Towards the heterogeneous multiscale method for simulation of the trajectory of a particle submitted to non-uniform electric field and strong magnetic field (2D-2V)

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Overview

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Towards the HMM for simulation of the trajectory of the charged particle in non-uniform electric field and strong magnetic field: 2D-2V

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The Heterogeneous Multiscale Method (HMM)

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Vlasov equation

\[
\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_x f + \left( \mathbf{E}(\mathbf{x}, t) + \frac{1}{\varepsilon} \mathbf{v} \times \mathbf{B}(\mathbf{x}, t) \right) \cdot \nabla_v f = 0 \quad (1)
\]


\[
f(\mathbf{x}, \mathbf{v}, t = 0) = f_0(\mathbf{x}, \mathbf{v})
\]

\(\mathbf{E}\): non-uniform electric field

\(\mathbf{B}\): strong (non-uniform) magnetic field

\(\mathbf{x} = (x_1, x_2), \ \mathbf{v} = (v_1, v_2)\)

\(\mathbf{E} = (E_1, E_2), \ \mathbf{B}(\mathbf{x}, t) = \Phi(\mathbf{x}, t) e_3\)

\(t \in (0, T], \ T \in \mathbb{R}^+\)

2d-2v-1t
A singularly perturbed system of ODEs

\[
\begin{align*}
\frac{dX}{dt} &= V \\
\frac{dV}{dt} &= E(X, t) + \frac{1}{\varepsilon} \Phi(X, t) V^\perp \\
\text{i.e.: } \frac{dV}{dt} &= E(X, t) + \frac{1}{\varepsilon} \Phi(X, t) \mathbf{M}V
\end{align*}
\]
The Heterogeneous Multiscale Method (HMM) - 1

Towards the HMM for simulation of the trajectory of the charged particle in non-uniform electric field and Strong Magnetic Field: 2D-2V

\[ \mathcal{O}^\varepsilon u^\varepsilon = f, \quad \Omega \subset \mathbb{R}^d, 0 < \varepsilon \ll 1 \] (4)

In the HMM:
- Microscopic model (4) with microscopic variables \( u^\varepsilon \)
- Macroscopic model ("Effective" model) with macroscopic variables \( U \)

\[ \mathcal{O} U = \bar{f} \] (5)
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The pseudo-period of the particle's trajectory.

Kernel $\mathcal{K}$:

a) Symmetric kernel

b) Nonsymmetric kernel

$[t_n - \eta, t_n + \eta]$: pseudo-period of the particle's trajectory.
Reconstruction: At $T = t_n$: $u^{\varepsilon(n)} = R^{\varepsilon} (U^{(n)})$

\[
X^{(n)} = \zeta_1^{(n)}
\]

\[
V^{(n)} = e^{\frac{t_n}{\varepsilon} \Phi(\zeta_1^{(n)}, t_n)} M \zeta_2^{(n)}
\]
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- Microscopic model:

\[
\frac{dX}{dt} = F_1(X, V, t) \quad (8)
\]

\[
\frac{dV}{dt} = F_2(X, V, t) \quad (9)
\]

\[
F_1(X, V, t) = V, \quad F_2(X, V, t) = E(X, t) + \frac{1}{\varepsilon} \Phi(X, t) MV
\]

- Macroscopic model:

\[
\frac{d\zeta_1}{dt} = \bar{F}_1(\zeta_1, \zeta_2, t) \quad (10)
\]

\[
\frac{d\zeta_2}{dt} = \bar{F}_2(\zeta_1, \zeta_2, t) \quad (11)
\]
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Additional condition: $K(t) = K(-t)$

The average kernel:

$K_\eta$: a symmetric, compactly supported kernel with support in $[-\eta, \eta]$

$$K_\eta(.) = \frac{1}{\eta} K\left(\frac{.}{\eta}\right) \tag{12}$$

$K \in K^{p,q}(I)$: $K \in \mathbb{C}_c^q(\mathbb{R})$, supp ($K$) = $I$, and

$$\int_{\mathbb{R}} K(t) t^i dt = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } 1 \leq i \leq p, \end{cases}$$
The Heterogeneous Multiscale Method (HMM)

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\[ K^{\text{exp}} \text{ kernel:} \]

\[ K^{\text{exp}} (t) = C_0 \chi_{[-1,1]} (t) \exp \left( \frac{5}{t^2 - 1} \right) \]  

\[ C_0 \approx 211.075339092667055 \]

\[ K^{\text{cos}} \text{ kernel:} \]

\[ K^{\text{cos}} (t) = \frac{1}{2} \chi_{[-1,1]} (t) (1 + \cos (\pi t)) . \]  

\[ \chi_{[-1,1]} (t) = \begin{cases} 
1, & \text{if } -1 \leq t \leq 1, \\
0, & \text{otherwise.} 
\end{cases} \]
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The Heterogeneous Multiscale Method (HMM) - 7

At $T = t_n$:

$$
\begin{pmatrix}
\vec{F}_1^{(n)} \\
\vec{F}_2^{(n)}
\end{pmatrix} = \begin{pmatrix}
(K_\eta \ast \vec{F}_1)(t_n) \\
(K_\eta \ast \vec{F}_2)(t_n)
\end{pmatrix} = \begin{pmatrix}
(K_\eta \ast \frac{dX}{dt})(t_n) \\
(K_\eta \ast \frac{dV}{dt})(t_n)
\end{pmatrix}
$$ (15)

$$(K_\eta \ast f)(t_n) = \int_{t_n-\eta}^{t_n+\eta} K_\eta(t_n-t)f(t)\,dt$$ (16)

$$\Rightarrow \begin{pmatrix}
\vec{F}_1^{(n)} \\
\vec{F}_2^{(n)}
\end{pmatrix} = \begin{pmatrix}
\left(-\frac{dK_\eta}{dt} \ast \vec{X}\right)(t_n) \\
\left(-\frac{dK_\eta}{dt} \ast \vec{V}\right)(t_n)
\end{pmatrix}$$ (17)
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In practice:

\[ \vec{F}_{1}^{(n)} = e^{\frac{t_n}{\varepsilon} \Phi(\zeta_{1}^{(n)}, t_n)} \mathbf{M} \zeta_{2}^{(n)} \iff \text{Reduce computational cost} \]

\[ \Rightarrow \]

\[ \begin{pmatrix} \vec{F}_{1}^{(n)} \\ \vec{F}_{2}^{(n)} \end{pmatrix} = \begin{pmatrix} e^{\frac{t_n}{\varepsilon} \Phi(\zeta_{1}^{(n)}, t_n)} \mathbf{M} \zeta_{2}^{(n)} \\ -\frac{dK_{n}}{dt} \star \mathbf{V}(t_n) \end{pmatrix} \] (18)

Macro-solver: Forward Euler method (FE)
Micro-solver: 4-order Runge-Kutta method (rk4)
\[ \Rightarrow \text{HMM-FE-rk4} \]
Test cases
Test case 1: \( \mathbf{E}(\mathbf{x}, t) = \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix}, \quad \Phi(\mathbf{x}, t) = 1 \)

\( \varepsilon = 0.01 \)
\( T_0 = 0, \quad T = 1 \)
\( H = 0.0001 \)
\( \eta = \pi \varepsilon \)
\( h = \eta/500 \)
\( K^{\text{exp}} \)

Initial conditions:
\( x_0 = (0; 0), \quad v_0 = (0.0001; 0.0001) \)
Test case 1: \( \mathbf{x}_0 = (0; 0), \mathbf{v}_0 = (0.0001; 0.0001) \)

**Figure 1: Global micro solver: The evolution of position**

Euclidean Error \( = 2.8905 \times 10^{-4} \)
Test case 1: \( \mathbf{x}_0 = (0; 0), \mathbf{v}_0 = (0.0001; 0.0001) \)

Figure 2: The evolution of position: Analytic solution and HMM-FE-rk4 (micro) solution

Euclidean Error = \( 2.9006 \times 10^{-4} \)
Test case 1: $x_0 = (0; 0), \ v_0 = (0.0001; 0.0001)$

Figure 3: The evolution of position with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution
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Figure 4: The evolution of velocity with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution

Test case 1: \( x_0 = (0; 0), v_0 = (0.0001; 0.0001) \)
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Test case 2
Test case 2:

\[
\begin{align*}
\mathbf{E}(x, t) &= \begin{pmatrix} 2x_1 + x_2 \\ x_1 + 2x_2 \end{pmatrix} \\
\Phi(x, t) &= 1 + 0.5 \cos(\sqrt{x_1^2 + x_2^2}) \\
x_0 &= (0; 0), \quad \mathbf{v}_0 = (0.0001; 0.0001)
\end{align*}
\]

\[
\begin{align*}
\varepsilon &= 0.01 \\
T_0 &= 0, \quad T = 1 \\
H &= 0.0001 \\
\eta &= 0.02 \\
h &= \eta/500 = 0.00004 \\
K^{\text{exp}}
\end{align*}
\]
Test case 2:

**Figure 5:** The evolution of position: Global micro solution (rk4) and HMM-FE-rk4 (micro) solution

Euclidean error $= 2.5005 \times 10^{-5}$
Test case 2:

**Figure 6:** The evolution of position with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution
Test case 2:

Figure 7: The evolution of velocity with time: HMM-FE-rk4 (micro) solution and HMM-FE-rk4 (macro) solution
The pseudo-period of the particle’s trajectory
The pseudo-period of the particle’s trajectory - 1

Algorithm:

**Step 1:** \( v_0 \rightarrow v_0 \parallel \)

**Step 2:** Compute dot products: \( v_0 \parallel \cdot v_i, \ i = 1, 2, \ldots \)

**Step 3:** Find indices \( i \) where \( v_0 \parallel \cdot v_i \) in Step 2 changes sign i.e \( v_0 \parallel \cdot v_i > 0, v_0 \parallel \cdot v_{i+1} < 0 \), or \( v_0 \parallel \cdot v_i < 0, v_0 \parallel \cdot v_{i+1} > 0 \)

**Step 4:** Find indices in Step 3 nearest to the time when the particle finishes a period.

**Step 5:** Find time \( t_i^* \) when the particle finishes a period:
\[
t_i^* = \frac{t_i + t_{i+1}}{2}
\] or linear interpolation method.

**Step 6:** Pseudo-period = \( t_i^* - t_{i^* - 1} \)
The pseudo-period of the particle’s trajectory

Test case: $\mathbf{v} = (\cos(t), \sin(t))$
$t \in [0, 10\pi], \ h = 10\pi/1000$

Pseudo period $= 6.28318530717958 \approx 2\pi$

Figure 8: The pseudo-period of the particle’s trajectory
**Test case 1**

Pseudo-period \( \approx 2\pi \varepsilon, \varepsilon = 0.01 \)
Test case 2
Pseudo-period ≈ 0.04184, $\varepsilon = 0.01$
Conclusion

1. HMM-FE-rk4 ⇒ Try other methods for micro-solver and macro-solver.
2. Build a database to have a better understanding of the influence of the electric field and magnetic field on the trajectory of the particle (the pseudo-period of the particle).
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