RELAXATION SCHEME FOR THE SIMULATION OF PLASMAS IN TOKAMAKS

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INTRODUCTION

We study the guiding-centre model which describes the drift of the plasma in a Tokamak :

$$\begin{cases} \partial_t \rho(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x}) \cdot \nabla \rho(t,\mathbf{x}) = 0 \\ -\Delta \phi(t,\mathbf{x}) = \rho(t,\mathbf{x}) \end{cases} \quad (1)$$

where $\mathbf{v}(t,\mathbf{x}) = E(t,\mathbf{x})^{\perp} = (-\nabla \phi(t,\mathbf{x}))^{\perp}$, ρ is the ion density, E is the electric field and ϕ the potential.

RELAXATION APPROXIMATION

We approximate the equation (1) with q transport equations at constant velocities $(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_q)$:

$$\partial_t \mathbf{f} + \sum_{k=1}^{d} \boldsymbol{\Lambda}_k \partial_{x_k} \mathbf{f} = \frac{1}{\varepsilon} \left(\mathbf{f}^{\mathrm{eq}}_{[\mathbf{v},\rho_{\mathbf{f}}]} - \mathbf{f} \right)$$

where $\mathbf{f}(t,\mathbf{x}) = (f_1(t,\mathbf{x}), f_2(t,\mathbf{x}), \dots, f_q(t,\mathbf{x})) \in \mathbb{R}^q$, $\boldsymbol{\Lambda}_k = \mathrm{diag}((\boldsymbol{\lambda}_1)_k, \dots, (\boldsymbol{\lambda}_2)_k)$, $\varepsilon > 0$ is a relaxation parameter and $\mathbf{f}^{\mathrm{eq}}_{[\mathbf{v},\rho]}$ is checking the consistency conditions :

$$\rho = \sum_{i=1}^{q} f^{\mathrm{eq}}_{[\mathbf{v},\rho],i} \quad \text{and} \quad \rho \mathbf{v} = \sum_{i=1}^{q} \boldsymbol{\lambda}_i f^{\mathrm{eq}}_{[\mathbf{v}\rho],i}.$$

In the limit $\varepsilon \to 0$, $\rho = \sum_{i=1}^{q} f_i$ tends to the solution of (1). We have to solve q equations instead of one, but they are linear, with constant velocities and can be parallelized.

SPLITTING STEPS

- **Transport step :**

$$f_i^*(t+\Delta t, \mathbf{x}) = f_i(t, \mathbf{x} - \Delta t \boldsymbol{\lambda}_i), \quad \forall i \in \{1, \dots, q\},$$

We can use a Fourier solver or a DG method for this step. We note : $\mathbf{f}^*(t+\Delta t,.) = T(\Delta t)\mathbf{f}(t,.)$.

- **Poisson step :**

We solve : $-\Delta \phi(t+\Delta t,.) = \sum_{i=1}^{q} \mathbf{f}^*(t+\Delta t,.)$.

- **Relaxation step :**

$$\mathbf{f}(t+\Delta t,.) = \mathbf{f}^*(t+\Delta t,\mathbf{x}) + \omega \left(\mathbf{f}^{\mathrm{eq}}_{[\mathbf{v}(t+\Delta t,.),\rho_{\mathbf{f}^*}(t+\Delta t,.)]} - \mathbf{f}^*(t+\Delta t,\mathbf{x}) \right),$$

with $\omega \in [1,2]$ the relaxation parameter. $\omega = 1$ correspond to the projection onto the equilibria set and $\omega = 2$ to the over-relaxation.

We note : $\mathbf{f}(t+\Delta t,.) = R_\omega \mathbf{f}^*(t+\Delta t,.)$.

KINETIC VELOCITIES

In two dimensions, we have several possible choices for the velocities :

- In [D2Q4], the four velocities are :

$$\boldsymbol{\lambda}_1 = \begin{bmatrix} \lambda \\ 0 \end{bmatrix}, \boldsymbol{\lambda}_2 = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}, \boldsymbol{\lambda}_3 = \begin{bmatrix} -\lambda \\ 0 \end{bmatrix}, \boldsymbol{\lambda}_4 = \begin{bmatrix} 0 \\ -\lambda \end{bmatrix},$$

The equilibrium vector is, $\forall i \in \{1,2,3,4\}$:

$$f^{\mathrm{eq}}_{[\mathbf{v},\rho],i} = \frac{\rho}{4} + \frac{\rho(\mathbf{v} \cdot \boldsymbol{\lambda}_i)}{4\lambda^2}.$$

We have the sub-characteristic condition :

$$\lambda > \max \|\mathbf{v}(t,\mathbf{x})\|.$$

- In [D2Q5], we add a fifth central null velocity to better control the dissipation introduced by the relaxation method :

$$\boldsymbol{\lambda}_5 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The equilibrium vector is :

$$f^{\mathrm{eq}}_{[\mathbf{v},\rho],i} = \rho \, (\boldsymbol{\lambda}_i \cdot \mathbf{v})_+ \, , \forall i \in \{1,2,3,4\},$$

$$f^{\mathrm{eq}}_{[\mathbf{v},\rho],5} = \rho - \sum_{i=1}^{4} f^{\mathrm{eq}}_{[\mathbf{v},\rho],i} \, .$$

PALINDROMIC COMPOSITION

For each time step, we do :

$$\mathbf{f}(t+\Delta t,.) = M(\Delta t)\mathbf{f}(t,.)$$

where M is a combination of transport and relaxation steps. Let's notice that we need to do a Poisson step before each relaxation step. We can choose :

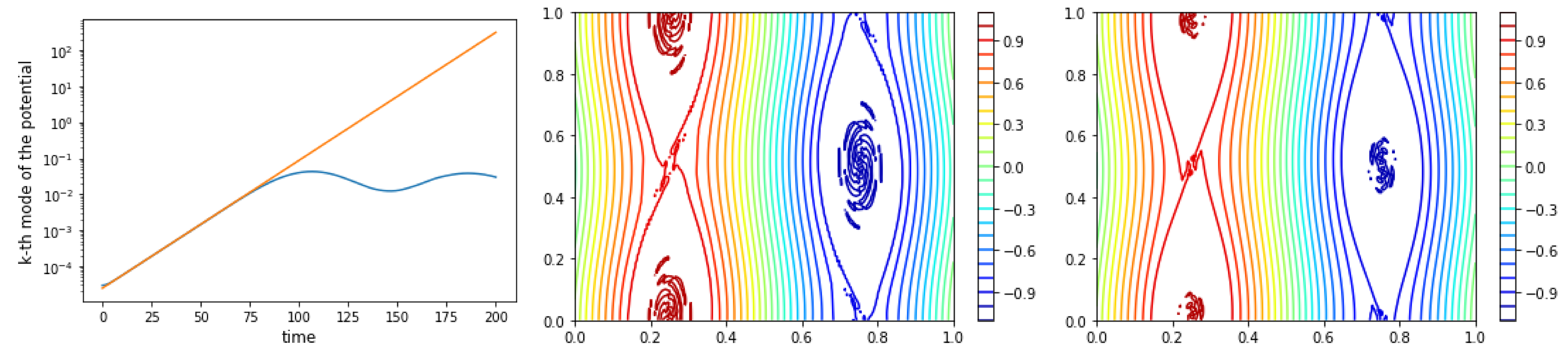
- $M_1(\Delta t) = \Big(R_\omega \circ T(\Delta t) \Big)$ is a first-order approximation for $\omega < 2$ and a second-order approximation for $\omega = 2$.
- $M_2(\Delta t) = \left(T\left(\frac{\Delta t}{4}\right) \circ R_2 \circ T\left(\frac{\Delta t}{2}\right) \circ R_2 \circ T\left(\frac{\Delta t}{4}\right) \right)$ is a second-order time-symmetric operator.
- $M_p(\Delta t) = M_2(s_0 \Delta t) \circ M_2(s_2 \Delta t) \circ \dots \circ M_2(s_p \Delta t)$, where $s_i = s_{p-i}$, for $i = 0, \dots, p$ are palindromic coefficients, like those of Suzuki, which give us a fourth order accuracy or those of Kahan-Li, which give us a sixth order accuracy.

KELVIN-HELMHOLTZ TEST-CASE

We use a Fourier method to solve the transport step. We initialize the density with :

$$\rho_0(x,y) = \sin x + \epsilon \cos(ky),$$

for $(x,y) \in [0,2\pi] \times [0,2\pi/k]$, with periodic boundary conditions, and where k is the perturbation wave number. We use the parameters : $k = 0.5$, $\epsilon = 10^{-4}$ and $\lambda = 2.02$.



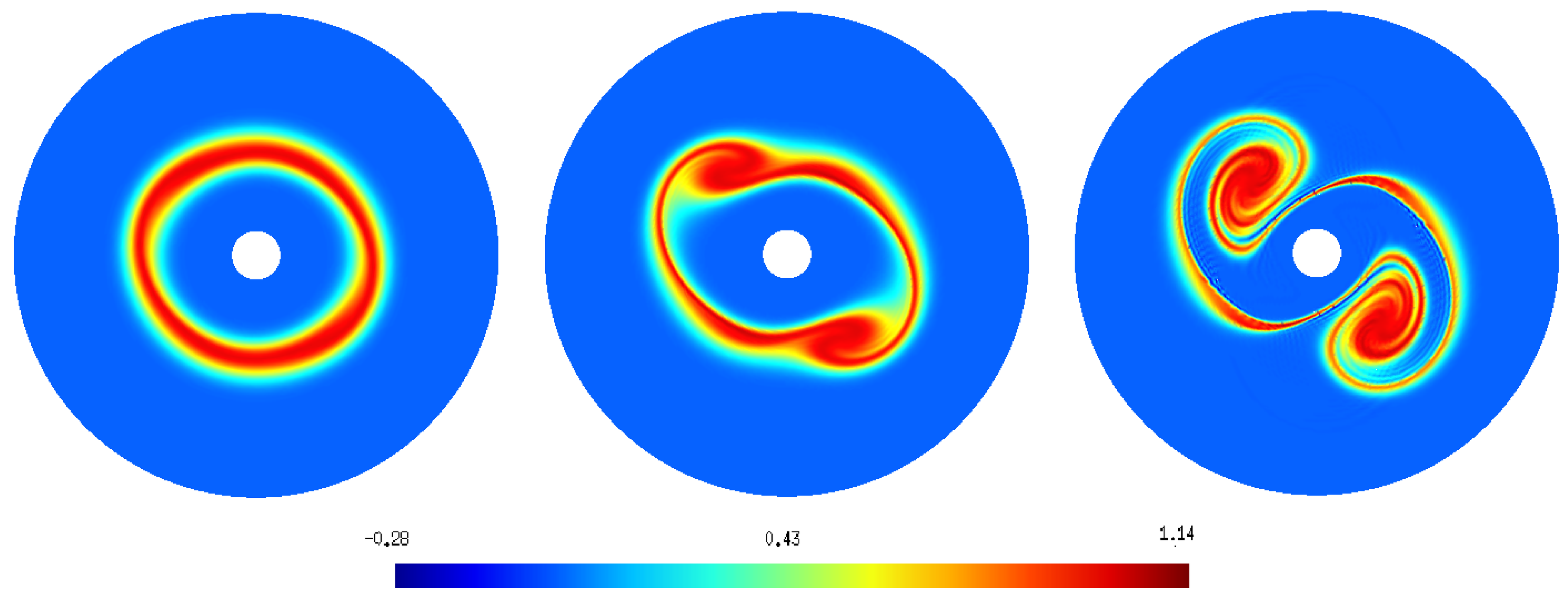
Left : The slope of the k^{th} Fourier mode of the potential (in blue) is fitting the theoretical instability rate (in orange) with Kahan-Li, $\omega = 2$, which confirms our computing.
Middle and right : Contour lines of the density at final time $T = 200$ with $\omega = 2$, Kahan-Li (middle) and $\omega = 1.95$, M_2 (right). We can notice the difference in the small structures between these two schemes.

DIOCOTRON TEST-CASE

This test-case has been done in **chukrut**, a C library for solving hyperbolic equation with Discontinuous Galerkin method. We initialize the density with :

$$\rho_0(r,\theta) = e^{-\frac{(r-r_0)^2}{2\sigma^2}} \times (1 + \epsilon \cos(k\theta)).$$

We choose a ring geometry $\Omega = \{(r\cos(\theta), r\sin(\theta)) \mid r_{min} \leq r \leq r_{max}, 0 \leq \theta \leq 2\pi\}$, with homogeneous Dirichlet boundary conditions on the potential.



Densities obtained at time $t = 80$, $t = 90$ and $t = 100$ with the parameters : $r_0 = 4.5$, $\sigma = 0.5$, $r_{min} = 1$, $r_{max} = 10$, $\epsilon = 10^{-6}$, $\omega = 1.999$, $\Delta_t = 0.0125$, $k = 2$ and $\lambda = 7$.

CONCLUSION

Thanks to the relaxation method, we can transform the equation (1) by several linear transport equations with constant velocities. We have to extend this method to three dimensions, and apply it in a toroidal geometry to model the plasma in a Tokamak.