Tiling Problems in Music

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Abstract

The research project is based on the interaction between mathematics and music. The study concerns particular types of canons, which are polyphonic pieces whose voices lead the same melody with different delays. A rhythmic canon is the one whose tone onsets result in a regular pulse train with no simultaneous tone onsets at a time. Tiling in mathematics is covering an area (e.g., a square) by disjoint equal figures. In that sense, a rhythmic canon tiles the time, providing a covering of a regular pulse train by disjoint equal rhythmic patterns.

Tiling Rhythmic Canons

A Musical Definition

A *Tiling Rhythmic Canon* is a rhythmic and counterpoint composition characterized by

1. same theme, translated over time;
2. cyclical theme;
3. complementary voices.

Example:

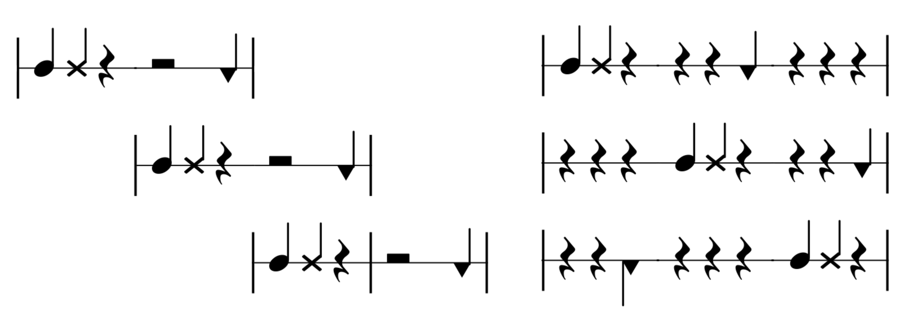


Figure 1: Finite representation and infinite representation

A First Model: Factorization of Cyclic Groups

A Mathematical Definition

A *Tiling Rhythmic Canon* is a factorization of a cyclic group with two subsets:

$$A \oplus B = \mathbb{Z}/n\mathbb{Z},$$

with $A :=$ *internal rhythm*, $B :=$ *external rhythm*.

Example:

$$\{0,1,5\} \oplus \{0,3,6\} \equiv \{0,1,\dots,8\} \mod 9$$

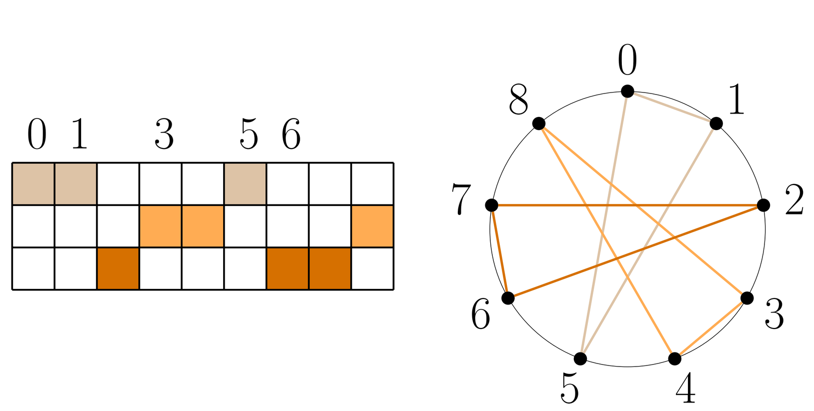


Figure 2: Grid representation (T.U.B.S.) and circular representation

Periodic and Aperiodic Canons

Hajós Groups and Vuza Canons

$$\mathbb{Z}/n\mathbb{Z}$$

Hajós group

$$\Updownarrow$$

$\forall A, B \subset \mathbb{Z}/n\mathbb{Z}$ such that $A \oplus B = \mathbb{Z}/n\mathbb{Z}$, A or B is periodic.

$$A \oplus B = \mathbb{Z}/n\mathbb{Z}$$

Vuza canon

$$\Updownarrow$$

A and B are not periodic

Characterization of Hajòs groups

$$\mathcal{V} := \left\{ N \in \mathbb{N} \mid N = nmk, \begin{cases} (n,m) = 1, \\ n = n_1 n_2, m = m_1 m_2, \\ n_1, n_2, m_1, m_2, k > 1. \end{cases} \right\}$$

$$\mathcal{H} := \{p^\alpha, p^\alpha q, p^2q^2, pqr, p^2qr, pqrs \mid \alpha \in \mathbb{N}, \; p, q, r, s \text{ dinstinct primes}\}$$

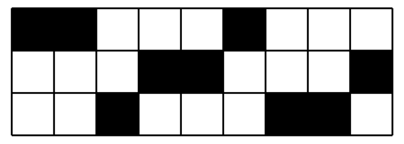
1. $\forall N \in \mathcal{V}$, there exists a Vuza canon $A \oplus B = \mathbb{Z}/N\mathbb{Z}$ (de Bruijn, 1953 - Vuza, 1991).
2. $\forall n \in \mathcal{H}, \mathbb{Z}/n\mathbb{Z}$ is Hajós (Hajós, Rédei, de Bruijn, Sands, 1941 → 1957).
3. $\mathcal{V} \cup \mathcal{H} = \mathbb{N} \setminus \{0\}$ (Sands, 1957).

A Second Model: The Polynomial Representation

Cyclotomic Polynomials

Let $0 \in A \subset \mathbb{N}$ be finite.
$A(x) := \sum_{a \in A} x^a$ is the *polynomial associated* with A.

Example:



$$\{0,1,5\} \oplus \{0,3,6\} = \mathbb{Z}/9\mathbb{Z}$$

$$(1+x+x^5)(1+x^3+x^6) \equiv 1+x+\dots+x^8 \mod (x^9-1)$$

$$A \oplus B = \mathbb{Z}/n\mathbb{Z}$$
$$\Updownarrow$$
$$A(x)B(x) \equiv 1 + x + x^2 + \dots + x^{n-1} \mod (x^n - 1)$$
$$\equiv \prod_{d|n, d>1} \Phi_d(x) \mod (x^n - 1)$$

.

The Coven-Meyerowitz Conditions

Conditions (T_1) and (T_2)

Let $0 \in A \subset N$ finite. If

- $R_A := \{d \in \mathbb{N}^+ : \Phi_d(x) | A(x)\}$;
- $S_A := \{d \in R_A : d = p^\alpha, p \text{ prime}, \alpha \in \mathbb{N}^+\}$,

then

T_1: $A(1) = \prod_{p\alpha \in S_A} p$

T_2: $p_1^{a_1}, \dots, p_k^{a_k} \in S_A$, dinstinct primes $\implies$ $p_1^{a_1} \dots p_k^{a_k} \in R_A$.

Coven-Meyerowitz Theorem (1999):

1. A satisfies (T_1) and (T_2) $\implies$ A tiles;
2. A tiles $\implies$ A satisfies (T_1);
3. A tiles and $|A| = p^\alpha q^\beta$ $\implies$ A satisfies (T_2).

Condition (T_2): an Open Problem

If $n \in \mathcal{H}$ then

$$A \oplus B = \mathbb{Z}/n\mathbb{Z} \implies A \text{ and } B \text{ verify } (T_2).$$

But is (T_2) condition necessary in all cases?

The Fuglede Conjecture

Fuglede Conjecture for tiling rhythmic canons:
Let $0 \in A \subset \mathbb{N}$ finite.

$$A \text{ tiles} \iff A \text{ is spectral.}$$

Łaba Theorem (2007):

1. A satisfies (T_1) and (T_2) $\implies$ A is spectral.
2. A satisfies (T_2) $\implies$ A is spectral.
3. A is spectral $\implies$ A satisfies (T_1).

.

Corollary

$$(T_1) + (T_2) \implies \text{Fuglede}$$

1. $|A| = p^\alpha q^\beta$ and A tiles $\implies$ A is spectral $\implies$ Fuglede.
2. $A \oplus B = \mathbb{Z}/n\mathbb{Z}$ $\implies$ A and B spectral $\implies$ Fuglede.

Open Problems

1. A tiles $\implies$ A verifies (T_2)?
2. A is spectral $\implies$ A verifies (T_2)?
3. A tiles $\iff$ A is spectral?

Transformations of Vuza Canons

Theorem (Amiot, 2004):
If a canon does not satisfy (T_2) then it collapses to a Vuza canon that does not satisfy (T_2).

Theorem (Tijdeman, 1995):
For any canon $A \oplus B = \mathbb{Z}/n\mathbb{Z}$, for any affine transformation $f : x \mapsto ax + b \mod n$ (meaning a is coprime with n), the affine transformation of A by f still tiles with B.

Other Open Questions

Let $A \oplus B = \mathbb{Z}/n\mathbb{Z}$ be a Vuza canon.

- What are the non-cyclotomic factors of $A(x)$ and $B(x)$ doing?
- What is the meaning of the repetition of some cyclotomic polynomials?
- How are the neutral cyclotomic polynomials (not in (T_1) or (T_2)) dispatched between A or B?
- How does one ensure that a ct of such polynomials has only 0 or 1 as coefficients?

First Results

In the particular case of palindromic Vuza canons so far classified, we have observed that within each affine orbit it is possible to find a representative rhythm which is the product of only cyclotomic polynomials, where at least one is repeated.
We observed also that we can construct palindromic tiling Vuza rhythms in some geometrical ways:

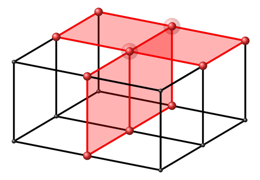


Figure: $n = 180$. $S_A = \{3, 5\}$. $A(x) = \Phi_3(x^{20})\,\Phi_5(x^{18})$.