Towards the heterogeneous multiscale method for simulation of the trajectory of a particle submitted to non-uniform electric field and strong magnetic field (2D-2V)

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Overview

Vlasov equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \left(\mathbf{E}(\mathbf{x}, t) + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}(\mathbf{x}, t) \right) \cdot \nabla_{\mathbf{v}} f = 0 \qquad (1)$$

$f(\mathbf{x}, \mathbf{v}, t = 0) = f_0(\mathbf{x}, \mathbf{v})$

$\mathbf{E}$: non-uniform electric field

$\mathbf{B}$: strong (non-uniform) magnetic field

$\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$

$\mathbf{E} = (\mathbf{E}_1, \mathbf{E}_2)$, $\mathbf{B}(\mathbf{x}, t) = \Phi(\mathbf{x}, t)\, e_3$

$t \in (0, T]$, $T \in \mathbb{R}^+$

2d-2v-1t

A singularly perturbed system of ODEs

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{V} \tag{2}$$

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{E}(\mathbf{X}, t) + \frac{1}{\varepsilon}\Phi(\mathbf{X}, t)\mathbf{V}^{\perp} \tag{3}$$

$$\text{i.e.: } \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{E}(\mathbf{X}, t) + \frac{1}{\varepsilon}\Phi(\mathbf{X}, t)\mathbf{M}\mathbf{V}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

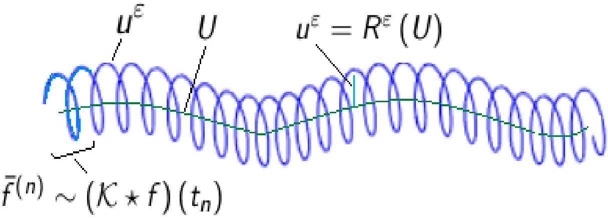
The Heterogeneous Multiscale Method (HMM) - 1

$$\mathcal{O}^{\varepsilon} u^{\varepsilon} = f, \quad \Omega \subset \mathbb{R}^d, 0 < \varepsilon \ll 1 \tag{4}$$

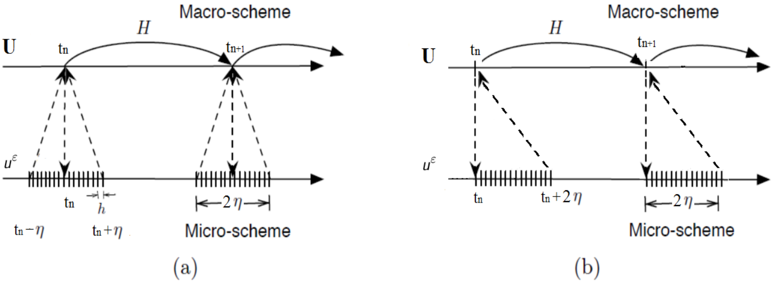
In the HMM:

- Microscopic model (4) with microscopic variables u^{ε}
- Macroscopic model ("Effective" model) with macroscopic variables U

$$\mathcal{O} U = \bar{f} \tag{5}$$



The Heterogeneous Multiscale Method (HMM) - 2



$[t_n - \eta, t_n + \eta]$: pseudo-period of the particle's trajectory.

Kernel $\mathcal{K}$:

a) Symmetric kernel

b) Nonsymmetric kernel

The Heterogeneous Multiscale Method (HMM) - 3

Reconstruction: At $T = t_n$: $u^{\varepsilon(n)} = R^{\varepsilon}\left(U^{(n)}\right)$

$$\mathbf{X}^{(n)} = \zeta_1^{(n)} \tag{6}$$

$$\mathbf{V}^{(n)} = e^{\frac{t_n}{\varepsilon}\mathbf{\Phi}\left(\zeta_1^{(n)}, t_n\right)\mathbf{M}}\zeta_2^{(n)} \tag{7}$$

The Heterogeneous Multiscale Method (HMM) - 4

- Microscopic model:

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t} = \mathbf{F}_1(\mathbf{X}, \mathbf{V}, t) \tag{8}$$

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F}_2(\mathbf{X}, \mathbf{V}, t) \tag{9}$$

$$\mathbf{F}_1(\mathbf{X}, \mathbf{V}, t) = \mathbf{V}, \quad \mathbf{F}_2(\mathbf{X}, \mathbf{V}, t) = \mathbf{E}(\mathbf{X}, t) + \tfrac{1}{\varepsilon}\Phi(\mathbf{X}, t)\,\mathbf{M}\mathbf{V}$$

- Macroscopic model:

$$\frac{\mathrm{d}\zeta_1}{\mathrm{d}t} = \bar{\mathbf{F}}_1(\zeta_1, \zeta_2, t) \tag{10}$$

$$\frac{\mathrm{d}\zeta_2}{\mathrm{d}t} = \bar{\mathbf{F}}_2(\zeta_1, \zeta_2, t) \tag{11}$$

The Heterogeneous Multiscale Method (HMM) - 5

The average kernel:

K_η: a symmetric, compactly supported kernel with support in $[-\eta, \eta]$

$$K_\eta(.) = \frac{1}{\eta} K\left(\frac{.}{\eta}\right) \quad (12)$$

$K \in \mathbb{K}^{p,q}(I)$: $K \in \mathbb{C}_C^q(\mathbb{R})$, $\operatorname{supp}(K) = I$, and

$$\int_{\mathbb{R}} K(t)\, t^i dt = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } 1 \leq i \leq p, \end{cases}$$

Additional condition: $K(t) = K(-t)$

The Heterogeneous Multiscale Method (HMM) - 6

$K^{\exp}$ kernel:

$$K^{\exp}(t) = C_0 \chi_{[-1,1]}(t) \exp\left(\frac{5}{t^2-1}\right) \tag{13}$$

$$C_0 \approx 211.075339092667055$$

$K^{\cos}$ kernel:

$$K^{\cos}(t) = \frac{1}{2}\chi_{[-1,1]}(t)\left(1+\cos(\pi t)\right). \tag{14}$$

$$\chi_{[-1,1]}(t) = \begin{cases} 1, & \text{if } -1 \leq t \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

The Heterogeneous Multiscale Method (HMM) - 7

At $T = t_n$:

$$\begin{pmatrix} \bar{\mathbf{F}}_1^{(n)} \\ \bar{\mathbf{F}}_2^{(n)} \end{pmatrix} = \begin{pmatrix} (K_\eta \star \mathbf{F}_1)(t_n) \\ (K_\eta \star \mathbf{F}_2)(t_n) \end{pmatrix} = \begin{pmatrix} \left(K_\eta \star \frac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}\right)(t_n) \\ \left(K_\eta \star \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t}\right)(t_n) \end{pmatrix} \tag{15}$$

$$(K_\eta \star f)(t_n) = \int_{t_n-\eta}^{t_n+\eta} K_\eta(t_n - t) f(t)\, \mathrm{d}t \tag{16}$$

$$\Rightarrow \begin{pmatrix} \bar{\mathbf{F}}_1^{(n)} \\ \bar{\mathbf{F}}_2^{(n)} \end{pmatrix} = \begin{pmatrix} \left(-\frac{\mathrm{d}K_\eta}{\mathrm{d}t} \star \mathbf{X}\right)(t_n) \\ \left(-\frac{\mathrm{d}K_\eta}{\mathrm{d}t} \star \mathbf{V}\right)(t_n) \end{pmatrix} \tag{17}$$

The Heterogeneous Mutiscale Method (HMM) - 8

In practice:

$$\bar{\mathbf{F}}_1^{(n)} = e^{\frac{t_n}{\varepsilon}\mathbf{\Phi}\left(\zeta_1^{(n)}, t_n\right)\mathbf{M}}\zeta_2^{(n)} \Leftarrow \text{Reduce computational cost}$$

$\Rightarrow$

$$\begin{pmatrix} \bar{\mathbf{F}}_1^{(n)} \\ \bar{\mathbf{F}}_2^{(n)} \end{pmatrix} = \begin{pmatrix} e^{\frac{t_n}{\varepsilon}\mathbf{\Phi}\left(\zeta_1^{(n)}, t_n\right)\mathbf{M}}\zeta_2^{(n)} \\ \left(-\frac{\mathrm{d}K_\eta}{\mathrm{d}t} \star \mathbf{V}\right)(t_n) \end{pmatrix} \tag{18}$$

Macro-solver: Forward Euler method (FE)
Micro-solver: 4-order Runge-Kutta method (rk4)
$\Rightarrow$ HMM-FE-rk4

Test cases

Test case 1: $\mathbf{E}(\mathbf{x}, t) = \begin{pmatrix} 2\mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}_1 + 2\mathbf{x}_2 \end{pmatrix}$, $\mathbf{\Phi}(\mathbf{x}, t) = 1$

$\varepsilon = 0.01$
$T_0 = 0,\ T = 1$
$H = 0.0001$
$\eta = \pi\varepsilon$
$h = \eta/500$
$K^{\exp}$
Initial conditions:
$x_0 = (0; 0),\ v_0 = (0.0001; 0.0001)$

Test case 1: $\mathbf{x}_0 = (0; 0)$, $\mathbf{v}_0 = (0.0001; 0.0001)$

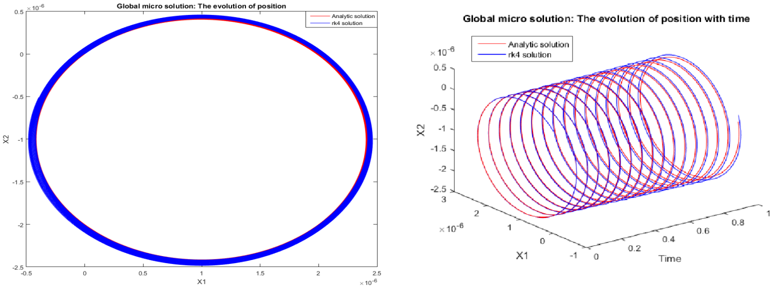


Figure 1: Global micro solver: The evolution of position

Euclidean Error $= 2.8905 \times 10^{-4}$

Test case 1: $\mathbf{x}_0 = (0; 0)$, $\mathbf{v}_0 = (0.0001; 0.0001)$

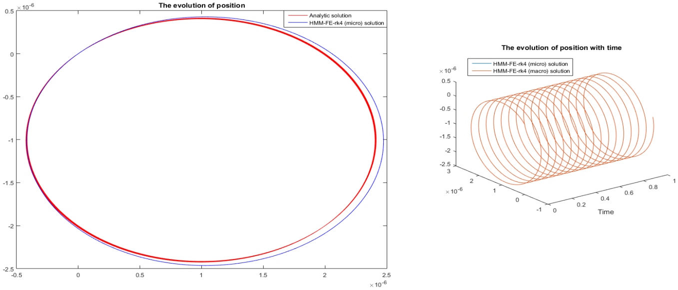


Figure 2: The evolution of position: Analytic solution and HMM-FE-rk4 (micro) solution

Euclidean Error $= 2.9006 \times 10^{-4}$

Test case 1: $\mathbf{x}_0 = (0; 0)$, $\mathbf{v}_0 = (0.0001; 0.0001)$

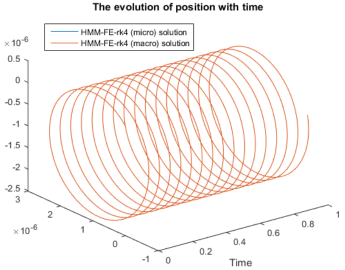


Figure 3: The evolution of position with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution

Test case 1: $\mathbf{x}_0 = (0; 0)$, $\mathbf{v}_0 = (0.0001; 0.0001)$

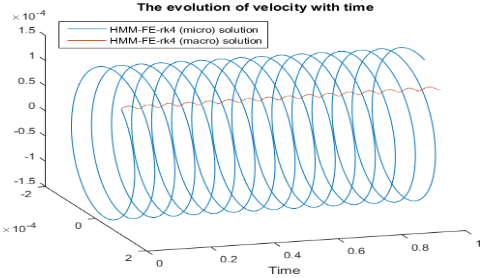


Figure 4: The evolution of velocity with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution

Test case 2

Test case 2:

$$\mathbf{E}(\mathbf{x}, t) = \begin{pmatrix} 2\mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}_1 + 2\mathbf{x}_2 \end{pmatrix}$$

$$\mathbf{\Phi}(\mathbf{x}, t) = 1 + 0.5\cos(\sqrt{\mathbf{x}_1^2 + \mathbf{x}_2^2})$$

$$\mathbf{x}_0 = (0; 0), \ \mathbf{v}_0 = (0.0001; 0.0001)$$

$\varepsilon = 0.01$
$T_0 = 0, T = 1$
$H = 0.0001$
$\eta = 0.02$
$h = \eta/500 = 0.00004$
K^{exp}

Test case 2:

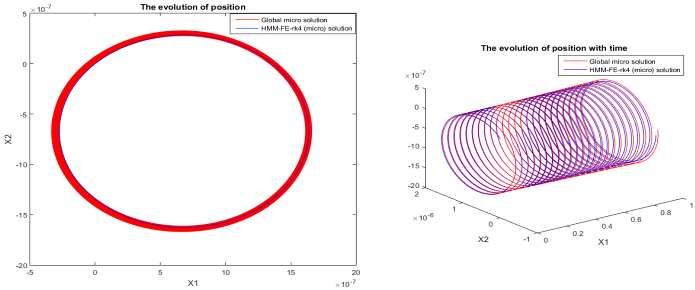


Figure 5: The evolution of position: Global micro solution (rk4) and HMM-FE-rk4 (micro) solution

Euclidean error $= 2.5005 \times 10^{-5}$

Test case 2:

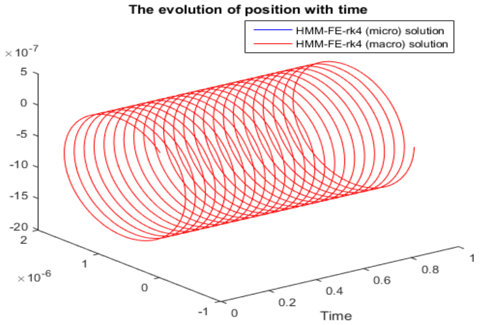


Figure 6: The evolution of position with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution

Test case 2:

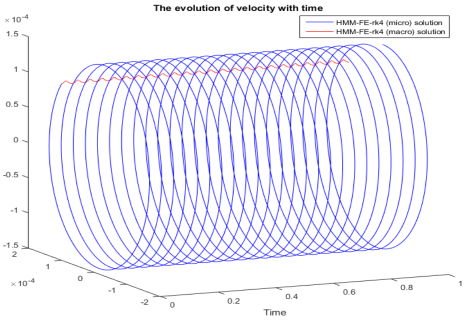


Figure 7: The evolution of velocity with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution

The pseudo-period of the particle's trajectory

The pseudo-period of the particle's trajectory - 1

Algorithm:

Step 1: $v_0 \Rightarrow v_0^\perp$

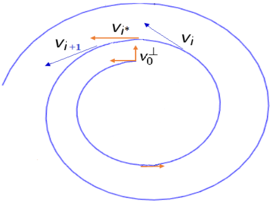
Step 2: Compute dot products: $v_0^\perp \cdot v_i$, $i = 1, 2, \dots$

Step 3: Find indices i where $v_0^\perp \cdot v_i$ in Step 2 changes sign i.e $v_0^\perp \cdot v_i > 0, v_0^\perp \cdot v_{i+1} < 0$, or $v_0^\perp \cdot v_i < 0, v_0^\perp \cdot v_{i+1} > 0$

Step 4: Find indices in Step 3 nearest to the time when the particle finishes a period.

Step 5: Find time $t_{i^\star}$ when the particle finishes a period: $t_{i^\star} = \frac{t_i + t_{i+1}}{2}$ or linear interpolation method.

Step 6: Pseudo-period $= t_{i^\star} - t_{i^\star - 1}$



The pseudo-period of the particle's trajectory - 2

Test case: $\mathbf{v} = (\cos(t), \sin(t))$
$t \in [0, 10\pi]$, $h = 10\pi/1000$
Pseudo period $= 6.28318530717958 \approx 2\pi$

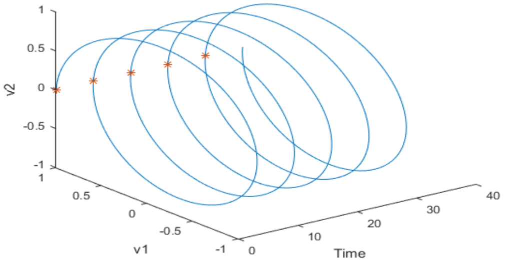


Figure 8: The pseudo-period of the particle's trajectory

The pseudo-period of the particle's trajectory - 3

Test case 1

Pseudo-period $\approx 2\pi\varepsilon$, $\varepsilon = 0.01$

The pseudo-period of the particle's trajectory - 4

Test case 2

Pseudo-period ≈ 0.04184, $\varepsilon = 0.01$

Conclusion

1. HMM-FE-rk4 $\Rightarrow$ Try other methods for micro-solver and macro-solver.
2. Build a database to have a better understanding of the influence of the electric field and magnetic field on the trajectory of the particle (the pseudo-period of the particle).

THANK YOU