf{D}^+(\mathcal{C})$ is the category of complexes up to quasi-isomorphisms: objects are as in $\mathbf{K}^+(\mathcal{C})$ and morphisms are given by

$$\begin{array}{ccc} K'^\bullet & & \\ {\scriptstyle \mathrm{q-is}}\downarrow & \searrow & \\ K^\bullet & \dashrightarrow & L^\bullet. \end{array}$$

If $\mathcal{C}$ has enough injectives and $\mathcal{I}$ is the full subcategory of all injective objects, then

$$\mathbf{K}^+(\mathcal{I}) \simeq \mathbf{D}^+(\mathcal{C})$$

(see Proposition 1.7.10 of [4]). We may then define the derived functor of F by making commutative the following diagram:

$$\begin{array}{ccc}
\mathbf{D}^+(\mathcal{C}) & \xrightarrow{\mathrm{R}F} & \mathbf{D}^+(\mathcal{C}') \\
\simeq \uparrow & & \uparrow \\
\mathbf{K}^+(\mathcal{I}) & & \\
\downarrow & & \\
\mathbf{K}^+(\mathcal{C}) & \xrightarrow{F} & \mathbf{K}^+(\mathcal{C}').
\end{array}$$

We will write

$$\mathrm{R}^n F K^\bullet := H^n(\mathrm{R}F K^\bullet).$$

Note that any object E of $\mathcal{C}$ may be seen as a complex concentrated in degree 0 and RFE will be a well defined object of $\mathbf{D}^+(\mathcal{C}')$.

Example

If $\mathcal{C}$ is a site, we may consider the global section functor on $X \in \mathcal{C}$:

$$\begin{aligned} \mathbf{Ab}(\mathcal{C}) &\longrightarrow \mathbf{Ab} \\ \mathcal{F} &\longmapsto \Gamma(X, \mathcal{F}) := \mathcal{F}(X) \end{aligned}$$

If $\mathcal{F}^\bullet$ is a complex of abelian group on $\mathcal{C}$, we write

$$\mathrm{R}^n\Gamma(X, \mathcal{F}^\bullet) =: H^n(X, \mathcal{F}^\bullet).$$

Assume X is a locally contractile metric space. If $\mathbf{Z}_X$ denotes the sheaf associated to the constant presheaf $\mathbf{Z}$ on X, we have

$$H^n(X, \mathbf{Z}_X) \simeq H^n_{\mathrm{sing}}(X).$$

Now, we have the necessary background in order to formally define overconvergent isocrystals and their cohomology.

overconvergent varieties

We let K be a non trivial complete ultrametric field with valuation ring $\mathcal{V}$, maximal ideal $\mathfrak{m}$ and residue field k.

We want to define rigid cohomology as the cohomology of a topos, generalizing in some sense the crystalline topos of Berthelot and the convergent topos of Ogus.

An *overconvergent variety* over $\mathcal{V}$ is a diagram

$$X \hookrightarrow P \xleftarrow{\mathrm{sp}} P_K \xleftarrow{\lambda} V$$

where the first map is a locally closed embedding of a k-algebraic variety into formal $\mathcal{V}$-scheme and λ is a morphism from a good analytic variety to P_K.

We will then consider the *tube* $]X[_V$ which is an analytic domain inside V and the embedding $i_X :]X[_V \hookrightarrow V$. We recall the following fundamental example:

Example (MW)

Built out of a $\mathcal{V}$-algebra A of finite presentation, we have

$$X = \mathrm{Spec}\ A_k \hookrightarrow \widehat{\mathbf{P}^n_{\mathcal{V}}} \longleftarrow \mathbf{P}^{n,\mathrm{an}}_K \hookleftarrow V = (\mathrm{Spec}\ A_K)^{\mathrm{an}}$$

and

$$]X[_V = (\mathrm{Spf}\ \hat{A})_K = \mathcal{M}(\hat{A}_K) = V \cap \mathbf{B}^n_K(0, 1^+).$$

Formal morphisms

A *formal morphism* of overconvergent varieties is a commutative diagram

$$\begin{array}{ccccccc} X' & \hookrightarrow & P' & \longleftarrow & P'_K & \xleftarrow{\lambda'} & V' \\ \downarrow f & & \downarrow v & & \downarrow v_K & & \downarrow u \\ X & \hookrightarrow & P & \longleftarrow & P_K & \xleftarrow{\lambda} & V. \end{array}$$

It will induce a map

$$]f[_u :]X'[_{V'} \to]X[_V$$

on the tubes.

Overconvergent varieties and formal morphisms form a category $\mathbf{An}_{/\mathcal{V}}$.

Example

A very simple but important example of formal morphism is given by the zero section:

$$\begin{array}{ccccccc} X & \hookrightarrow & \widehat{\mathbf{A}}^n_P & \leftarrow & \mathbf{B}^n_{P_K}(0,1^+) & \leftarrow & \mathbf{B}^n_V(0,1^+) \\ \| & & \downarrow & & \downarrow & & \downarrow \\ X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V \end{array}$$

that will induce

$$]X[_V \times_K \mathbf{B}^n_{P_K}(0,1^-) \to]X[_V$$

on the tubes.

Strict neighborhoods

A *strict neighborhood* is a formal morphism of overconvergent varieties

$$\begin{array}{ccccccc}
X & \hookrightarrow & Q & \leftarrow & Q_K & \leftarrow & W \\
\| & & \downarrow & & \downarrow & & \downarrow \\
X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V
\end{array}$$

where the first map is an equality and the last map is a (Berkovich) open immersion inducing an equality $]X[_W =]X[_V$ on the tubes.

Example (MW)

In the Monsky-Washnitzer situation, we may replace $\widehat{\mathbf{P}^n_{\mathcal{V}}}$ with the completion of the Zariski closure Q of $\mathrm{Spec}\, A$ in $\mathbf{P}^n_{\mathcal{V}}$ and/or V with

$$V_\lambda := V \cap \mathbf{B}^n(0, \lambda^+)$$

and obtain a strict neighborhood

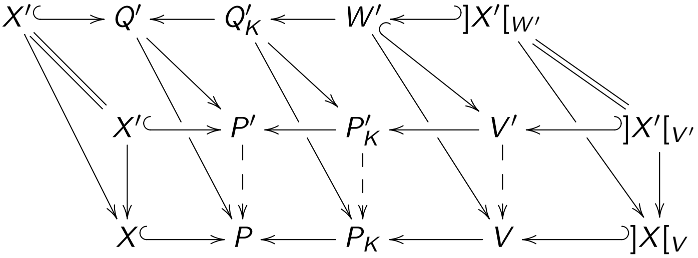
$$\begin{array}{ccccccc}
X & \hookrightarrow & \widehat{Q} & \leftarrow & Q_K^{\mathrm{an}} & \hookleftarrow & V_\lambda \\
\| & & \downarrow & & \downarrow & & \downarrow \\
X & \hookrightarrow & \widehat{\mathbf{P}^n_{\mathcal{V}}} & \leftarrow & \mathbf{P}_K^{n,\mathrm{an}} & \hookleftarrow & V.
\end{array}$$

Recall that the link with Monsky-Washnitzer theory is given by

$$A_K^\dagger = \varinjlim A_\lambda \quad \text{with} \quad V_\lambda = \mathcal{M}(A_\lambda).$$

Morphisms of overconvergent varieties

A *morphism of overconvergent varieties* over $\mathcal{V}$ is a formal morphism up to a strict neighborhood:



More precisely, we may consider the category $\mathbf{An}_{/\mathcal{V}}$ of overconvergent varieties and formal morphisms, and then define $\mathbf{An}^{\dagger}_{/\mathcal{V}}$ as the quotient category on the right with respect to the family of all strict neighborhoods.

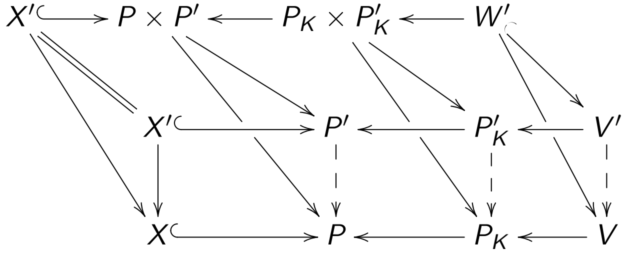
Any morphism in $\mathbf{An}^{\dagger}_{/\mathcal{V}}$ will provide a pair of morphisms

$$(f : X' \to X, \quad u : W' \to V)$$

where W' is a neighborhood of $]X'[_{V'}$ in V'. Moreover, they will satisfy

$$\mathrm{sp}(\lambda(u(x'))) = f(\mathrm{sp}(\lambda'(x'))$$

for $x' \in]X'[_{V'}$. And the same property holds after any isometric extension of K. One can show that, conversely, any such data determines a unique morphism of overconvergent varieties:



Taking this into account, we will usually denote by (X, V) and object of $\mathbf{An}^{\dagger}_{/\mathcal{V}}$ (neglecting P unless it plays a particular role).

Example (MW)

Assume that we are given two $\mathcal{V}$-algebras of finite type A and B and a morphism $\varphi : A^{\dagger} \to B^{\dagger}$. On one hand, we may consider the reduction

$$\varphi_k : A_k \to B_k$$

of φ mod $\mathfrak{m}$ and the corresponding morphism

$$f : Y = \mathrm{Spec}\ B_k \to X = \mathrm{Spec}\ A_k.$$

But we can also work on the generic fibre side.

Example (continuing)

We consider the weak completion

$$\varphi_K^\dagger : A_K^\dagger = \varinjlim A_\lambda \to B_K^\dagger = \varinjlim B_\mu$$

that will induce $A_\lambda \to B_\mu$ for suitable λ, μ and therefore

$$u : W_\mu \to V_\lambda \hookrightarrow V$$

And we get a morphism

$$\begin{array}{ccccccc}
Y = \mathrm{Spec}\ B_k & \hookrightarrow & \widehat{\mathbf{P}^m_{\mathcal{V}}} & \longleftarrow & \mathbf{P}_K^{m,\mathrm{an}} & \hookleftarrow & W = (\mathrm{Spec}\ B_K)^{\mathrm{an}} \\
\downarrow f & & & & & & \downarrow u \\
X = \mathrm{Spec}\ A_k & \hookrightarrow & \widehat{\mathbf{P}^n_{\mathcal{V}}} & \longleftarrow & \mathbf{P}_K^{n,\mathrm{an}} & \hookleftarrow & V = (\mathrm{Spec}\ A_K)^{\mathrm{an}}
\end{array}$$

which is not a formal morphism.

Theorem (Strong fibration theorem)

Let

$$
\begin{array}{ccccccc}
 & & P' & \longleftarrow & P'_K & \longleftarrow & V' \\
 & \nearrow & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle v_K} & & \big\downarrow{\scriptstyle u} \\
X & \searrow & P & \longleftarrow & P_K & \longleftarrow & V
\end{array}
$$

be a formal morphism such that

1. *v is smooth in the neighborhood of X,*
2. *v induces a proper morphism from the Zariski closure of X in P' to (the Zariski closure of X in) P,*
3. *$V' = P'_K \times_{P_K} V$.*

Then, locally for the Zariski topology on X and the Tate topology on V, we have in $\mathbf{An}^\dagger(\mathcal{V})$,

$$(X, V') \simeq (X, \mathbf{B}^n_V(0, 1^+)) \quad \text{over } (X, V).$$

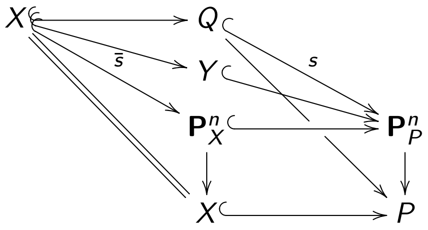
Proof.

We assume first that the result is known in the étale case and work locally on X and V.

Since we use Tate topology, the question is also local on P.

Blowing up outside X will induce an isomorphism in $\mathbf{An}^{\dagger}(\mathcal{V})$. Thus, we may assume that v is projective (Chow's lemma).

We embed the closure Y of X in P' into some $\mathbf{P}^n_P$ and lift the regular section $\overline{s}$ as follows



Proof. (the end)

Then, the map $Q \to P$ *is étale, and we may therefore assume that* $Q = P$*, and in particular that* Y *is also the closure of* X *in* P*. The question is now local on* Y*. We may therefore assume that* P' *is affine.*

Since P' *is smooth over* P *around* X*, we may assume that the conormal sheaf of* X *into* P'_X *is free and lift a basis to sections of* P' *defining an étale map* $P' \to \mathbf{A}^n_{\mathcal{V}}$*. And we are done.*

It still remains to do the étale case which says that $(X, V') \simeq (X, V)$ *(global result).*

Using the weak fibration theorem, it is sufficient to prove that the morphism $V' \to V$ *is (Berkovich) étale in the neighborhood of the tube. Formally étale follows from hypothesis 1 and boundaryless follows from hypothesis 2.*

The overconvergent site

We endow $\mathbf{An}_{/\mathcal{V}}$ with the coarsest topology making the forgetful functor

$$\begin{array}{ccc} \mathbf{An}_{/\mathcal{V}} & \longrightarrow & \mathbf{An}_{/K} \\ (X \subset P \leftarrow V) & \longrightarrow & V \end{array}$$

cocontinuous. Then, we endow $\mathbf{An}^{\dagger}_{\mathcal{V}}$ with the finest topology making the canonical functor $\mathbf{An}_{/\mathcal{V}} \to \mathbf{An}^{\dagger}_{/\mathcal{V}}$ continuous.

Thus, the topology of $\mathbf{An}^{\dagger}_{\mathcal{V}}$ is induced by the pretopology whose coverings are all formal morphisms

$$\{(X \subset P_i \leftarrow V_i)\}_{i \in I} \to (X \subset P \leftarrow V)$$

where $\{V_i\}_{i \in I}$ is a (Berkovich) open covering of a neighborhood of $]X[_V$ in V.

Restricted site

An *overconvergent presheaf* on $\mathcal{V}$ is a presheaf T on $\mathbf{An}^\dagger_{\mathcal{V}}$. We will be interested in the restricted site $\mathbf{An}^\dagger{}_{/T}$ of objects over T. In particular, if (X, V) is any overconvergent variety on $\mathcal{V}$, we may consider the site

$$\mathbf{An}^\dagger{}_{/X,V}$$

of overconvergent varieties over (X, V).

There exists a pair of morphisms of toposes

$$\widetilde{\mathbf{An}^\dagger{}_{/X,V}} \underset{\psi_{X,V}}{\overset{\varphi_{X,V}}{\rightleftarrows}} \widetilde{\mathbf{Open}(]X[_V)}$$

with

$$\varphi_{X,V} \circ \psi_{X,V} = \mathrm{Id} \quad \text{and} \quad \varphi_{X,V*} = \psi^{-1}_{X,V}.$$

It is characterized by

$$\varphi_{X,V}^{-1}(]X[_{V'}) = (X, V')$$

if V' is an open domain in V.

If $\mathcal{F}$ is an overconvergent sheaf over T and (X, V) an overconvergent variety over T, then the *realization* of $\mathcal{F}$ on (X, V) is

$$\mathcal{F}_{X,V} := \varphi_{X,V*}\mathcal{F}_{/X,V} \quad \text{on }]X[_V.$$

Realizations induce an equivalence between the category of sheaves on $\mathbf{An}^{\dagger}_{/T}$ and compatible families of sheaves on $]X[_V$ for all (X, V) over T. it is given by

$$\mathcal{F}(X, V) = \Gamma(]X[_V, \mathcal{F}_{X,V}).$$

Overconvergent site over a variety

If (C, O) is an overconvergent variety over $\mathcal{V}$, the forgetful functor

$$\begin{array}{rcl}\mathbf{An}^{\dagger}_{/C,O} & \longrightarrow & \mathbf{Sch}_{/C} \\ (X, V) & \longmapsto & X\end{array}$$

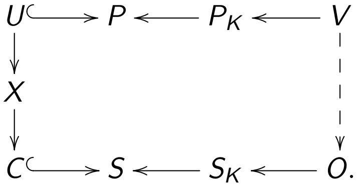
is left exact and continuous giving rise to a morphism of toposes

$$I : \widetilde{\mathbf{Sch}_{/C}} \to \widetilde{\mathbf{An}^{\dagger}_{/C,O}}.$$

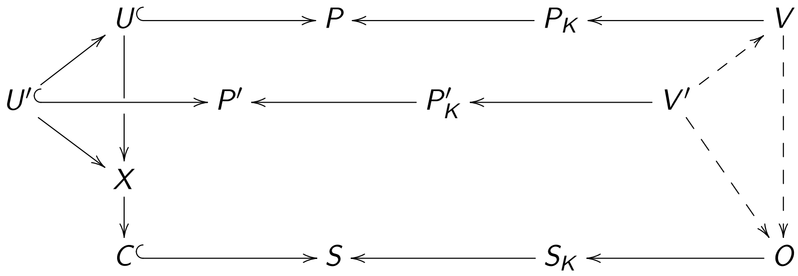
If X is a variety over C, we will write $X/O := I_*X$ and consider the overconvergent site $\mathbf{An}^{\dagger}_{/X/O}$ over X/O.

It can be described as follows:

An object of $\mathbf{An}^{\dagger}_{/X/O}$ is just an overconvergent variety (U, V) over (C, O) together with a fixed factorization $U \to X$ over C:



A morphism in $\mathbf{An}^{\dagger}_{/X/O}$ is a morphism of overconvergent varieties $(U', V') \to (U, V)$ over (C, O) where $U' \to U$ is defined over X:



Overconvergent modules

The presheaf

$$\mathcal{O}^\dagger_{\mathcal{V}} : (X, V) \mapsto \Gamma(]X[_V, i_X^{-1}\mathcal{O}_V)$$

is a sheaf of rings on $\mathbf{An}^\dagger_{\mathcal{V}}$. Its realization on (X, V) is $i_X^{-1}\mathcal{O}_V$.

More generally, if T is an overconvergent presheaf on $\mathcal{V}$, we may consider the restriction $\mathcal{O}^\dagger_{/T}$ of the structural sheaf.

Now, any morphism $(f, u) : (X', V') \to (X, V)$ of overconvergent varieties over $\mathcal{V}$ will induce a morphism of *ringed spaces*

$$(u_*, u^\dagger) : (]X'[_{V'}, i_{X'}^{-1}\mathcal{O}_{V'}) \to (]X[_V, i_X^{-1}\mathcal{O}_V)$$

with

$$u^\dagger\mathcal{F} := i_{X'}^{-1}u^* i_{X*}\mathcal{F}.$$

Giving an $\mathcal{O}^{\dagger}_{/T}$-module E is equivalent to giving the family of $i_X^{-1}\mathcal{O}_V$-module $E_{X,V}$ on $]X[_V$, for each (X,V) over T and compatible transition maps

$$u^{\dagger}E_{X,V} \to E_{X',V'}$$

for all morphism $(f,u) : (X',V') \to (X,V)$.

We will call E a *crystal* if all these maps are isomorphisms. Note that this is automatic if E is (locally) finitely presented.

We are interested into deriving the composed functor

$$p_{X/O} : \widetilde{\mathbf{An}^{\dagger}_{/X/O}} \to \widetilde{\mathbf{An}^{\dagger}_{/C,O}} \to \widetilde{\mathbf{Open}(]C[_O)}.$$

and apply it to crystals.

Overconvergent isocrystals again

Assume that we are given a morphism of overconvergent varieties $(X, V) \to (C, O)$. We may consider the diagonal embedding

$$\Delta : (X, V) \to (X, V^2).$$

as well as the projections

$$p_1, p_2 : (X, V^2) \to (X, V).$$

If E is an overconvergent module on X/O, we may consider the transition maps

$$p_2^\dagger E_{X,V} \to E_{X,V^2} \leftarrow p_1^\dagger E_{X,V}$$

If E is a crystal all these maps are isomorphisms.

Actually, if E is finitely presented, we obtain isomorphisms of *coherent* modules. In other words, $E_{X,V}$ becomes an *overconvergent isocrystal* on $(X, V)/O$.

Theorem ([7])

Let

$$\begin{array}{ccccccccc}
X & \hookrightarrow & P & \leftarrow & P_K & \leftarrow & V & \hookleftarrow &]X[_V \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow p \\
C & \hookrightarrow & S & \leftarrow & S_K & \leftarrow & O & \hookleftarrow &]C[_O
\end{array}$$

be a formal morphism such that

1. *P is smooth over S in the neighborhood of X,*
2. *the Zariski closure of X in P is proper over (the Zariski closure of C in) S,*
3. *$V = P_K \times_{S_K} O$.*

Then, we have

1. $\mathcal{O}_{/X/O}-\mathbf{Mod}^{\mathrm{fp}} \simeq \mathbf{Isoc}^{\dagger}(X, V/O)$
2. $\mathrm{R}p_{X/O*}E \simeq \mathrm{R}p_{\mathrm{rig}}E_{X,V}$.

Note that the left hand side does not depend on V. In particular, we obtain as corollary, the independence theorem of Berthelot.

The arguments used to prove the second assertion (*strong fibration theorem* and *overconvergent Poincaré lemma*) may be used to prove the first one. But these arguments require to work locally for the Tate topology of V (and the Zariski topology of X). Alternatively, one can use the following result.

Theorem (Covering)

In the situation of the above theorem, (X, V) *is a covering of* X/O*.*

Proof.

The idea is to build local sections. $\square$

Essential surjectivity in assertion 1 follows directly from this theorem. And so does Full faithfulness as we shall see right now.

Proof (full faithfulness).

It follows from the *covering theorem* that, if E is finitely presented, we have

$$p_{X/O*}E \simeq p_* E_{X,V}^{\nabla=0}.$$

Also, one easily checks that

$$\mathcal{H}om(E,F)_{X,V} = \mathcal{H}om_{i_X^{-1}\mathcal{O}_V}(E_{X,V},F_{X,V})$$

for finitely presented modules.

Thus, we get

$$p_{X/O*}\mathcal{H}om(E,F) = p_*\mathcal{H}om_\nabla(E_{X,V},F_{X,V})$$

and taking global sections gives

$$\mathrm{Hom}(E,F) = \mathrm{Hom}_\nabla(E_{X,V},F_{X,V}). \quad \square$$

Proof. (Assertion 2)

We consider the de Rham complex of $E_{X,V}$

$$\mathcal{F}^\bullet = E_{X,V} \otimes_{i_X^{-1}\mathcal{O}_V} i_X^{-1}\Omega_V^\bullet$$

and the derived linearization operator

$$\mathrm{R}L\mathcal{F}^\bullet = \mathrm{R}j_{X,V*}\varphi_{X,V}^*\mathcal{F}^\bullet.$$

with $j_{X,V} : \mathbf{An}^\dagger_{/X,V} \to \mathbf{An}^\dagger_{X/O}$. It is completely formal to check that

$$\mathrm{R}p_{X/O*}\mathrm{R}L\mathcal{F}^\bullet \simeq \mathrm{R}p_*\mathcal{F}^\bullet \quad (= \mathrm{R}p_{\mathrm{rig}}E_{X,V}).$$

In order to finish the proof of the theorem, it is therefore sufficient to verify that

$$E \simeq \mathrm{R}L\mathcal{F}^\bullet.$$

This can be checked on any (Y,W) over X/O.

Proof. (continuiing)

We use the diagonal embedding

$$
\begin{array}{ccccccc}
Y & \hookrightarrow & Q & \longleftarrow & Q_K & \longleftarrow & W \\
\Vert & & \uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle p_1} & & \uparrow{\scriptstyle p_1} \\
Y & \hookrightarrow & Q \times P & \longleftarrow & Q_K \times P_K & \longleftarrow & W \times V \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_2} \\
X & \hookrightarrow & P & \longleftarrow & P_K & \longleftarrow & V
\end{array}
$$

One easily computes

$$(\mathrm{R}L\mathcal{F}^\bullet)_{Y,W} = \mathrm{R}p_{1*}p_2^\dagger\mathcal{F}^\bullet.$$

Thus, we have to show that $E_{Y,W} = \mathrm{R}p_{1*}p_2^\dagger\mathcal{F}^\bullet$.

Proof. (the end)

We have

$$p_2^\dagger \mathcal{F}^\bullet = p_2^\dagger E_{X,V} \otimes_{i_Y^{-1}\mathcal{O}_{W\times V}} i_X^{-1}\Omega^\bullet_{W\times V/W}$$

$$= p_1^\dagger E_{Y,W} \otimes_{i_Y^{-1}\mathcal{O}_{W\times V}} i_X^{-1}\Omega^\bullet_{W\times V/W}$$

because E is a crystal.

Therefore, we just have to show that if $\mathcal{G}$ is a coherent $i_Y^{-1}\mathcal{O}_W$-module, the canonical map

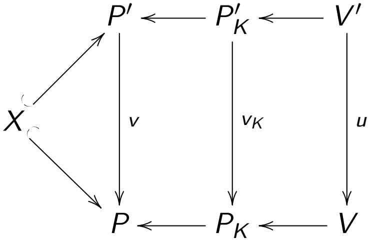
$$\mathcal{G} \to \mathrm{R}p_{1*}p_1^\dagger\mathcal{G} \otimes_{i_Y^{-1}\mathcal{O}_{W\times V}} i_X^{-1}\Omega^\bullet_{W\times V/W}$$

is a quasi-isomorphism and apply it to the case $\mathcal{G} = E_{Y,W}$.

Our theorem will therefore follow from the overconvergent Poincaré lemma below. More precisely, we will apply it to the embedding of Y into the product $Q \times P$ and to the first projection.

Theorem (Overconvergent Poincaré Lemma)

Let



be a formal morphism such that

1. *v is smooth in the neighborhood of X,*
2. *v induces a proper morphism from the Zariski closure of X in P' to (the Zariski closure of X in) P,*
3. *$V' = P'_K \times_{P_K} V$.*

If $\mathcal{F}$ is a coherent $i_X^{-1}\mathcal{O}_V$-module, there is a quasi-isomorphism

$$\mathcal{F} \simeq \mathrm{R}u_* u^\dagger \mathcal{F} \otimes_{i_X^{-1}\mathcal{O}_{V'}} i_X^{-1}\Omega^\bullet_{V'/V}.$$

Proof.

The question is local for the Zariski topology on X and the Tate topology on V. Therefore, using the strong fibration theorem, we may assume that

$$(X, V') = (X, \mathbf{B}^n_V(0, 1^+)).$$

Using an induction argument, we may actually reduce to the case $n = 1$.

The theorem therefore follows from its local one dimensional version below. □

Theorem (Local Overconvergent Poincaré lemma)

Let

$$X \hookrightarrow P \leftarrow P_K \leftarrow V$$

be an overconvergent variety. Denote by $\overline{X}$ *the Zariski closure of* X *in* P*. Assume that*

1. P *is affine*
2. *The complement of* X *in* $\overline{X}$ *is a hypersurface*
3. V *is affinoid*
4. *We have* $]\bar{X}[_V = V$

Then, if $\mathcal{F}$ *is a coherent* $i_X^{-1}\mathcal{O}_V$*-module, we have*

$$\Gamma(]X[_V, \mathcal{F}) \simeq \mathrm{R}\Gamma(]X[_V \times \mathbf{D}(0, 1^-), p^\dagger\mathcal{F} \stackrel{\partial/\partial t}{\to} p^\dagger\mathcal{F}).$$

Proof.

Shrinking V if necessary, we may assume that $\mathcal{F} = i_V^{-1}\mathcal{M}$ with $\mathcal{M}$ coherent on V. If

$$X = \{x \in \overline{X}, \quad \overline{g}(x) \neq 0\}$$

we will write for $\lambda < 1$,

$$V^\lambda = \{x \in V, \quad |g(x)| \geq \lambda\}.$$

Then, the LHS of the equality is

$$M = \varinjlim M^\lambda \quad \text{with} \quad M^\lambda = \Gamma(V^\lambda, \mathcal{M}).$$

Now, we set for $\eta_k \overset{<}{\to} 1$,

$$M_k := \Gamma(]X[_V \times \mathbf{D}(0, \eta_k^+), p^*\mathcal{M}).$$

We have $M_k = \varinjlim M_k^\lambda$ and $M_k^\lambda := M^\lambda \hat{\otimes} K\{t/\eta_k\}$.

Proof. (continuing)

Now, one can check that the RHS of the equality is the cohomology of the bicomplex

$$\begin{array}{ccc} \prod M_k & \xrightarrow{\partial} & \prod M_k \\ \big\downarrow \mathrm{d} & & \big\downarrow \mathrm{d} \\ \prod M_k & \xrightarrow{\partial} & \prod M_k \end{array}$$

with

$$\mathrm{d}(s_k) = s_{k+1} - s_k \quad \text{with} \quad \partial(s_k) = \partial/\partial t(s_k)$$

Then, we consider the integration map

$$\begin{aligned} \textstyle\int : M_k^{\lambda} &\longrightarrow M_{k-1}^{\lambda} \\ t^i &\longmapsto \tfrac{t^{i+1}}{i+1} \end{aligned}$$

and extend it to $\int : \prod M_k \to \prod M_k$.

Proof. (the end)

One easily checks that

$$\mathrm{d} \circ \int = \int \circ \, \mathrm{d} \quad \text{and} \quad \partial \circ \int = \int \circ \, \partial + \text{constant} = \mathrm{Id} + d.$$

It follows that the bicomplex has no cohomology except in degree 0 where we get M. And we are done.

We can check the middle part for example: given $s, s' \in \prod M_k$ such that $\partial(s) = \mathrm{d}(s')$, we have to find $s'' \in \prod M_k$ such that $\mathrm{d}(s'') = s$ and $\partial(s'') = s'$. If s has no constant term, we may just choose $s'' = \int s' - s$. When s is constant and $s' = 0$, we may just set $s'' = h(s)$ where h is the section defined by

$$0 \longrightarrow M \longrightarrow M^{\mathbf{N}} \xrightarrow{\ \mathrm{d}\ } M^{\mathbf{N}} \longrightarrow 0.$$

$$\left(-\sum_0^{k-1} s_i\right) \xleftarrow[h]{} (s_k)$$

Conclusion

We can finally give a definition of rigid cohomology and overconvergent isocrystals that is completely independent of the choices.

Definition

Let X be an algebraic variety over C, $C \hookrightarrow S$ a formal embedding and $O \to S_K$ any morphism of analytic varieties with O good.

1. An *overconvergent isocrystal* on X/O is a finitely presented module on the ringed site $\mathbf{An}^{\dagger}_{/X/O}$.
2. If E is an overconvergent isocrystal on X/O, its *rigid cohomology* is $Rp_{rig}E := Rp_{X/O*}E$.

In other words, these objects (overconvergent isocrystals and rigid cohomology) do exist and we can compute them the way we want !

– Thank you –

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