Ground State of 1-d Anderson Model

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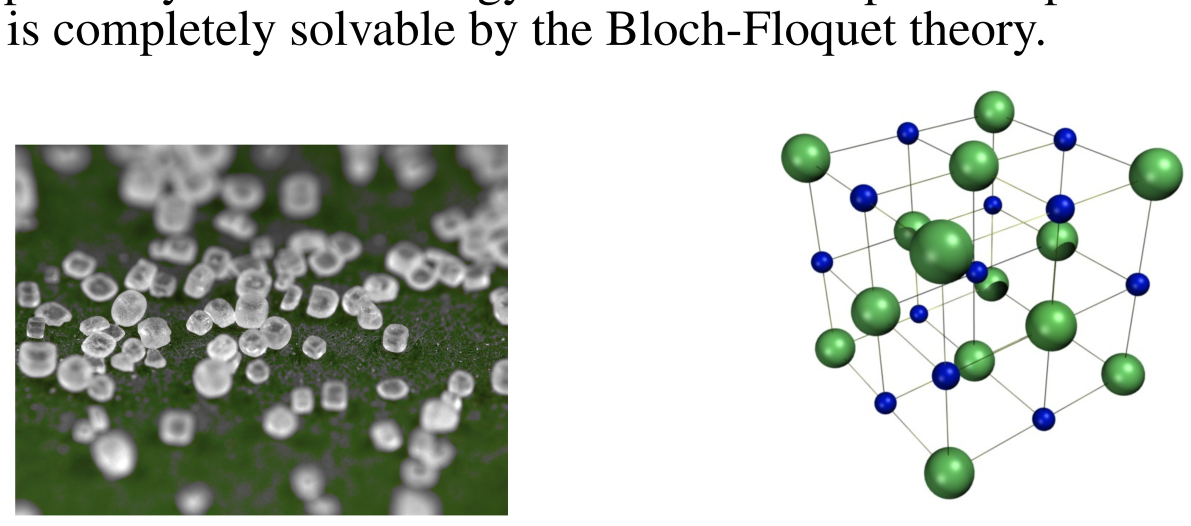
Abstract

We define the Anderson model on $\mathbb{N}$ and its finite volume restrictions. We establish the leading order asymptotic of the ground state energy of the latter and present the main technical tools for the proof.

Introduction

Solids occur in nature in various forms, sometimes they are (almost) totally ordered. In crystals the atoms or nuclei are distributed on a periodic lattice (say the lattice $\mathbb{Z}^d$ for simplicity) in a completely regular way. A quantum particle moving through a perfect crystal is described by its Hamiltonian, which is composed by a kinetic energy term $-\Delta$ and a periodic potential V_{per}. Such model is completely solvable by the Bloch-Floquet theory.



Crystal structure of Na-Cl.
Taken from: https://ast.wikipedia.org/wiki/Sal

Unfortunately, no real crystal is perfect. They have impurities or defects that deviates them from the ideal model while keeping the physical properties constant from sample to sample. The theory of random Schrödinger operators accounts for such impurities by adding random perturbation to the Hamiltonian; and looks for objects or properties that are somehow deterministic.

The Anderson Model

Philip Warren Anderson (1923 – 2020) was an American theoretical physicist who first proposed a model for impurities [And58] back in 1958. He was later awarded the Novel Prize in Physics in 1977 for his research on this topic.

Definition. Let $\{V_\omega(j)\}_{j\in\mathbb{N}}$ be a sequence of independent and identically distributed, real random variables defined over a probability space $(\Omega,\mathcal{F},\mathbb{P})$. The *Anderson model* on $\mathbb{N}$ is a random Schrödinger operator indexed by $\omega\in\Omega$

$$\begin{aligned} H_\omega := -\Delta + V_\omega : \ell^2(\mathbb{N}) &\longrightarrow \ell^2(\mathbb{N}) \\ \phi &\longmapsto H_\omega\phi(j) := -\Delta\phi(j) + V_\omega(j)\phi(j) \end{aligned}$$

where $\Delta\phi(j) = -2\phi(j)+\phi(j-1)+\phi(j+1)$ with the Dirichlet boundary condition $\phi(0)=0$.

The independence of the potential guaranties this operator to have a deterministic spectrum

$$\sigma(H_\omega) = [0,4] + \operatorname{supp}\operatorname{dist} V(1), \quad \mathbb{P}\text{-a.s.}$$

But again, no real crystal is infinite. They have a really large (Avogadro$\approx$ 6×10^{26}) number of atoms. For this reason we introduce $H_{n,\omega}$, the restriction of H_ω to n (very large) points with an extra Dirichlet boundary condition at $n+1$, i.e. $H_{n,\omega}$ is the random matrix:

$$H_{n,\omega} = -\begin{pmatrix} -2 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{pmatrix} + \begin{pmatrix} V_\omega(1) & & & \\ & \ddots & & \\ & & \ddots & \\ & & & V_\omega(n) \end{pmatrix}.$$

For each $\omega\in\Omega$, $H_{n,\omega}$ is a symmetric $n\times n$ matrix so it has n real eigenvalues counted with multiplicities. We denote the smallest eigenvalue by $E_{n,\omega}$ and its unique eigenvector $\psi_{n,\omega}$ which can be chosen to be real, non-negative and normalized.

The asymptotics of $E_{n,\omega}$ and $\psi_{n,\omega}$ are the main topic of our project.

(Partial) Results

From now on, we assume $V(1)$ is bounded from below, hence by adding a constant to H_ω we can achieve $0 = \inf\operatorname{supp}\operatorname{dist} V(1)$.

We denote the cumulative distribution function of $V(1)$ by $F(x) = \mathbb{P}[V(1)\le x]$.

Theorem 1. *Suppose $F(0)>0$, then*

$$\lim_{n\to\infty} E_{n,\omega} \Big/ \left(\frac{\pi\log(F(0))}{\log(n)}\right)^2 = 1, \quad \mathbb{P}\text{-a.s.}$$

Theorem 2. *Suppose there exist $\alpha,\beta,\gamma>0$ and an interval $(0,a)$ in which F is differentiable and satisfies*

$$F(x) = \gamma x^\alpha + \mathcal{O}(x^{\alpha+\beta}) \quad \text{and} \quad F'(x) = \alpha\gamma x^{\alpha-1} + \mathcal{O}(x^{\alpha-1+\beta}) \quad \text{as } x\to 0,$$

then we have almost surely

$$1 \le \liminf_{n\to\infty} E_{n,\omega} \Big/ \left(\frac{\alpha\pi\log\log(n)}{\log(n)}\right)^2 \le \limsup_{n\to\infty} E_{n,\omega} \Big/ \left(\frac{\alpha\pi\log\log(n)}{\log(n)}\right)^2 \le 4.$$

The proof of these theorems consist in deriving appropriate upper and lower bounds for E_n using connected subsets in which V is uniformly bounded.

Auxiliary Lemmas

Lemma 1 (First obtained by [BW12]). *Consider the operator $A = -\Delta + W$ on $\ell^2(\{1,\dots,n\})$ with Dirichlet boundary conditions at $j=0,n+1$, where W only takes the values 0 and $W_+>0$. Let μ be the principal eigenvalue of A, L the size of the maximal connected component of 0's of W, and $C\ge 1$, then we have*

$$L \ge 1 \vee \frac{3\pi^2 C^2}{W_+} \implies 4\sin^2\left(\frac{\pi}{2(L+1)}\right)\left(1-\frac{2}{C}\right) \le \mu.$$

Lemma 2. *Let $\{p_n\}_{n\in\mathbb{N}} \subseteq (0,1)$ be a decreasing sequence such that $\left\{\frac{\log(n)}{|\log(p_n)|}\right\}_{n\in\mathbb{N}}$ is an increasing one with $\lim_{n\to\infty}\frac{\log(n)}{|\log(p_n)|} = \infty$. Let $\{X_{m,n}\}_{m,n\in\mathbb{N}}$ be random variables defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$ that satisfy:*

i) $\{X_{m,n}\}_{n\in\mathbb{N}}$ are I.D.D. following a Bernoulli(p_m) distribution.

ii) $\{X_{m,n}\}_{m\in\mathbb{N}}$ is a decreasing sequence.

Define L_n to be the size of the longest success run in $\{X_{n,j}\}_{j=1}^n$, then we have

$$\mathbb{P}\left[\lim_{n\to\infty}\frac{L_n}{\log(n)/|\log(p_n)|} = 1\right] = 1.$$

Remark. When $X_{m,n} = X_{1,n}$ for all $m,n\in N$ this reduces to the celebrated result of Erdös and Rényi [ER70] on the largest success run on n independents Bernoulli trials of parameter p:

$$\mathbb{P}\left[\lim_{n\to\infty}\frac{L_n}{\log(n)/|\log(p)|} = 1\right] = 1.$$

Forthcoming Research

- Extend Theorems 1 and 2 to higher dimensions (Method of enlargement of obstacles [Szn98]).
- Find limit or limit distribution in Theorem 2.
- Is the mass of $\psi_{n,\omega}$ concentrated?
- Eigenvalue gap.

References

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