One century of *p*-adic geometry
From Hensel to Berkovich and beyond

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Writing numbers

Example

$$3103.59 = 9.10^{-2} + 5.10^{-1} + 3.10^{0} + 0.10^{1} + 1.10^{2} + 3.10^{3}$$

Theorem

1. *Any decimal number can be uniquely written*

$$\sum_{-\infty < n < \infty} a_n p^n \quad \text{with} \quad 0 \leqslant a_n < p \quad \text{and} \quad p = 10.$$

2. *Given any fixed $p \geqslant 2$, any whole number can be uniquely written*

$$\sum_{0 \leqslant n < \infty} a_n p^n \quad \text{with} \quad 0 \leqslant a_n < p.$$

Hensel, 1905

Definition

The *p-adic integers* and the *p-adic numbers* are defined by

$$\mathbf{Z}_p := \{\sum_{n \geqslant 0} a_n p^n \quad \text{with} \quad 0 \leqslant a_n < p\}$$

$$\mathbf{Q}_p := \{\sum_{n \geqslant -\infty} a_n p^n \quad \text{with} \quad 0 \leqslant a_n < p\}$$

Operations are defined by truncation/operation/truncation (and shift):

Example

How to compute a^2 for $p = 10$ and $a = \cdots 9999$?

$9^2 = 81 \Rightarrow a^2 = ?1,$

$99^2 = 9801 \Rightarrow a^2 = ?01,$

$999^2 = 998001 \Rightarrow a^2 = ?001$

Theorem

$$\mathbf{Z}_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z} \quad \text{and} \quad \mathbf{Q}_p = \mathbf{Z}_p\left[\frac{1}{p}\right]$$

(as topological rings).

Warning: $\mathbf{Z}_{10} \simeq \mathbf{Z}_2 \times \mathbf{Z}_5$ and $\mathbf{Q}_{10} \simeq \mathbf{Q}_2 \times \mathbf{Q}_5$ (Chinese Remainder Theorem): there exists zero divisors in $\mathbf{Q}_{10}$!

Theorem

If p is prime, $\mathbf{Z}_p$ is an integral domain and $\mathbf{Q}_p$ is its fraction field.

Example

In $\mathbf{Z}_2$, we have $1/3 = \sum(-1)^n 2^n$ and $-1 = \sum 2^n$.

We will only consider prime numbers p from now on.

p-adic absolute value

Definition

If $a = \frac{m}{n} p^r \in \mathbf{Q}$ with m, n not divisible by p, the *p-adic valuation* of a is $v_p(a) := r$ and the *p-adic absolute value* of a is $|a|_p = p^{-r}$.

We will write $|a|_\infty$ for the *usual* absolute value on $\mathbf{Q}$.

Theorem (Product formula)

$$\forall a \in \mathbf{Q}, \qquad \prod_{p \text{ prime or } p=\infty} |a|_p = 1$$

The p-adic absolute value extends naturally to $\mathbf{Q}_p$:

Definition

If $a = \sum_{n \geqslant r} a_n p^n \in \mathbf{Q}_p$ with $a_r \neq 0$, the *p-adic valuation* of a is $v_p(a) := r$ and the *p-adic absolute value* of a is $|a|_p = p^{-r}$

Ostrowski, 1918

Definition

An *absolute value* on a field K is a map $|\ | : K \to 0 \cup \mathbf{R}_{>0}$ such that

1. $\forall a \in K, \quad |a| = 0 \Leftrightarrow a = 0$
2. $\forall a, b \in K, \quad |a + b| \leqslant |a| + |b|$
3. $\forall a, b \in K, \quad |ab| = |a||b|$

It is said to be *ultrametric* if

$$\forall a, b \in K, \quad |a + b| \leqslant \max(|a|, |b|).$$

For example, $|\ |_p$ is an ultrametric absolute value and $|\ |_\infty$ is a non-ultrametric absolute value.

Theorem (Ostrowski)

Any absolute value on $\mathbf{Q}$ is of the form $|x| = |x|_p^\alpha$ for some p prime or infinity and $\alpha \in \mathbf{R}$

Ultrametric geometry

There is a *metric* $d(x,y) = |y - x|$ associated to the absolute value.

Theorem

If K is ultrametric, any triangle in K is isosceles.

Theorem

If K is ultrametric, any two discs are either disjoints, or one is contained in the other.

We will write

$$D(c,r) = \{x \in K, |x-c| \leqslant r\} \quad \text{and} \quad D(c,r^-) = \{x \in K, |x-c| < r\}.$$

Example

Inside $\mathbf{Q}_2$, we have $\mathbf{Z}_2 = D(0,1) = D(0,1^-) \coprod D(1,1^-)$.

Completions

Definition

A metric space is said to be *complete* if any decreasing intersection of closed discs whose radii go to 0 is non empty. It is said to be *maximally complete* if any decreasing intersection of closed discs is non empty.

Theorem

1. *Any valued field has a completion which is valued field.*
2. *The absolute value on a complete valued field extends uniquely to any finite field extension.*
3. *Any valued field is a contained in a maximally complete one.*

Example

The completion of $\mathbf{Q}$ for $|\ |_p$ is $\mathbf{Q}_p$ (and for $|\ |_\infty$, it is $\mathbf{R}$). They are maximally complete. The algebraic closure of $\mathbf{Q}_p$ is not complete (unlike $\mathbf{C}$) but its completion $\mathbf{C}_p$ is algebraically closed.

Krasner, 1954

Let K be an ultrametric field. As a topological space, it is *totally disconnected* (no other connected subsets other than points and $\varnothing$): there is no hope for a local theory of functions. Assume K is non trivial, complete, algebraically closed ($\mathbf{C}_p$ for example).

Definition

Let U be a bounded infinite subset of $K \cup \{\infty\}$. An *analytic element* on U is a uniform limit of rational functions with no pole on U. They form a K-algebra with a well defined norm $\|f\|_U = \max_{x \in U} |f(x)|$.

Example

The ring of analytic elements on $D(c, r)$ is the set of functions

$$f := \sum_{n \geqslant 0} a_n (T - c)^n, \quad |a_n| r^n \to 0$$

with $\|f\|_D = \max |a_n| r^n$.

Tate, 1961

Let K be a non trivial complete ultrametric field.

Definition

An *affinoid algebra* is a quotient A of the *Tate algebra*

$$K\{T_1, \ldots, T_m\} := \{\sum_{\underline{n}} a_{\underline{n}} \underline{T}^{\underline{n}}, \quad a_{\underline{n}} \in K, \quad |a_{\underline{n}}| \to 0\}$$

(endowed with the topology induced by the *Gauss norm* $\|f\| = \max |a_{\underline{n}}|$). The set $X := \mathrm{Spm}(A)$ of all maximal ideals of A is a *rigid affinoid variety*.

Example

The following map is a bijection when K is algebraically closed:

$$\begin{aligned} D(0,1) &\longrightarrow \mathrm{Spm}(K\{T\}). \\ a &\longmapsto (x-a) \end{aligned}$$

Acyclicity theorem

If $x \in \mathrm{Spm}(A)$, then $K(x) := A/x$ is a finite extension field of K and has a natural absolute value. We denote by $f(x)$ the image of f inside $K(x)$. If $f_1, \ldots, f_r$ are some generators of the unit ideal of A, then the subsets

$$X_i = \{x \in X, \forall j, |f_j(x)| \leqslant |f_i(x)\}$$

are affinoid, say $X_i = \mathrm{Spm} A_i$. And the same holds for $X_{ij} := X_i \cap X_j \ldots$

Theorem (Tate)

The sequence

$$0 \longrightarrow A \longrightarrow \oplus A_i \longrightarrow \oplus A_{ij} \longrightarrow \cdots$$

is exact.

One can then define *rigid analytic varieties* by pasting rigid affinoid varieties.

Topology

Rigid analytic varieties over K are not endowed with a usual topology but with a *Grotendieck topology*: There are some *admissible subsets* and *admissible coverings* and among them, affinoid ones.

Example

Assume K algebraically closed, then

1. An affinoid subset of $D(0,1)$ is a finite union of non trivial closed discs minus a finite union of open discs.
2. Any finite covering of an affinoid subset by affinoid subsets is admissible.
3. An open disc is admissible and $D(c, r^-) = \cup_{s<r} D(c, s)$ is an admissible covering.
4. The rigid projective line $\mathbf{P}^{1,\mathrm{rig}}$ is obtained by glueing two closed discs $D(0,1)$ and $D(\infty, 1)$ along the annulus $D(0,1) \backslash D(0,1^-)$.
5. Same as (1) for $\mathbf{P}^{1,\mathrm{rig}}$ or $\mathbf{A}^{1,\mathrm{rig}} := \mathbf{P}^{1,\mathrm{rig}} \backslash \{\infty\}$

Raynaud, 1974

If K is a non trivial complete ultrametric field, then $\mathcal{V} := D(0,1)$ is a complete local ring with maximal ideal $\mathfrak{m} := D(0,1^-)$. If $\mathcal{A}$ is a quotient of

$$\mathcal{V}\{T_1,\ldots,T_m\} := \{\sum_{\underline{n}} a_{\underline{n}} \underline{T}^{\underline{n}}, \quad a_{\underline{n}} \in \mathcal{V}, \quad a_{\underline{n}} \to 0\},$$

one can define the *affine formal schemes* $\mathrm{spf}\mathcal{A}$ (made of open primes) and paste them in order to get *formal schemes*. Formal schemes come with a usual topology. Any formal scheme has a *generic fiber* $\mathcal{X}_K$ which is a rigid analytic variety.

Example

The generic fibre of $\widehat{\mathbf{A}}^1_{\mathcal{V}} := \mathrm{Spf}(\mathcal{V}\{T\})$ is $\mathbf{D}^{\mathrm{rig}}(0,1) := \mathrm{Spm}(K\{T\})$.

Theorem (Raynaud)

Generic fiber induces an equivalence between qc formal schemes, up to formal blowing up, and qcqs rigid analytic varieties.

Berkovich, 1990

Definition

A *multiplicative semi-norm* on an affinoid algebra A is a map $|\ | : A \to 0 \cup \mathbf{R}_{>0}$ such that

1. $\forall a \in K, f \in A, \quad |af| = |a||f|$
2. $\forall f, g \in A, \quad |f + g| \leqslant \max(|f|, |g|)$
3. $\forall f, g \in A, \quad |fg| = |f||g|$

Definition

The set $\mathcal{M}(A)$ of continuous multiplicative semi-norms on A is called an *affinoid variety* (or space).

Theorem

For the topology of simple convergence, $\mathcal{M}(A)$ is compact Hausdorff locally simply connected and locally arcwise connected. And $\dim \mathcal{M}(A) = \dim A$.

The Berkovich disc

Example

There is a *generic point* map

$$\{\text{closed discs}\} \subset D(0,1) \longrightarrow \mathcal{M}(K\{T\}) =: \mathbf{D}^{\text{an}}(0,1).$$
$$D \longmapsto \xi := |\ |_D := \|\ \|_D$$

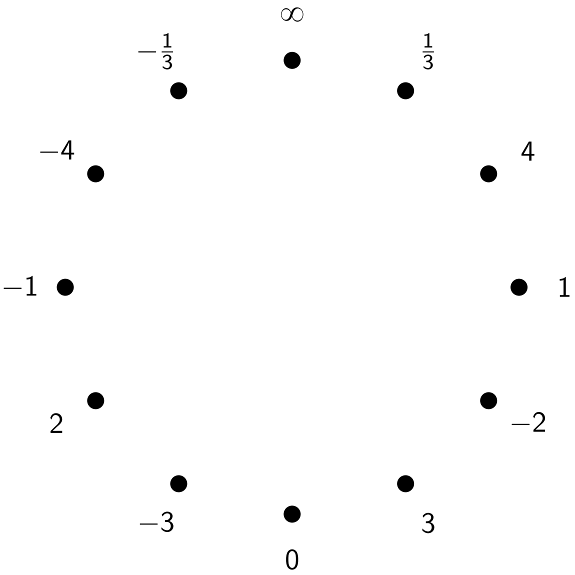
We will denote by $\xi_{c,r}$ the generic point of $D(c,r)$. If K is algebraically closed and maximally complete, this is a bijection.

In general, there is an natural inclusion $\mathrm{Spm}(A) \hookrightarrow \mathcal{M}(A)$ sending x to the semi norm $f \mapsto |f(x)|$. For example, we will identify $a \in D(0,1)$ with $\xi_{a,0}$. An *analytic variety* is obtained by "pasting" affinoid varieties.

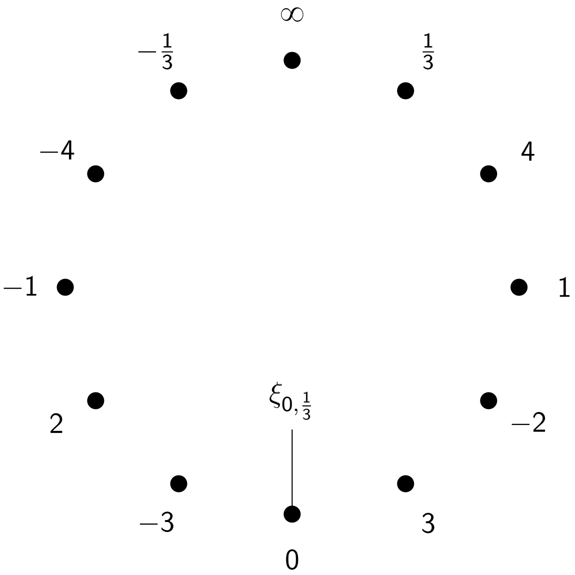
Example

If K is algebraically closed and maximally complete, $\mathbf{A}^{1,\text{an}}$ is in bijection with the set of all closed discs inside K.

The Berkovich projective line for $p = 3$



The Berkovich projective line for $p = 3$



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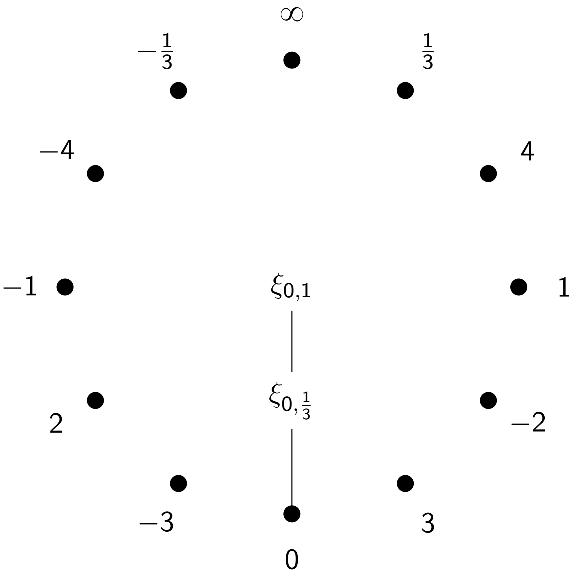

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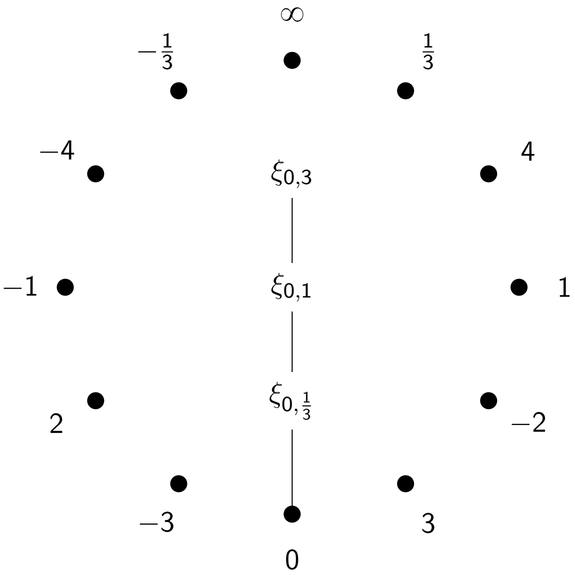

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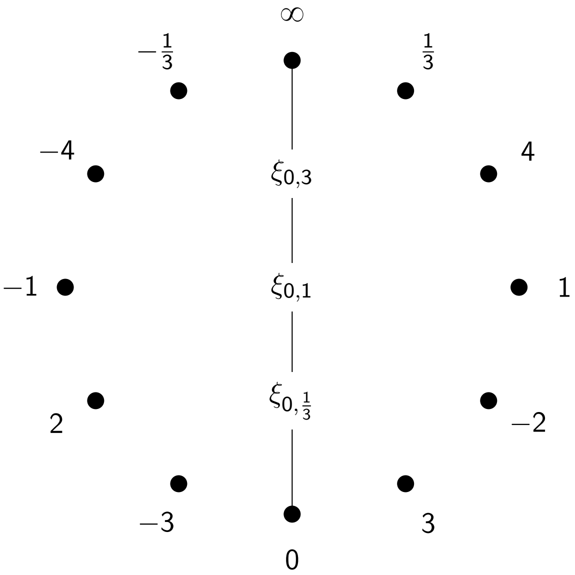

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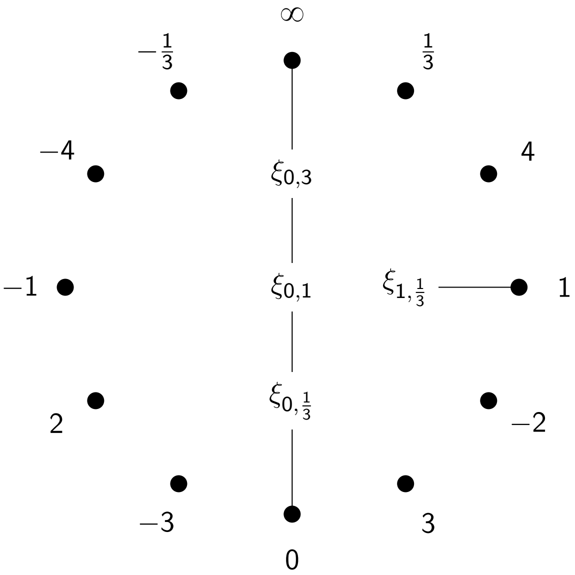

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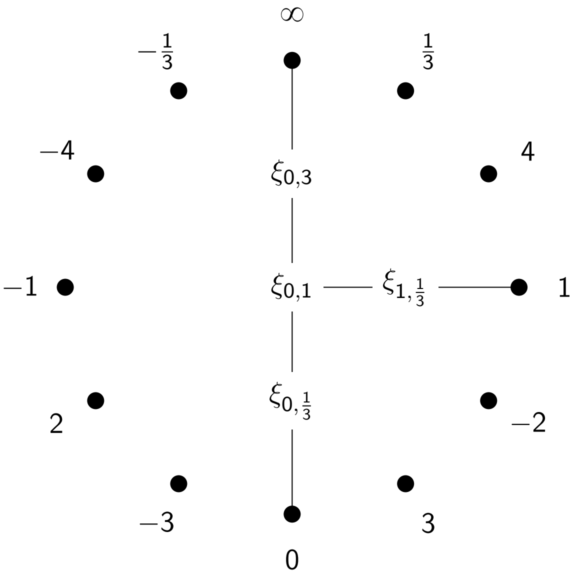

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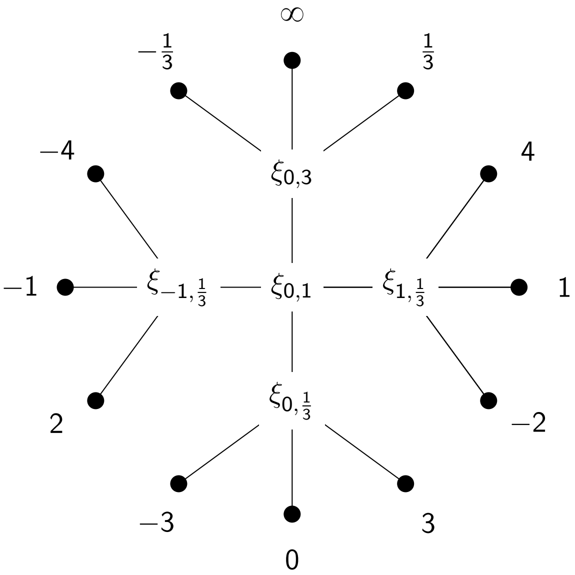


Huber, 1994

Definition

A *valuation* on an affinoid algebra A is a map $|\ | : A \to 0 \cup \Gamma$, where Γ is a totally ordered abelian group, such that

1. $|0| = 0, \quad |1| = 1 \quad \text{and} \quad |A^0| \subset [0,1]$
2. $\forall f, g \in A, \quad |f + g| \leqslant \max(|f|, |g|)$
3. $\forall f, g \in A, \quad |fg| = |f||g|$

Here A^0 denotes the set of power-bounded elements of A.

Definition

The set $\mathcal{P}(A)$ of all continuous valuations on A, up to equivalence, is called an *adic affinoid variety* (or space). It contains $\mathcal{M}(A)$.

The topology is generated by the subsets $X_i = \{x \in X, \forall j, |f_j(x)| \leqslant |f_0(x)\}$ where $f_0, \dots, f_r$ generate the unit ideal of A (not Hausdorff).

The Huber line

Example

Let $\Gamma_r = \mathbf{R}_{>0} \times (r^-)^{\mathbf{Z}}$, totally ordered with the rule $r^- < \alpha \Leftrightarrow r \leqslant \alpha$. We set

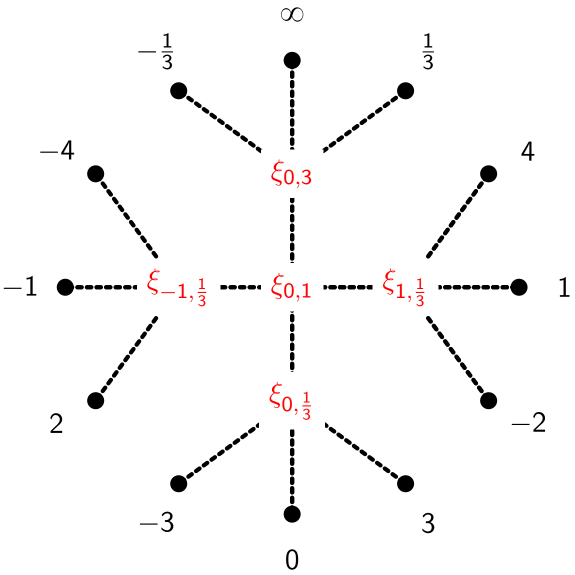
$$|f|_D := \max_n |a_n|(r^-)^n \quad \text{if} \quad D := D(c, r^-) \quad \text{and} \quad f := \sum_n a_n(T-c)^n.$$

Then, there is a *generic point* map

$$\begin{array}{ccc} \{\text{all discs}\} \subset K \cup \infty & \longrightarrow & \mathbf{P}^{1,\mathrm{ad}} \\ D & \longmapsto & \xi = |\ |_D \end{array}$$

(the complement of a disc of K is also a disc in $K \cup \infty$). Trivial and open discs are sent to closed points. Non trivial closed discs with radius in $|K^\times|$ are sent to non closed points. All the points of the adic line are obtained in this way when K is algebraically closed and maximally complete.

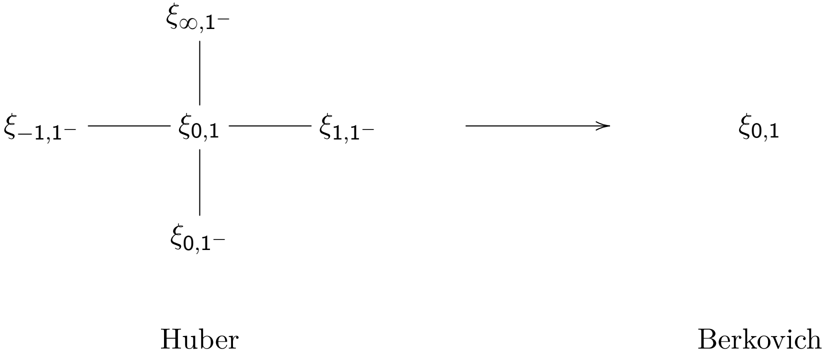
The Huber projective line for $p = 3$



Berkovich versus Huber

When A is an affinoid algebra, there inclusion $\mathcal{M}(A) \hookrightarrow \mathcal{P}(A)$ is not continuous but it has a unique continuous retraction. Any Huber point is adherent to a unique Berkovich point.

Example



Conclusion

There are 4 flavors of p-adic geometry, leading to equivalent categories.

1. TATE: The underlying set of an object is simple enough but this is not a usual topological space (not enough points).
2. RAYNAUD: Objects do not come with an underlying space but the whole theory derives from classical algebraic geometry.
3. BERKOVICH: Objects have a Hausdorff underlying space with many new points. But the categories of sheaves are not the same.
4. HUBER: Objects are sober topological spaces with still more points (not necessarily closed).

Note that all four theories treat more general cases that are not always compatible with the others.

More: Deligne, 1992

Definition

If $\mathcal{X}$ is a qc formal scheme, the *Zariski-Riemann space* of $\mathcal{X}$ is

$$\mathcal{P}(\mathcal{X}) := \lim_{\substack{\mathcal{X} \longleftarrow \mathcal{X}' \\ \text{blowing up}}} \mathcal{X}'.$$

Theorem

Let $\mathcal{X}$ be a qc formal scheme, then

1. *$\mathcal{P}(\mathcal{X})$ is a sober topological space and the topos of $\mathcal{P}(\mathcal{X})$ is equivalent to the topos of $\mathcal{X}_K$.*
2. *The Hausdorff quotient $\mathcal{M}(\mathcal{X})$ of $\mathcal{P}(\mathcal{X})$ is a (Berkovich) analytic variety.*

Actually, $\mathcal{P}(\mathcal{X})$ is an adic (Huber) variety. Moreover, if $\mathcal{X} = \mathrm{Spec}(\mathcal{A})$, we have $\mathcal{P}(\mathcal{X}) = \mathcal{P}(\mathcal{A})$ and $\mathcal{M}(\mathcal{X}) = \mathcal{M}(\mathcal{A})$.

More: van der Put/Schneider, 1995

Let X be a rigid analytic variety over K.

Definition

A *filter* on X is a non empty family of admissible subsets which is stable under finite intersections and containment. It is said to be *prime* if given any admissible covering of a subset of the filter, then some element of the covering must belong to the filter.

We denote by $\mathcal{P}(X)$ (resp. $\mathcal{M}(X)$) the set of all prime (resp. maximal) filters of X.

Theorem

1. *$\mathcal{P}(X)$ is a sober topological space, and the "inclusion" $X \hookrightarrow \mathcal{P}(X)$ induces an equivalence of topos.*
2. *$\mathcal{M}(X)$ is the maximal Hausdorff quotient of $\mathcal{P}(X)$ (for X pcqs).*
3. *For $X = \mathrm{Spm}(A)$, we have $\mathcal{P}(X) = \mathcal{P}(A)$ and $\mathcal{M}(X) = \mathcal{M}(A)$*

Ultimate: Abbes, 2010 (and beyond)

(see also Bosch/Lütkebohmert/Raynaud and Fujiwara/Kato)

Definition

The *category of qcqs rigid varieties* is “the” general category of qc formal schemes and tft morphisms, localized at formal blowing-ups.
The *category of qs rigid varieties* is the category of sheaves on the previous one that are locally representable by open immersions.

The topology is induced by finite coverings for the Zariski topology of formal schemes.

One can define *rigid points* of a rigid variety X, *adic points* as well as *analytic points*.

This way, one recovers Tate geometry, Berkovich geometry as well as Huber geometry.

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