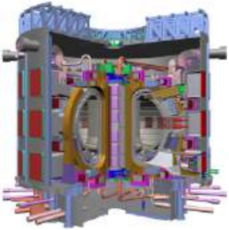
Numkin 2016

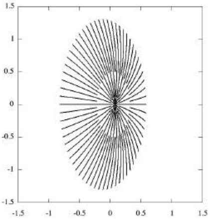


Targeting Realistic Geometry in Tokamak code Gysela

CEMRACS 2016 project
(Mathematic summer school in Marseille)

–

Collaborators of **CEA**:
Virginie Grandgirard, Guillaume Latu



Collaborators of **INRIA** and **University of Strasbourg**:
Michel Mehrenberger, Nicolas Bouzat

Collaborators of **IPP Garching** (Germany):
Camilla Bressan

–

Interdisciplinary collaboration
(mathematics, physics, computer science)

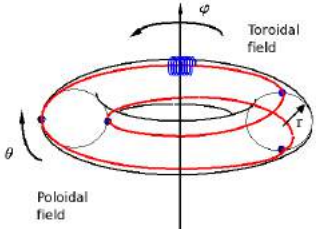
Outline

Improvement of the poloidal mesh

Tokamak geometry

Tokamak kinetic model: 6D phase space

- 3D in space:
  $\hookrightarrow$ toric geometry (r, θ, φ)
- 3D in velocity: $(v_{\parallel}, v_{\perp}, \varphi_c)$



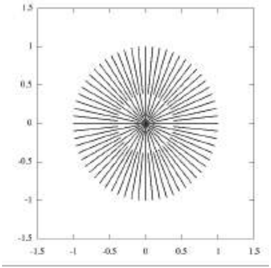


Figure: Circular shape (current reactors).

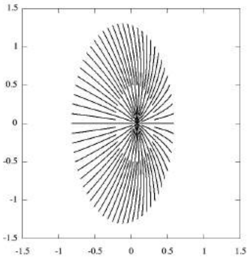
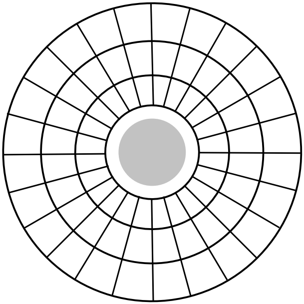


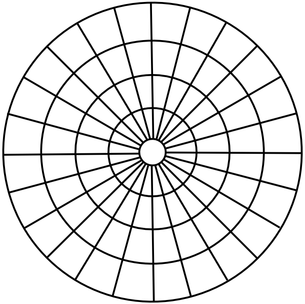
Figure: D shape (reactors under development).

Improvement of the poloidal mesh



Old grid:

- *hole* in the center;
- uniform in both directions.

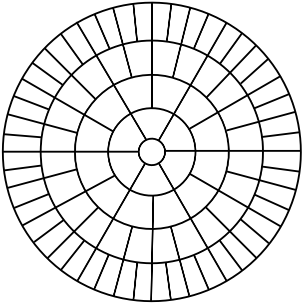


New uniform grid:

- $r_{min} = \frac{\Delta r}{2}$.

$\rightarrow$ 2D Poisson solver available (G. Latu talk).

(Ming-Chih LAI. *A note on finite difference discretizations for Poisson equation on a disk.*)



New non-uniform grid:

- Variable number of points in θ direction.

Uses less points for the same precision.

Assets of improved polar mesh

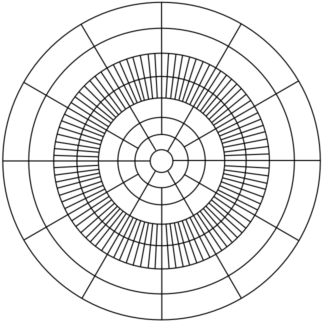


Figure: Improved polar mesh with high resolution on an annulus.

- Reduced number of points will allow for more physics (kinetic electrons) without loss of accuracy;

- Variable number of points in θ direction will allow to focus on specific structures developping at certain *radii*.
  High loss of accuracy on the rest of the plane.

Final goal: D-shape mapping

Culham mapping (analytic)

$$x = R_0 - \delta_0 r^2 + (1 - E_0) r \cos(\theta), \; y = (1 + E_0) r \sin(\theta)$$
$$E_0 = 0.3, \; R_0 = 0.08, \; \delta_0 = 0.2$$

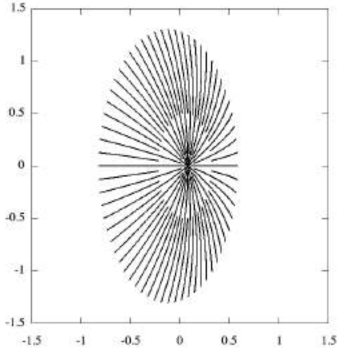


Figure: D shape poloidal plane.

Adapatable number of points in θ allows to easily introduce mappings (Culham, Miller) to achieve the desired shape for the plasma.

Gysela 5D: parallel domain decomposition

Input : *Physics parameters*, $\bar{f}^0$
Output : *Diagnostics*

for *time step* $n \geq 0$ **do**

- Integration: $\mathcal{N}_i^n(r,\theta,\varphi) = \int\int$ **Gyroaverage of** $\bar{f}^n \; B(r,\theta) \; dv_\parallel d\mu$;
- Solve **Field equation**: $\mathcal{N}_i^n(r,\theta,\varphi) \rightarrow$ **Gyroaverage of** $\Phi^n(r,\theta,\varphi)$;
- Broadcast of **Gyroaverage of** Φ^n to all processors;
- Diagnostics for time step n;
- 1D Advection in $v_\parallel$ $\quad (\forall(\mu,r,\theta) = [local], \forall(\varphi, v_\parallel) = [*])$;
- 1D Advection in φ $\quad (\forall(\mu,r,\theta) = [local], \forall(\varphi, v_\parallel) = [*])$;
- Transposition of $\bar{f}$;
- **2D Advection in** (r,θ) $\quad (\forall(\mu,\varphi,v_\parallel) = [local], \forall(r,\theta) = [*])$;
- Transposition of $\bar{f}$;
- 1D Advection in φ $\quad (\forall(\mu,r,\theta) = [local], \forall(\varphi, v_\parallel) = [*])$;
- 1D Advection in $v_\parallel$ $\quad (\forall(\mu,r,\theta) = [local], \forall(\varphi, v_\parallel) = [*])$;

- Gyroaverage, Poisson, 2D advection act on poloidal plane
- Aim: improve the poloidal mesh

Integration of new operators in Gysela

Input : $\bar{f}^{\star}(r,\theta,\varphi,v_{\parallel},\mu)$
Output : $\bar{f}^{\diamond}(r,\theta,\varphi,v_{\parallel},\mu)$

for μ // `MPI` **do in parallel**
  for $v_{\parallel}$ // `MPI+OpenMP` **do in parallel**
    for φ // `MPI` **do in parallel**

Advection operator

for θ *and* r **do**
  $(\Delta r, \Delta\theta) \leftarrow Compute\_displacement(Electric\ field, \Delta t)$

$Prepare\_interpolation$ (spline coeff if needed)

for θ *and* r **do**
  $\bar{f}^{\diamond}(r,\theta,\varphi,v_{\parallel},\mu) = interpolate(\bar{f}^{\star}(r-\Delta r, \theta-\Delta\theta, \varphi, v_{\parallel}, \mu))$

Lagrangian interpolation

2D-Lagrangian interpolation

- Gysela based on splines but non local -> Lagrange simpler;
- Two simple Lagrangian interpolations on θ then r;
- **Degree** choice has to be studied to achieve the desired accuracy.
- Stencil size $\leq$ (degree+1)^2

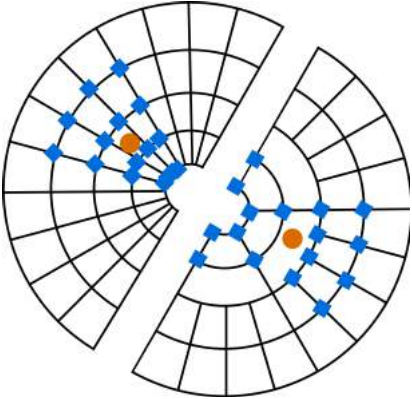


Figure: Lagrange of degree 3; stencil of 4 points. Uniform mesh (top) and non-uniform mesh (bottom).

Building test cases

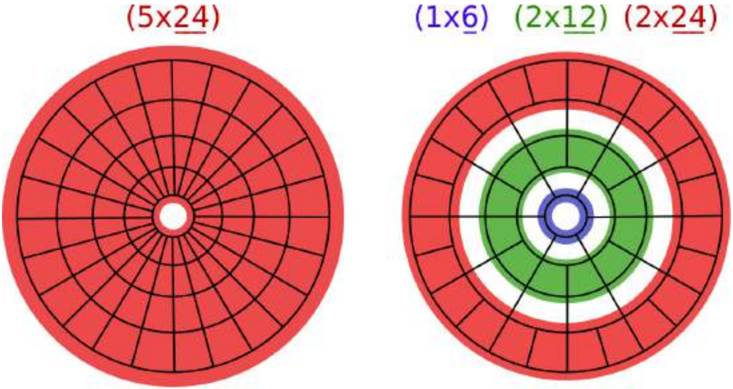


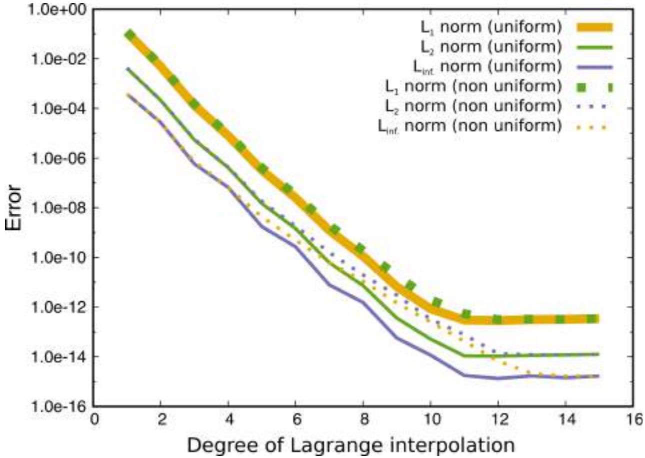
Figure: Corresponding uniform and non-uniform meshes used in tests.

Typical non-uniform mesh: $(2 \times \underline{16})$, $(8 \times \underline{32})$, $(16 \times \underline{64})$, $(102 \times \underline{128})$

2D-Lagrangian interpolation

Convergence in degree

Test case : $f(x,y) = cos(mx) * sin(ny)$

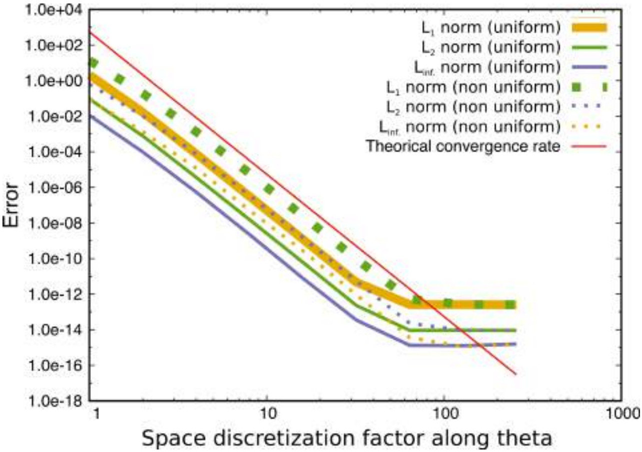


Uniform mesh : 65k points

Non-uniform mesh : 56k points

2D-Lagrangian interpolation

Convergence in space along θ (degree 7)

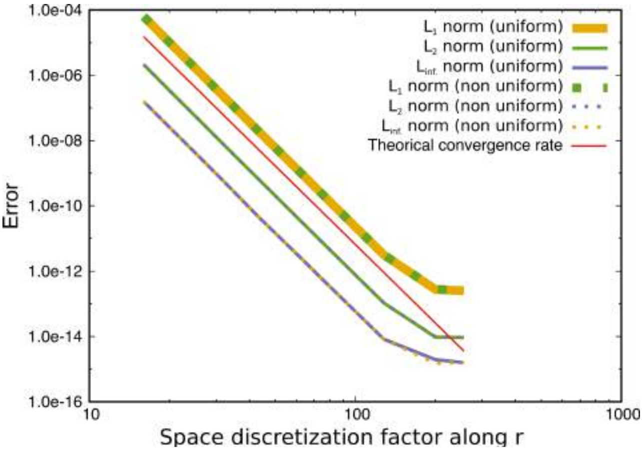


Uniform mesh : 4k points $\rightarrow$ 1 048k points

Non-uniform mesh : 3.5k points $\rightarrow$ 900k points

2D-Lagrangian interpolation

Convergence in space along r (degree 7)

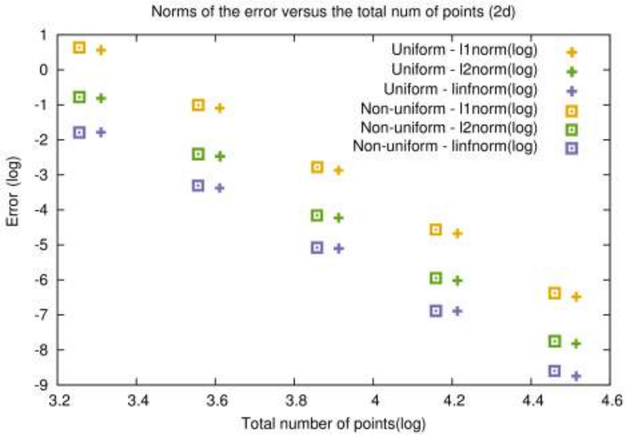


Uniform mesh : 4k points → 1 048k points

Non-uniform mesh : 3.5k points → 900k points

2D-Lagrangian interpolation

Number of points at constant error

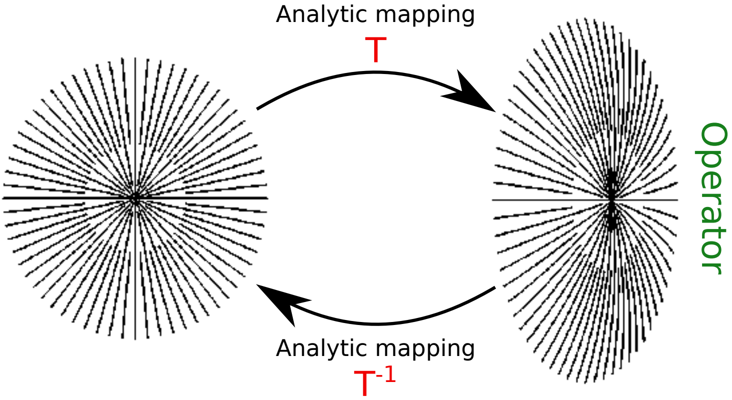


For a given accuracy the non-uniform mesh uses 15% less points.

2D-Lagrangian interpolation

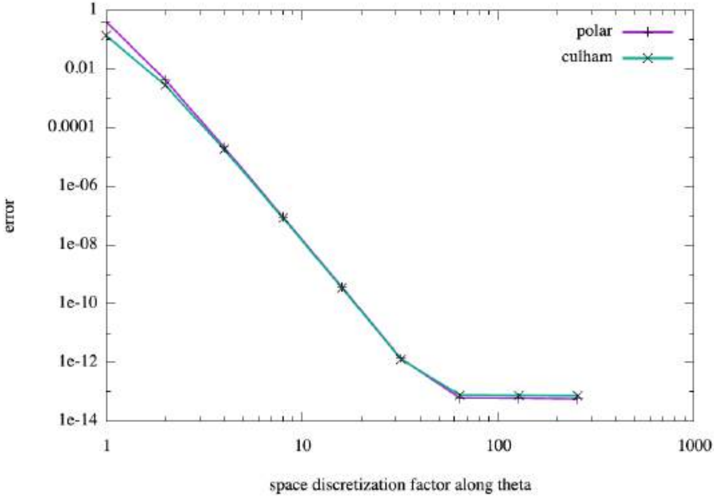
Culham mapping (analytic)

$$x = R_0 - \delta_0 r^2 + (1 - E_0) r \cos(\theta), \;\; y = (1 + E_0) r \sin(\theta)$$
$$E_0 = 0.3, \;\; R_0 = 0.08, \;\; \delta_0 = 0.2$$



2D-Lagrangian interpolation

Mapping: convergence in θ



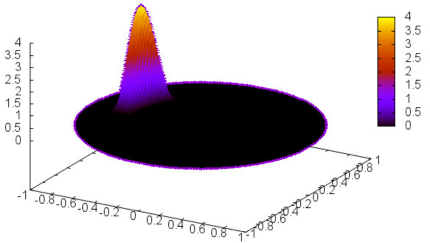
The mapping impact only slightly the accuracy of the operator.

Advection

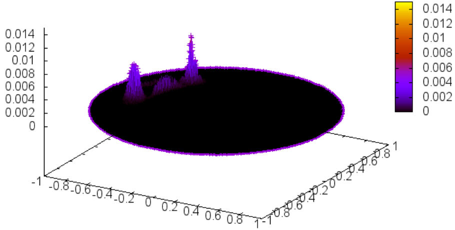
Advection along θ

Error evaluation on uniform mesh (16 384 points)

$$f(r, \theta, t + \Delta t) = f(r + v_r \Delta t, \theta + v_\theta \Delta t, t)$$



Reference function



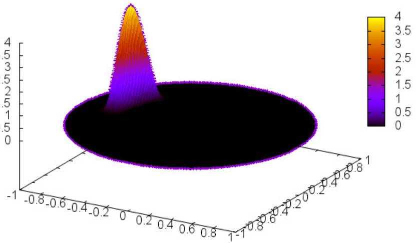
Error
(ref. value - interpolated value)

Lagrange interpolation of order 7.

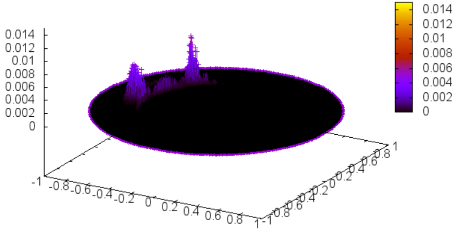
Advection along θ

Error evaluation on non uniform mesh (14 368 points)

Number of points by radius : $2 \times \underline{16}$, $8 \times \underline{32}$, $16 \times \underline{64}$, $102 \times \underline{128}$



Reference function

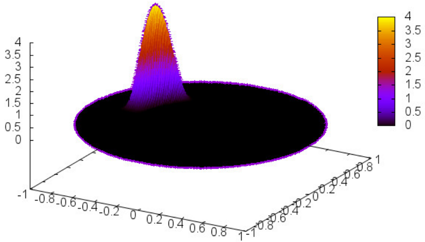


Error
(ref. value - interpolated value)

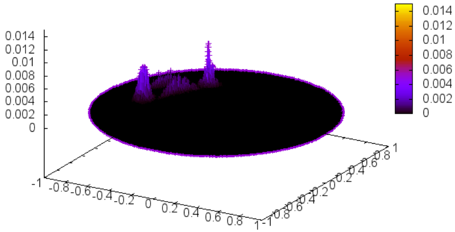
Lagrange interpolation of order 7.

Advection through the center

Error evaluation on uniform mesh (16 384 points)



Reference function



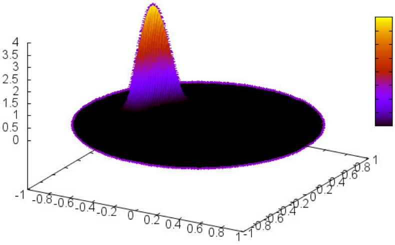
Error
(ref. value - interpolated value)

Lagrange interpolation of order 7.

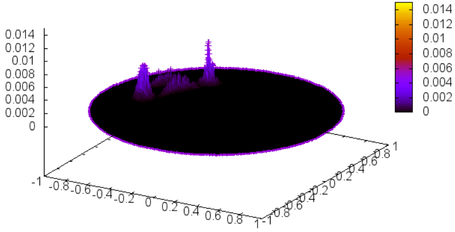
Advection through the center

Error evaluation on non uniform mesh (14 368 points)

Number of points by radius : 2×16, 8×32, 16×64, 102×128



Reference function



Error
(ref. value - interpolated value)

Lagrange interpolation of order 7.

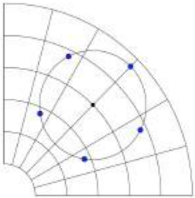
Gyroaverage operator

Gyroaverage operator

Given the Larmor radius ρ, the gyroaverage $\mathcal{J}_\rho$ of a function f at a point $\mathbf{x_g}$ can be written as:

$$\mathcal{J}_\rho(f)(\mathbf{x_g}) = \frac{1}{2\pi}\int_0^{2\pi} f(\mathbf{x_g} + \mathbf{a})d\alpha,$$

with $\mathbf{a} = \rho cos(\alpha)\mathbf{e_x} + \rho sin(\alpha)\mathbf{e_y}$ being the gyro-radius.



Different approaches are available. We have explored a method based on interpolation:

$$\mathcal{J}_\rho(f)(r_i,\theta_j) = \frac{1}{N}\sum_{k=0}^{N-1} P(f)(r_i cos(\theta_j) + \rho cos(\alpha_k), r_i sin(\theta_k) + \rho sin(\alpha_k))$$

where P is a Lagrange or a cubic spline interpolator and $\alpha_l = l \cdot \frac{2\pi}{N}$.

Gyroaverage operator

Testcases and parameters considered

Different testcases $f_k(r,\theta)$ have been considered:

- $J_m\left(j_{m,k} \cdot \frac{r}{r_{\max}}\right) e^{im\theta}$, with m as the Bessel function order and $j_{m,k}$ its k-th zero.

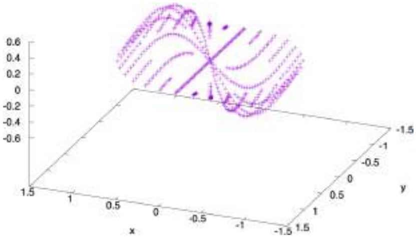


Figure: Bessel function of order 1.

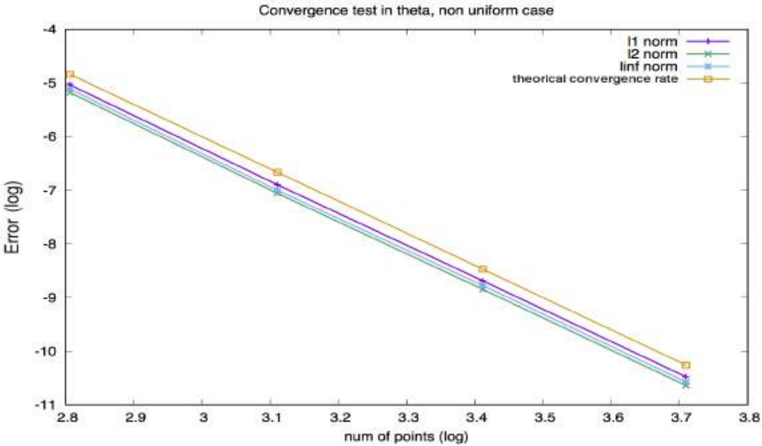
Gyroaverage is analytically known:

$$\mathcal{J}_\rho(f_k)(r_0,\theta_0) = J_0\left(j_{m,k} \cdot \frac{\rho}{r_{\max}}\right) f_k(r_0,\theta_0),$$

with ρ as the gyroradius.

Gyroaverage operator

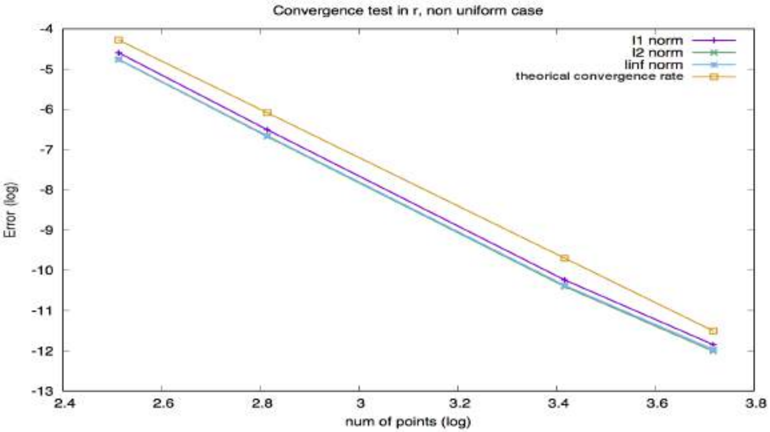
Convergence in θ direction, non uniform grid



Lagrange degree = 5. Parameters of the Bessel function testcase: $\rho = 0.1$, $n_r = 256$, $n_{\text{gp}} = 128$.

Gyroaverage operator

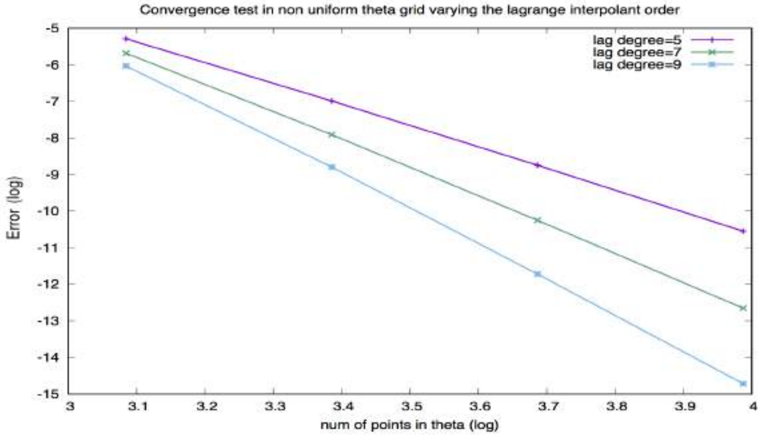
Convergence in r direction, non uniform grid



Lagrange degree = 5. Parameters of the Bessel function testcase: $\rho = 0.1$, $n_\theta = 256$, $n_{\text{gp}} = 128$.

Gyroaverage operator

Convergence on Lagrange order, non uniform grid



Parameters of the Bessel function testcase: $\rho = 0.1$, $n_r = 256$, $n_{\text{gp}} = 128$.

Poisson solver

Poisson solver for non-circular, non-uniform mesh

Currently in GYSELA, Poisson solver in toroidal geometry (r, θ, φ) is solved by using:

- In toroidal direction φ: Fourier projection
- In poloidal plan (r, θ): Finite Differences of 2nd order in radial direction and Fourier projection in poloidal direction

☺ Easy to implement and accurate
  - Parallelized in φ direction
  - Poloidal plan (r, θ) is treated sequentially

☹ Not adapted for a non-circular geometry and a non-uniform mesh

$$\Downarrow$$

idea: Keep Fourier projection in φ and use a 2D Finite Differences of 2nd order for (r, θ) direction.

First step: 2D Finite Differences Poisson solver for a disk

Let us consider the Poisson equation in polar coordinates on a disk $\Omega = \{(r, \theta) : 0 < r < r_{\max} \text{ with } r_{\max} \in \mathbb{R} \text{ and } 0 \leq \theta \leq 2\pi\}$,

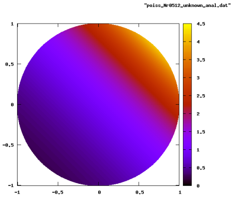
$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r}\frac{\partial^2 f}{\partial \theta^2} = R(r, \theta)$$

with Dirichlet boundary conditions $f(r = r_{\max}, \theta) = g(\theta)$ on $\partial\Omega$.

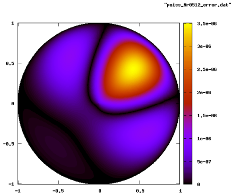
- We use the same trick than proposed by M.C Lai to treat the singularity at $r = 0$. *[Lai M.C, Notes]*
- The solver is tested for $R(r, \theta) = 2 * \exp(r(\cos(\theta) + \sin(\theta)))$ on the polar domain $\Omega = \{(r, \theta) : 0 < r < 1 \text{ and } 0 \leq \theta \leq 2\pi\}$.
- The linear sparse matrix system is solved by using the Intel PARDISO solver.

Poisson solver of 2nd order on a disk

Case $N_r = 512$ and $N_\theta = 1024$: Relative error compared to analytical solution $f(r,\theta) = \exp(r(\cos(\theta) + \sin(\theta)))$ of the order of 10^{-6}.

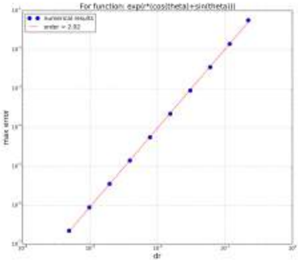


Analytic solution



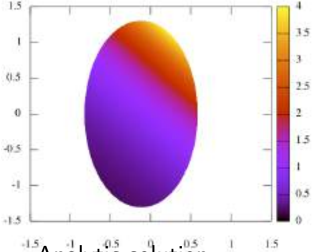
Relative error

2D Finite Differences scheme of second order

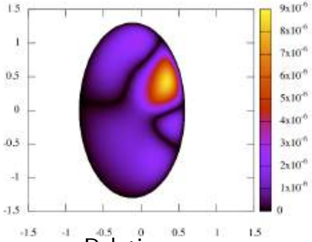


Poisson solver of 2nd order for Culham mapping

Previous algorithm modified to take into account Culham mapping



Analytic solution



Relative error

⇒ Scheme of second order

N	Relative error	Order
4	1.038D-01 (9.249D-02)	
8	2.941D-02 (2.385D-02)	1.82 (1.96)
16	7.777D-03 (6.089D-03)	1.92 (1.97)
32	2.201D-03 (1.626D-03)	1.82 (1.90)
64	5.592D-04 (4.118D-04)	1.98 (1.98)
128	1.407D-04 (1.037D-04)	1.99 (1.99)

- Next step: 2D Finite Differences for non-uniform Culham mesh

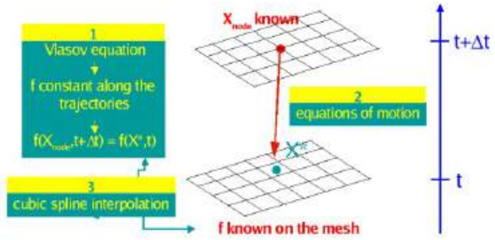
Final words

Conclusion, Perspectives

- Done: reduction of the number of points of 15%, analytic mapping for D-shape (Culham).
- Work in progress: cubic splines along theta direction, integration in Gysela and execution times study.

Results are engaging for further addition of kinetic electrons to Gysela and adjustable mesh structure allows for specific spatial solving. Results made it up to the expectation in term of performance.

The Semi-Lagrangian method (backward version)



- *f* conserved along characteristics
- Find the origin of the characteristics ending at the grid points
- Interpolate value at origin $X^\star$ of characteristics from known grid values → Interpolation method needed

- Typical interpolation schemes:
  - Cubic spline
  - Lagrange polynomial (high order)