Averaging and simulating highly-oscillatory transport equations with varying frequency

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Problems with varying frequency: time-space oscillations

Fisrt Part

- Propose a general strategy (the phase method) to solve numerically a class of oscillatory problems. A toy-model is the following transport equation on $u(t,x) \in \mathbb{C}$:

$$\partial_t u + c(x) \cdot \partial_x u = \frac{i}{\varepsilon} E(x) u + R(u), \qquad u(0,x) = u_0(x), \quad x \in \mathbb{R}^d.$$

- extends to intermediate semi-classical models derived from the Liouville (Wigner) equation, Chai, Jin, Li, Morandi (2015). High-frequency waves (seismology, propagation in optic media ..): very much developed in the hyperbolic community. Majda (85), Joly-Métivier-Rauch (95) ...
- Goal: design uniformly accurate shemes with respect to ε.

Problems with varying frequency: time-space oscillations

Second Part

- Combine the phase method with a suitable averaging technique to derive asymptotic models in a systematic way for the Vlasov equation with strong magnetic field in the 3D case.

$$\boxed{\partial_{\mathbf{t}} \mathbf{f} + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathbf{f} + (\mathbf{E} + \mathbf{v} \times \frac{\mathbf{B}}{\varepsilon}) \cdot \nabla_{\mathbf{v}} \mathbf{f} = \mathbf{0}}$$

  $f(t, x, v)$: density at time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$, velocity $v \in \mathbb{R}^3$.

  E, B: electric, magnetic fields given by Maxwell equations in general. Here we shall assume that **E and B are given, but may depend on (t, x)**.

- Average with respect to a suitable phase and not only with repsect to the gyro-angle as usual.
- The approach is also designed to prepare the construction of uniformly accurate numerical schemes with respect to ε.

Examples with time oscillations only

$$
\begin{aligned}
&i\partial_t u^\varepsilon = \frac{1}{\varepsilon} D u^\varepsilon + R(u^\varepsilon), \qquad t \geq 0, \quad u^\varepsilon(t,\cdot) \in X, \\
&u^\varepsilon(0,\cdot) = u_0 \in X,
\end{aligned}
$$

D is a self-adjoint operator on X.

- Mono-frequency time oscillation: D generates a periodic semi-group $\exp(i\tau D)$ in τ: the spectrum of D is a subset of $\lambda\mathbb{Z}$, $\lambda > 0$. Examples are non-linear Schrödinger equations: $D = -\Delta$ on $\mathbb{T}^d$, or $D = -\Delta + V_{ext}$ on $\mathbb{R}^d$. Work with Chartier, Crouseilles and Méhats.

  Another class of models with time oscillations only: Vlasov kinetic equations where the strong fields are constant: Crouseilles, L, Méhats, Zhao: Highly oscillatory Vlasov-Poisson, gyro-kinetic limit (2013 and 2016).

- Multi-frequency time oscillation: A has non-resonant eigenvalues. An example is the multicomponent Schrödinger equation (Bao, 04'):

$$
i\partial_t u_j = -\frac{\omega_j}{\varepsilon}\Delta u_j + f_j(u), \quad j = 1, 2, 3
$$

with periodic B.C. Work with Chartier and Méhats.

space-time oscillations (with Crouseilles and Jin)

A simplified semi-classical models

$$\partial_t u + c(x)\cdot\partial_x u - R(u) = \frac{i}{\varepsilon}E(x)u, \qquad t \geq 0; \quad x \in \mathbb{R}^d,$$
$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}^d,$$

- Inspired from nonlinear geometric optics, we seek U and S such that

$$u(t,x) = U(t, S(t,x)/\varepsilon, x)e^{iS(t,x)/\varepsilon}, \qquad \tau = \frac{S(t,x)}{\varepsilon}.$$

- Equation on the profile

$$\partial_t U + c(x)\cdot\partial_x U - e^{-i\tau}R(e^{i\tau}U) = \frac{E(x)}{\varepsilon}\partial_\tau U,$$
$U(0,x,\tau) =$ Suitable expansion in ε with $U(0,0,x) = u_0(x)$.

- Equation on the phase

$$\partial_t S + c(x)\cdot\partial_x S = E(t,x), \quad S(0,x) = 0.$$

"Another" strategy

- Immerse the problem into a problem with constant frequency

$$\partial_s v + \frac{1}{E(t,x)}\partial_t v + \frac{c(x)}{E(t,x)}\cdot\partial_x v - \frac{1}{E(x)}R(v) = \frac{i}{\varepsilon}v,$$
$$v(s=0,t,x) = u(t,x),$$

- Filter out the main oscillation: $\tilde{v} = e^{is/\varepsilon}$ and write a two-scale formulation in terms of $V(s,\tau = s/\varepsilon, t, x) = \tilde{v}(s,t,x)$

$$\partial_s V + \frac{1}{\varepsilon}\partial_\tau V + \frac{1}{E(x)}\cdot\partial_t V + \frac{c(x)}{E(x)}\partial_x V - \frac{1}{E(x)}e^{-i\tau}R(e^{-i\tau}V) = 0.$$

- We eliminate the variable s by writing an equation on

$$U(t,\tau,x) = V(S(t,x),\tau,t,x)$$

We find

$$\tfrac{1}{E(t,x)}\partial_t U + \tfrac{1}{\varepsilon}\partial_\tau U + \tfrac{c(x)}{E(x)}\cdot\partial_x U - \tfrac{1}{E(x)}e^{-i\tau}R(e^{-i\tau}U) =$$
$$\left[\frac{1}{E}\partial_t S + \frac{c}{E}\cdot\partial_x S - 1\right]\partial_s V.$$

- Two equations, for the profile U and the phase S

$$\partial_t U + c(x)\cdot\partial_x U - e^{-i\tau}R(e^{i\tau}U) = \frac{E(x)}{\varepsilon}\partial_\tau U, \quad U(0,x,\tau) = \text{CE expansion}.$$
$$\partial_t S + c(x)\cdot\partial_x S = E(x), \quad S(0,x) = 0.$$

- The following strong convergence holds

$$u^\varepsilon(t,x)e^{-iS(t,x)/\varepsilon} \to \overline{u}(t,x) \qquad \text{with}$$

$$\partial_t\overline{u} + c(x)\cdot\partial_x\overline{u} - \Pi\left[e^{-i\tau}R(e^{i\tau}\overline{u})\right] = 0.$$

- This is a suitable framework to design efficient numerical schemes: with uniform accuracy with respect to ε.

$$\partial_t U + c(x)\cdot\partial_x U - e^{-i\tau}R(e^{i\tau}U) = \frac{E(x)}{\varepsilon}\partial_\tau U$$

$$U(0,x,\tau) = u_0(x) + \frac{\varepsilon}{E(x)}\Big[G(0,u_0(x)) - G(\tau,u_0(x))\Big],$$

$$\text{with} \qquad G(\tau,\alpha) = \mathcal{L}^{-1}(\mathcal{I}-\Pi)[e^{-i\tau}R(e^{i\tau}\alpha(x))],$$

$$\Pi g = \frac{1}{2\pi}\int_0^{2\pi} g d\tau, \qquad \mathcal{L}^{-1}g = \int_0^{\tau} g(\sigma)d\sigma + \frac{1}{2\pi}\int_0^{2\pi}\sigma g(\sigma)d\sigma$$

Theorem

Let U be the solution of the augmented problem on $[0,T]$ with the above initial data and with periodic boundary condition in x and τ variables. Then, the time and spatial derivatives of U are bounded uniformly in $\varepsilon \in]0,1]$, that is, $\exists C>0$ independent of ε such that $\forall t\in[0,T]$

$$\|\partial_t^p U(t)\|_{L^\infty_{\tau,x}} \le C, \quad \textit{and} \quad \|\partial_x^p U(t)\|_{L^\infty_{\tau,x}} \le C, \ \textit{for } p=0,1,2,$$

and

$$\|\partial_{xt}^2 U(t)\|_{L^\infty_{\tau,x}} \le C, \quad \textit{and} \quad \|\partial_\tau U(t)\|_{L^\infty_{\tau,x}} \le C.$$

- The simplest scheme of order 1.

$$\frac{V_j^{n+1} - V_j^n}{\Delta t} + c\frac{V_j^n - V_{j-1}^n}{\Delta x} + e^{-i\tau} R(e^{i\tau} V_j^n) = -\frac{E(x_j)}{\varepsilon}\partial_\tau V_j^{n+1},$$
$$\frac{S_j^{n+1} - S_j^n}{\Delta t} + c\frac{S_j^n - S_{j-1}^n}{\Delta x} = E(x_j).$$

- At the final time $t_f = N\Delta t$ of the simulation

$$u(t_f, x_j) = V_j^N\left(\tau = \frac{S_j^N}{\varepsilon}\right). \tag{1}$$

In practice we use the pseudo-spectral method in x to solve the equation for S.

Theorem

Assume that $E : [0,1] \to \mathbb{R}$ is a $\mathcal{C}^2$ function satisfying $E(x) \geq E_0 > 0, \forall x \in [0,1]$, and assume the CFL condition $c\Delta t/\Delta x < 1$ Then $\exists C > 0$ independent of $\Delta t, \Delta x$ and ε, such that

$$\sup_{\varepsilon\in]0,1]} \|V(t_n, x_j) - V_j^n\|_{L^\infty_\tau} \leq C(\Delta t + \Delta x),$$

for all $n = 0, \ldots, N$, $n\Delta t \leq T$ and all $j = 0, \ldots, N_x$.

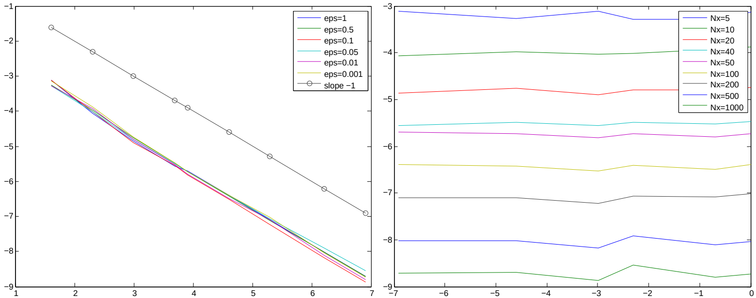

Figure: Plot of the ℓ^∞ error for the two-scale method *with* corrected initial correction and exact computation for S. Left: error as a function of N_{ts} ($N_{ts} = 5, \dots, 1000$) for different values of ε ($\varepsilon = 1, \dots, 10^{-3}$). Right: error as a function of ε for different N_{ts}.

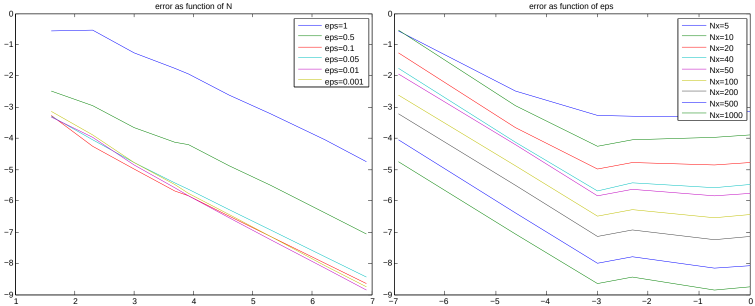

Figure: Plot of the ℓ^∞ error for the two-scale method *with* corrected initial correction and numerical approximation for S. Left: error as a function of N_{ts} ($N_{ts} = 5, \ldots, 1000$) for different values of ε ($\varepsilon = 1, \ldots, 10^{-3}$). Right: error as a function of ε for different N_{ts}.

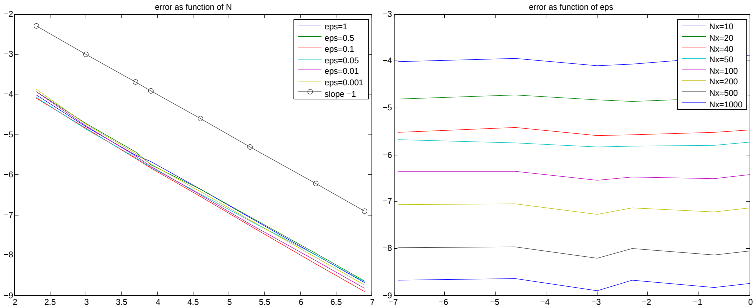

Figure: Plot of the ℓ^∞ error for the two-scale method *with* corrected initial correction and an improved numerical approximation for S (pseudo-spectral and Runge-Kutta 4). Left: error as a function of N_{ts} ($N_{ts} = 10, \ldots, 1000$) for different values of ε ($\varepsilon = 1, \ldots, 10^{-3}$). Right: error as a function of ε for different N_{ts}.

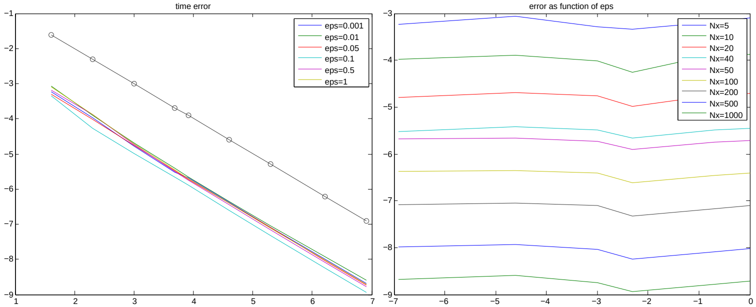

Figure: Plot of the ℓ^∞ error for the two-scale method *with* corrected initial correction and exact computation for S, in the case where a vanishes $(a(x) = 1 + \cos(2x))$. Left: error as a function of N_{ts} ($N_{ts} = 5, \dots, 1000$) for different values of ε ($\varepsilon = 1, \dots, 10^{-3}$). Right: error as a function of ε for different N_{ts}.

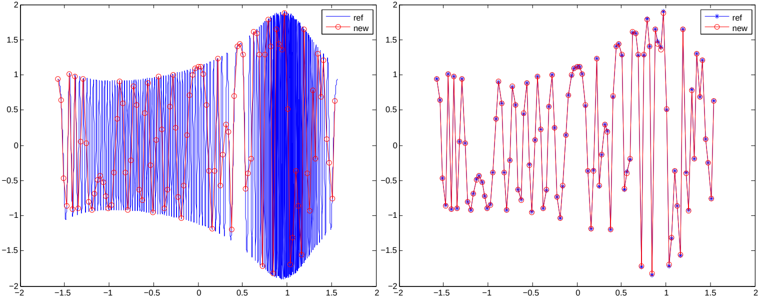

Figure: Comparison between a reference solution and the two-scale solution (*with* initial correction and exact computation for S), for $\varepsilon = 0.005$, $t_f = 1$. Left: space dependence of the real part of the unknown. Right: space dependence of the real part of the unknown on the mesh of the two-scale solution $N_{ts} = 100$.

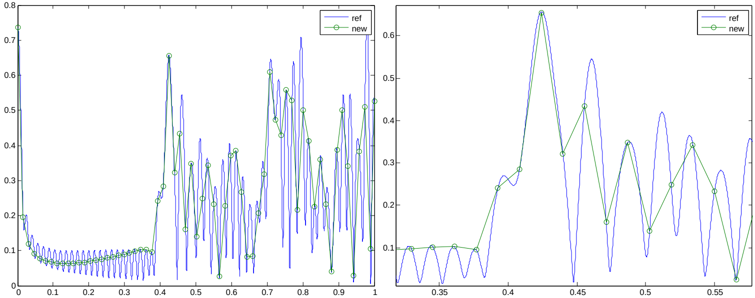

Figure: Time history of $\mathcal{R}$. Comparison between a reference solution and the two-scale solution (*with* initial correction and exact computation for S), for $\varepsilon = 0.005$, $t_f = 1$. The right part is a zoom of the left part.

$$\mathcal{R}(t) = \left| \int_I u(t,x)x \, \mathrm{d}x \right|.$$

Vlasov equation: intrinsic difficulties

$$\boxed{\partial_t \mathbf{f} + \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathbf{f} + (\mathbf{E} + \mathbf{v}\times\frac{\mathbf{B}}{\varepsilon})\cdot\nabla_{\mathbf{v}}\mathbf{f} = \mathbf{0}}$$

Main numerical and computational challenges come from

- the problem dimension (7D = 3D position + 3D velocity +time);
- the necessity to preserve some quantities: the energy, the measure ...
- the occurrence of various time-space scales when B is large.

Gyrokinetics is an appropriate theory to model plasma for large B.

The trajectory of particles is a helix composed of

- a slow motion along the field line;
- a fast circular motion around the field line, called gyromotion.

For most plasma behavior, gyromotion is irrelevant. **Gyrokinetics** it (R.G. LittleJohn 83') reduces the equations to 6 dimensions.

Main goals

1. Our first aim is to show how to derive **high-order asymptotic equations** in a systematic way, rather then the mere limit where ε tends to zero. This work is thus an attempt to re-derive equations of gyrokinetics;
2. The case of varying magnetic field $B(x)$ requires to detect the right oscillation phase $S(t, \tau, x)$. This allows to recover the distribution function f.
3. Ongoing work: based on these techniques, we will construct **uniformly accurate methods** for solving fast-oscillating kinetic equations.

The main tool used to reach this objective is to combine the phase approach with the theory of averaging for ordinary differential equations.

First step: models with contant frequency

In a first step, we consider simplified models with constant frequency:

(i) **2D-Vlasov equation with constant magnetic field**

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + \left(E(x) + \frac{1}{\varepsilon} J v \right) \cdot \nabla_v f(t,x,v) = 0$$

where $f : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}^2 \mapsto \mathbb{R}$ and $E(x)$ is the electric field.

(ii) **A paraxial model of the axisymmetric Vlasov equation**

$$\partial_t f(t,r,v) + \frac{v}{\varepsilon} \cdot \nabla_r f(t,r,v) + \left(E(r) - \frac{H(r)}{\varepsilon} r \right) \cdot \nabla_v f(t,r,v) = 0$$

where $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \mapsto \mathbb{R}$ and $E(r)$ is the electric field.

(iii) ...

Models

Denoting

(i) $y = (x, v)$ and

$$f(t,y) \equiv f(t,x,v),\;\; F^\varepsilon(y) \equiv \begin{pmatrix} v \\ \frac{1}{\varepsilon} Jv + E(x) \end{pmatrix},\;\; \nabla_y \equiv \begin{pmatrix} \nabla_x \\ \nabla_v \end{pmatrix},$$

for equation (i),

(ii) $y = (r, v)$ and

$$f(t,y) \equiv f(t,r,v),\;\; F^\varepsilon(y) \equiv \begin{pmatrix} \frac{v}{\varepsilon} \\ -\frac{H(r)r}{\varepsilon} + E(r) \end{pmatrix},\;\; \nabla_y \equiv \begin{pmatrix} \nabla_r \\ \nabla_v \end{pmatrix},$$

for equation (ii),

Vlasov equations (i) and (ii) can be recast in the form

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0.$$

An averaging technique for transport equations with constant frequency

We consider a transport equation of the form

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0$$

with initial condition $f(0,y) = f_0(y) \in \mathbb{R}$.

- ε is a small parameter;
- $y \mapsto F^\varepsilon(y)$ is split into two parts

$$F^\varepsilon(y) = \frac{1}{\varepsilon} G(y) + K(y)$$

  where the flow associated with the vector field $y \mapsto G(y)$ is assumed to be **periodic**, regardless of the specific trajectory;

- If the operators $G(y) \cdot \nabla_y$ and $K(y) \cdot \nabla_y$ commute, the solution splits as

$$f = f_0(\phi^G_{t/\varepsilon} \circ \phi^K_t(y))$$

- This commutation is equivalent to $[G,K] = 0$ where $[\cdot,\cdot]$ is the usual Lie-bracket.

Assuming the commutation

Consider a transport equation of the form

$$\boxed{\partial_t f(t,y) + F(y)\cdot\nabla_y f(t,y) = 0.}$$

where $F = \frac{1}{\varepsilon}G + K$ with $[G,K]=0$. We have

$$f(t,y) = \exp\left(-\frac{t}{\varepsilon}\mathcal{L}_G\right)\exp\left(-t\mathcal{L}_K\right) f_0(y) = \tilde{f}(t, t/\varepsilon, y).$$

The commutation of $\mathcal{L}_K$ and $\mathcal{L}_G$ allows to separate the fast and slow dynamics on $\tilde{f}(t,\tau,y)$

$$\partial_t \tilde{f} + K(y)\cdot\nabla_y \tilde{f} = 0, \qquad \partial_\tau \tilde{f} + G(y)\cdot\nabla_y \tilde{f} = 0.$$

with the initial condition $\tilde{f}(0,0,y) = f_0(y)$.
The solution f can be recovered by

$$f(t,y) = \tilde{f}(t, t/\varepsilon, y).$$

If non commutation: a normal form theorem

Consider the following differential equation with flow $\varphi_t^{F^\varepsilon}$

$$\dot{y} = F^\varepsilon(y) := \frac{1}{\varepsilon} G(y) + K(y)$$

1. the vector field G generates a 2π-**periodic flow** Φ_τ^G, regardless of the specific trajectory;
2. both G and K are smooth (say C^k at this formal level);

Theorem (e.g. Chartier, Murua and Sanz-Serna, FOCM 11', 12',15')

There exist two vector fields G^ε and K^ε such that

1. *both vector fields commute, i.e.* $[G^\varepsilon, K^\varepsilon] = 0$;
2. *vector field G^ε generates a* 2π*-periodic flow* $(y, \tau) \mapsto \Phi_\tau^\varepsilon(y)$;
3. $F^\varepsilon = \frac{1}{\varepsilon} G^\varepsilon + K^\varepsilon$, *up to* $\exp(-1/\varepsilon)$ *terms.*

red : oscillation part *blue : smooth long-term evolution*

Series Expansions

Consider the Fourier series of

$$K_\tau(y) = \left(\frac{\partial \Phi^G_{-\tau}}{\partial y} \circ \Phi^G_\tau\right)(y) \cdot (K \circ \Phi^G_\tau)(y) = \sum_{k\in\mathbb{Z}} e^{ik\tau} \hat{K}_k(y)$$

Theorem (Chartier, Murua, Sanz-Serna & Murua, Sanz-Serna)

The vector field K^ε of previous theorem admits the formal ε-series

$$K^\varepsilon = \sum_{r=1}^{+\infty} \frac{\varepsilon^{r-1}}{r} \sum_{(i_1,\dots,i_r)\in\mathbb{Z}^r} \overline{\beta}_{i_1\cdots i_r} [\dots[\hat{K}_{i_1}, \hat{K}_{i_2}], \hat{K}_{i_3}], \dots, \hat{K}_{i_r}]$$

where:

- $[.,.]$ *are Lie-brackets of vector fields, i.e.* $[f,g] := f'g - g'f$;
- *the coefficients* $\overline{\beta}$ *are universal (problem-independent): the* $\overline{\beta}$*'s can be computed once for all, independently of the form of* G *and* K.

Main statement

Corollary

Consider a highly-oscillating vector field $F^\varepsilon = \frac{1}{\varepsilon} G + K$ and its associated normal form $\frac{1}{\varepsilon} G^\varepsilon + K^\varepsilon$. Then the solution of the transport equation

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0, \qquad f(0,y) = f_0(y).$$

may be obtained as the **diagonal value** *(i.e. for the value $\tau = t/\varepsilon$) of the two-scale function $\tilde{f}(t,\tau,y)$, periodic in τ and satisfying the two equations*

$$(i) \quad \forall (t,\tau,y), \quad \partial_\tau \tilde{f}(t,\tau,y) + G^\varepsilon(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0,$$
$$(ii) \quad \forall (t,\tau,y), \quad \partial_t \tilde{f}(t,\tau,y) + K^\varepsilon(y) \cdot \nabla_y \tilde{f}(t,\tau,y) = 0.$$

$$\tilde{f}(0,0,y) = f_0(y),$$

Equation (ii) is the so-called averaged equation.

A constant-frequency constant-electric field problem

Consider the system

$$\partial_t f + v \cdot \nabla_x f + \left(\frac{1}{\varepsilon} J v + E \right) \cdot \nabla_v f = 0.$$

The characteristics write

$$\begin{array}{rcl} \dot{x}(t) & = & v(t) \\ \dot{v}(t) & = & \frac{1}{\varepsilon} J v(t) + E \end{array}$$

A constant-frequency constant-electric field problem (ii)

The corresponding flow (here $y = (x, v)$)

$$\varphi_t^{F^\varepsilon}(y_0) = \begin{pmatrix} x_0 - \varepsilon J(e^{t/\varepsilon J} - I)v_0 - \varepsilon^2(e^{t/\varepsilon J} - I)E + \varepsilon tJE \\ e^{t/\varepsilon J} v_0 - \varepsilon J(e^{t/\varepsilon J} - I)E \end{pmatrix}$$

can be decomposed into (with $\tau = t/\varepsilon$)

$$\Phi_\tau^\varepsilon(y_0) = \begin{pmatrix} x_0 - \varepsilon J(e^{\tau J} - I)v_0 - \varepsilon^2(e^{\tau J} - I)E \\ e^{\tau J} v_0 - \varepsilon J(e^{\tau J} - I)E \end{pmatrix}$$

and

$$\Psi_t^\varepsilon(y_0) = \begin{pmatrix} x_0 + \varepsilon tJE \\ v_0 \end{pmatrix}$$

A constant-frequency constant-electric field problem (iii)

The decomposition of vector field F^ε is thus $F^\varepsilon = \frac{1}{\varepsilon} G^\varepsilon + K^\varepsilon$ with

$$G^\varepsilon(y) = \begin{pmatrix} \varepsilon v - \varepsilon^2 JE \\ Jv + \varepsilon E \end{pmatrix} \qquad K^\varepsilon(y) = \begin{pmatrix} \varepsilon JE \\ 0 \end{pmatrix}.$$

The first equation for $\tilde{f}(t, \tau, x, v)$ (with again $\tau = t/\varepsilon$) is thus

$$\boxed{\partial_\tau \tilde{f} + (\varepsilon v - \varepsilon^2 JE) \cdot \nabla_x \tilde{f} + (Jv + \varepsilon E) \cdot \nabla_v \tilde{f} = 0}$$

while the second one (averaged equation) is

$$\boxed{\partial_t \tilde{f} + \varepsilon JE \cdot \nabla_x \tilde{f} = 0}$$

The case of a variable frequency

Consider the transport equation

$$\partial_t f(t,y) + F^\varepsilon(y) \cdot \nabla_y f(t,y) = 0$$

where the vector field F^ε is now of the form

$$\boxed{F^\varepsilon(y) = \frac{1}{\varepsilon}\omega(y) G(y) + K(y)}$$

with G still generating a 2π-periodic flow Φ^G_τ, independently of the initial condition. Previous theorem **does not directly apply**.

Immersion into an augmented problem

To deal with problems with constant frequency, we first immerse the model into the augmented problem on $g(s, t, y)$

$$\partial_s g(s, Y) + \frac{1}{\omega(y)} \partial_t f(t, y) + \frac{1}{\omega(y)} K(y) \cdot \nabla_y f(t, y) + \frac{1}{\varepsilon} G(y) \cdot \nabla_y f(t, y) = 0,$$
$$g(0, t, y) = f(t, y).$$

or

$$\partial_s g(s, Y) + \check{F}^\varepsilon(Y) \cdot \nabla_Y g(s, Y) = 0, \quad g(0, Y) = f(t, y).$$

where $Y = (t, y)$ and $\check{F}^\varepsilon(Y)$ an **augmented** vector field

$$\begin{aligned} \check{F}^\varepsilon(Y) &= \begin{pmatrix} \frac{1}{\omega(y)} \\ \frac{1}{\omega(y)} F^\varepsilon(y) \end{pmatrix} = \frac{1}{\varepsilon} \begin{pmatrix} 0 \\ G(y) \end{pmatrix} + \begin{pmatrix} \frac{1}{\omega(y)} \\ \frac{1}{\omega(y)} K(y) \end{pmatrix} \\ &:= \frac{1}{\varepsilon} \check{G}(Y) + \check{K}(Y). \end{aligned}$$

The vector field $\check{G}(Y)$ generates a periodic flow.

Averaging of the immersed problem

Now we apply the averaging process to

$$\partial_s g(s, Y) + \check{F}^\varepsilon(Y) \cdot \nabla_Y g(s, Y) = 0, \quad g(0, Y) = f(t, y).$$

and get two equations for $\tilde{g}(s, \tau, Y)$

$$\begin{aligned} (i) \quad & \partial_s \tilde{g}(s, \tau, Y) + \check{K}^\varepsilon(Y) \cdot \nabla_Y \tilde{g}(s, \tau, Y) = 0 \\ (ii) \quad & \partial_\tau \tilde{g}(s, \tau, Y) + \check{G}^\varepsilon(Y) \cdot \nabla_Y \tilde{g}(s, \tau, Y) = 0 \end{aligned}$$

If these two equations can be solved then we recover f by the relation

$$\tilde{g}(s, s/\varepsilon, Y) = g(s, Y) = f(t, y), \qquad \text{up to } e^{-1/\varepsilon} \text{ terms}$$

However, there is here **no initial condition at** $s = \tau = 0$, since $\tilde{g}(0, 0, Y) = f(t, y)$, i.e. the unknown of the original problem.

Eliminating the extra-variable s

Introducing a *phase-function* $S(t,\tau,y)$ and a *profile-function* $h(t,\tau,y)=\tilde{g}(S(t,\tau,y),\tau,t,y)$ and choosing S and h such that

$$\begin{aligned}
&\check{K}_1^\varepsilon \partial_t S + \check{K}_2^\varepsilon \cdot \nabla_y S = 1,\\
&\partial_\tau S + \check{G}_1^\varepsilon \partial_t S + \check{G}_2^\varepsilon \cdot \nabla_y S = 0,\\
&S(0,0,y) = 0
\end{aligned}$$

and

$$\begin{aligned}
&\check{K}_1^\varepsilon \partial_t h + \check{K}_2^\varepsilon \cdot \nabla_y h = 0,\\
&\partial_\tau h + \check{G}_1^\varepsilon \partial_t h + \check{G}_2^\varepsilon \cdot \nabla_y h = 0,\\
&h(0,0,y) = f_0(y)
\end{aligned}$$

one can eliminate the variable s. Here, the index $_1$ refers to the first component of $\check{G}^\varepsilon$ or $\check{K}^\varepsilon$. The index $_2$ to the others.

Recovering f

For any given (t, y) we define $\tau(t, y)$ (locally) such that

$$\varepsilon\tau(t, y) = S(t, \tau(t, y), y).$$

Then one can recover the distribution $f(t, y)$ from S and h:

$$\begin{aligned} h(t, \tau(t, y), y) &= \tilde{g}\left(S(t, \tau(t, y), y), \frac{S(t, \tau(t, y), y)}{\varepsilon}, t, y\right). \\ &= g(S(t, \tau(t, y), y), t, y) \\ &= f(t, y). \end{aligned}$$

Main result

Theorem

Let S and h be given by the two separate smooth Cauchy problems

$$\begin{aligned}
&\check{K}_1^\varepsilon \partial_t S + \check{K}_2^\varepsilon \cdot \nabla_y S = 1,\\
&\check{K}_1^\varepsilon \partial_\tau S + \left(\check{K}_1^\varepsilon \check{G}_2^\varepsilon - \check{G}_1^\varepsilon \check{K}_2^\varepsilon\right) \cdot \nabla_y S = -\check{G}_1^\varepsilon,\\
&S(0,0,y) = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\check{K}_1^\varepsilon \partial_t h + \check{K}_2^\varepsilon \cdot \nabla_y h = 0,\\
&\check{K}_1^\varepsilon \partial_\tau h + \left(\check{K}_1^\varepsilon \check{G}_2^\varepsilon - \check{G}_1^\varepsilon \check{K}_2^\varepsilon\right) \cdot \nabla_y h = 0,\\
&h(0,0,y) = f_0(y).
\end{aligned}$$

Then one has $f(t,y) = h(t,\tau(t,y),y)$, where $\tau(t,y)$ is implicitly defined (locally) by the relation $\varepsilon\tau(t,y) = S(t,\tau(t,y),y)$.

Application: 3D Vlasov with strong B(x)

$$\partial_t f + v \cdot \nabla_x f + \left[-\nabla_x U(x) + v \times \frac{B(x)}{\varepsilon} \right] \cdot \nabla_v f = 0,$$

$$b(x) = \|B(x)\|, \quad e(x) = \frac{B(x)}{\|B(x)\|}, \quad \mathcal{J}_e v = v \times e, \quad \mathcal{P}_e v = (e \cdot v) e.$$

$$\partial_t f + v \cdot \nabla_x f - \nabla_x U(x) \cdot \nabla_v f + \frac{1}{\varepsilon} b(x) \mathcal{J}_{e(x)} v \cdot \nabla_v f = 0,$$

with initial condition $f(0, x, v) = f_0(x, v)$. Note that $b(x) \neq 0$.

- This equation is of the form, with $y = (x, v)$

$$\partial_t f + \left[K(y) + \frac{1}{\varepsilon} \omega(y) G(y) \right] \cdot \partial_y f = 0,$$

  with

$$K = \begin{pmatrix} v \\ -\nabla_x U \end{pmatrix}, \quad G(x, v) = \begin{pmatrix} 0 \\ \mathcal{J}_{e(x)} v \end{pmatrix}, \quad \omega(y) = b(x).$$

- The flow generated by the vector field $G(y)$ is 2π-periodic : the period is independent of x, since the period of $\exp(\tau \mathcal{J}_e)$ in τ is always 2π, for any unit vector e.

So that we may apply our strategy.

Computing the averaged fields $\check{K}^\varepsilon$ and $\check{G}^\varepsilon$

- Compute the Fourier modes of

$$K_\tau(t,x,v) = \frac{1}{|B(x)|}\begin{pmatrix} 1 \\ Q_\tau v \\ -Q_{-\tau}R_\tau Q_\tau v + Q_{-\tau}E(x) \end{pmatrix}.$$

$$\begin{array}{ll} R_\tau & = (1-\cos\tau)\partial_x\left(\mathcal{P}_{e(x)}v\right) + (\sin\tau)\partial_x\left(\mathcal{J}_{e(x)}v\right) \\ Q_\tau & = (\cos\tau)I + (1-\cos\tau)\mathcal{P}_{e(x)} + (\sin\tau)\mathcal{J}_{e(x)} \end{array},$$

- Only modes k with $|k| \leq 3$ are not zero. If there is no curvature, then only modes k with $|k| \leq 1$ are not zero.
- Compute the expansion of the averaged field $\check{K}^\varepsilon$ using the Fourier modes of K_τ, and then $\varepsilon\check{G}^\varepsilon = \varepsilon(\check{F}^\varepsilon - \check{K}^\varepsilon) = \check{G} + \varepsilon(\check{K} - \check{K}^\varepsilon)$.

And the averaged transport equation at the $0th$ order

The averaged model at the 0^{th} order in ε:

$$\partial_t h + \left(\frac{B(x)}{|B(x)|}\cdot v\right)\frac{B(x)}{|B(x)|}\cdot\nabla_x h + (\hat{K}_0)_3(x,v)\cdot\nabla_v h = 0,$$
$$\partial_\tau h + \left(v\times\frac{B(x)}{|B(x)|}\right)\cdot\nabla_v h = 0,$$

$$\partial_t S + \left(\frac{B(x)}{|B(x)|}\cdot v\right)\frac{B(x)}{|B(x)|}\cdot\nabla_x S + (\hat{K}_0)_3(x,v)\cdot\nabla_v S = |B(x)|,$$
$$\partial_\tau S + \left(v\times\frac{B(x)}{|B(x)|}\right)\cdot\nabla_v S = 0,$$

where $(\hat{K}_0)_3 = (E\cdot e(x))e(x) +$ curvature terms. The initial conditions are: $h(0,0,x,v) = f_0(x,v)$ and $S(0,0,x,v) = 0$. In the particular case where $B(x)$ has a constant direction $B(x) = b(x)e_0 = (0,0,b(x))^T$, we get

$$\begin{cases} \partial_t h + \ v_{||}\partial_{x_{||}} h + E_{||}\partial_{v_{||}} h = 0, \\ \partial_\tau h + v^\perp\cdot\partial_{v_\perp} h = 0. \end{cases}$$

and

$$\begin{cases} \partial_t S + \ v_{||}\partial_{x_{||}} S + E_{||}\partial_{v_{||}} S = b(x), \\ \partial_\tau S + v^\perp\cdot\partial_{v_\perp} S = 0, \end{cases}$$

The averaged transport equation at the first order

Still in the particular case where $B(x)$ has a constant direction $B(x) = b(x)e_0 = (0, 0, b(x))^T$, we get, up to ε^2 terms

$$\partial_t h + v_\parallel \partial_{x_\parallel} h - \frac{\varepsilon}{2b}\left[|v_\perp|^2 \frac{\partial_{x_\perp} b}{b} - 2E^\perp\right] \cdot \partial_{x_\perp} h$$
$$+ \left[E_\parallel + \varepsilon E_\parallel J v_\perp \cdot \frac{\partial_{x_\perp} b}{b^2} + \varepsilon \partial_{x_\perp}\left(\frac{E_\parallel}{b}\right) \cdot v_\perp\right] \partial_{v_\parallel} h$$
$$+ \frac{\varepsilon}{2}\left[\left(\frac{\partial_{x_\perp} b}{b^2} \cdot E^\perp\right) v_\perp + 2 v_\parallel \partial_{x_\parallel}\left(\frac{E_\perp}{b}\right) - \left(\partial_{x_\perp} \cdot \frac{E_\perp}{b}\right) J v_\perp\right] \cdot \partial_{v_\perp} h = 0,$$

$$\partial_\tau S(t, \tau, y) + Jv \cdot \nabla_{v_\perp} S + \frac{\varepsilon}{b} v_\perp \cdot \nabla_{x_\perp} S + \varepsilon \frac{E_\perp}{b} \cdot \nabla_{v_\perp} S = 0$$

- This coincides with the terms obtained by Frénod & Sonnendrücker (2000) for the $2D \times 2D$ case and constant magnetic field.
- It also reproduces all the terms in the models recently derived by Bostan (2010) and Degond-Filbet (2016) for the $3D \times 3D$ case. But contains more terms since the dependence on the angle is still present.

Conclusion/Perspectives

1. The approach is systematic. It also allows to recover the whole distribution function at any time. Most of the other appraoches use averages with respect to the angle in the polar coordinates for the velocity.
2. This, together with the introduction of the right phase of the solution would ease the construction of uniformly accurate schemes for Vlasov with strong $B(x)$.
3. derive higher-order asymptotic equations for some relevant models from the literature: Finite-Larmor-radius asymptotics ...
4. Explore other geometric aspects: conservation of energy and divergence-free property.

THANK YOU FOR YOUR ATTENTION