Strasbourg, il y a cinquante ans.

Yves Meyer

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C'était en 1964. Nous étions encore dans les locaux prêtés par la Théologie Protestante, au Palais Universitaire.

Georges Reeb (1920-1993) me voit marcher à grands pas, dans le couloir. De son bureau, il me lance, avec son merveilleux accent alsacien, "*où cours-tu comme cela ?*" Je lui réponds calmement et lui dis ce que j'allais faire.

Il interrompt mes explications pour me dire: "*si tu te dépêches toujours autant, tu ne seras jamais un bon chercheur*".

Je le regarde, muet, attristé par ce verdict. Alors il continue: "*mais si tu t'arrêtes chaque fois qu'un idiot te parle, tu ne feras jamais de recherche*".

Éloge de la lenteur et de l'originalité.

Il y a cinquante ans, à Strasbourg, le mot d'ordre était de ne pas publier: "*Ne publie pas, écris plutôt aux 2 ou 3 personnes qui travaillent sur le même sujet. On ne publie que quand on a pris suffisamment de recul, quatre ou cinq ans après*".

La "nouvelle thèse" (née en 1984) n'existait pas encore. La préparation du Doctorat d'État pouvait durer une vie entière. Nous, assistants ou maître-assistants, disposions de postes permanents pour ce faire.

Le rôle du directeur de recherche était d'abord de savoir si le sujet dans lequel le jeune s'engageait avait déjà été traité et d'indiquer la bibliographie. Ensuite il conseillait, il répondait aux questions. Il ne dirigeait pas la thèse. Il pratiquait la maïeutique de Socrate.

La formule sommatoire de Poisson (Denis Poisson 1781-1840) est ce qui fonde mathématiquement la cristallographie par rayons X. La cristallographie par rayons X a permis de révéler la structure tridimensionnelle des grosses molécules de la biologie. Connaître la structure aide à comprendre les propriétés biologiques des macromolécules et à prédire l'efficacité de certains médicaments.

La formule sommatoire de Poisson est également importante en théorie des nombres (André Weil, etc.).

Très récemment, Nir Lev et Alexander Olevskii (travail en cours de publication) ont prouvé l'existence de nouvelles formules sommatoires de Poisson, sans pouvoir cependant donner d'exemples explicites et calculables.

En partant des travaux d'André-Paul Guinand (1912-1987) en théorie des nombres, nous avons découvert de nouveaux exemples, numériquement explicites, de formules sommatoires de Poisson. Ils seront décrits en détail dans cet exposé.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. The distributional Fourier transform of the Dirac comb $\mu = \text{vol}(\Gamma) \sum_{\gamma \in \Gamma} \delta_\gamma$ is the Dirac comb $\sum_{y \in \Gamma^*} \delta_y$ on the dual lattice Γ^*.

The dual lattice is defined by

$$\Gamma^* = \{y; \;\; y \cdot \gamma \in \mathbb{Z}, \forall \gamma \in \Gamma\}$$

We have :

$$\text{vol}(\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{y \in \Gamma^*} \hat{f}(y) \tag{1}$$

Poisson formula (1) is valid at least for all testing functions f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Abstract

In [4] Alexander Olevskii and Nir Lev proved the existence of a measure μ on $\mathbb{R}^n$ enjoying three conflicting but fortunately compatible properties: (a) μ is a sum of weighted Dirac masses on a locally finite set, (b) the Fourier transform $\hat{\mu}$ of μ is also a sum of weighted Dirac masses on a locally finite set, and (c) μ is not a generalized Dirac comb. We give surprisingly simple examples of such measures in this note. These unexpected patterns might give us a whole new insight into aperiodic order.

1 Introduction

The Dirac mass at $a \in \mathbb{R}^n$ is denoted by δ_a or $\delta_a(x)$. A *purely atomic measure* is a linear combination $\mu = \sum_{\lambda\in\Lambda} c(\lambda)\delta_\lambda$ of Dirac masses where the coefficients $c(\lambda)$ are real or complex numbers and $\sum_{\lambda\in B} |c(\lambda)|$ is finite for every bounded set B.

If $c(\lambda) \neq 0$, $\forall\lambda \in \Lambda$, then Λ is the *support* of μ.

A subset $\Lambda \subset \mathbb{R}^n$ is *locally finite* if $\Lambda \cap B$ is finite for every bounded set B. Equivalently $\Lambda = \{\lambda_j,\, j = 1, 2, \dots\}$ and $|\lambda_j|$ tends to infinity with j.

A measure μ is a *tempered distribution* if it has a polynomial growth at infinity in the sense given by Laurent Schwartz in [8]. For instance the measure $\sum_1^\infty k^3\delta_k$ is a tempered distribution, $\sum_1^\infty 2^k\delta_k$ is not a tempered distribution while $\sum_1^\infty 2^k[\delta_{(k+2^{-k})} - \delta_k]$ is a tempered distribution.

The *Fourier transform* $\mathcal{F}(f) = \hat{f}$ of a function f is defined by $\hat{f}(y) = \int_{\mathbb{R}^n} \exp(-2\pi\, ix \cdot y) f(x)\, dx$.

The *distributional Fourier transform* $\hat{\mu}$ of μ is defined by $\langle \hat{\mu}, \phi \rangle = \langle \mu, \hat{\phi} \rangle$ for every testing function ϕ belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

It will suffice for our purpose to define the distributional Fourier transform of a tempered measure μ by $\hat{\mu} = \lim_{\epsilon \to 0} \hat{\mu}_\epsilon$ where $\mu_\epsilon(x) = \exp(-\epsilon x^2)\mu(x)$, $\epsilon > 0$.

The *spectrum* S of μ is the support of its distributional Fourier transform.

Definition 1. *A purely atomic measure μ on $\mathbb{R}^n$ is a crystalline measure if:*

(a) the support Λ of μ is a locally finite set

(b) μ is a tempered distribution

(c) the distributional Fourier transform $\hat{\mu}$ of μ is a purely atomic measure which is also supported by a locally finite set S.

Let us denote by $\mathbb{Q}$ the field of rational numbers. The following theorem will be proved in this note:

Theorem 1. *There exists a crystalline measure μ on $\mathbb{R}^n$ such that*

(a) μ is odd in the last variable x_n

(b) the support Λ of μ is the union of $\Lambda_+ = \Lambda \cap \{x_n > 0\}$ and $\Lambda_- = \Lambda \cap \{x_n < 0\}$

(c) both Λ_+ and Λ_- are linearly independent over $\mathbb{Q}$

(d) the spectrum S of μ is the union of

$$S_+ = S \cap \{y_n > 0\} \text{ and } S_- = S \cap \{y_n < 0\}$$

(e) both S_+ and S_- are linearly independent over $\mathbb{Q}$.

André-Paul Guinand (1912-1987) pioneered this program in [1]. Nir Lev and Alexander Olevskii (Section 5 and [3]) recently proved the *existence* of crystalline measures μ which are not generalized Dirac combs (see Definition 2 below). The support of the measures constructed by Nir Lev and Alexander Olevskii are contained in a finitely generated $\mathbb{Z}-$module and the same is true for the spectrum.

Let $\Gamma \subset \mathbb{R}^n$ be a lattice. The distributional Fourier transform of the Dirac comb $\mu = \mathrm{vol}(\Gamma) \sum_{\gamma \in \Gamma} \delta_\gamma$ is the Dirac comb $\sum_{y \in \Gamma^*} \delta_y$ on the dual lattice Γ^*. We have :

$$\mathrm{vol}(\Gamma) \sum_{\gamma \in \Gamma} f(\gamma) = \sum_{y \in \Gamma^*} \hat{f}(y) \tag{1}$$

Poisson formula (1) is valid at least for all testing functions f belonging to the Schwartz class $\mathcal{S}(\mathbb{R}^n)$.

Definition 2. *Generalized Dirac combs are finite sums* $\mu = \mu_1 + \ldots + \mu_N$ *where* $\mu_j = P_j(x)\, \sigma_j$ *and* σ_j *is a Dirac comb supported by a coset* $x_j + \Gamma_j$ *of a lattice* Γ_j, $P_j(x)$ *being a trigonometric polynomial.*

The Fourier transform of a generalized Dirac comb is a generalized Dirac comb. Therefore a generalized Dirac comb is a crystalline measure.

Let μ be a crystalline measure. We then have $\mu = \sum_{\lambda \in \Lambda} a(\lambda) \delta_\lambda$ and $\hat{\mu} = \sum_{y \in S} b(y) \delta_y$ where $(a(\lambda))_{\lambda \in \Lambda}$ and $(b(y))_{y \in S}$ satisfy

$$a(\lambda) \neq 0,\ \lambda \in \Lambda, \qquad b(y) \neq 0,\ y \in S, \tag{2}$$

and Λ, S are two locally finite sets.

Following Guinand we then say that Λ and S are in *Fourier reciprocity.* Then for every testing function $f \in \mathcal{S}(\mathbb{R}^n)$ the following *generalized Poisson formula* holds

$$\sum_{\lambda \in \Lambda} a(\lambda)\hat{f}(\lambda) = \sum_{y \in S} b(y) f(y) \tag{3}$$

In Selberg trace formula or in the explicit identity connecting the zeros of the Riemann zeta function to the set of prime numbers an integral term appears in the right hand side of (3). Therefore these identities do not provide us with crystalline measures.

A locally finite set Λ is uniformly discrete if

$$\inf_{\{\lambda,\lambda' \in \Lambda;\ \lambda \neq \lambda'\}} |\lambda - \lambda'| = \beta > 0 \tag{4}$$

Nir Lev and Alexander Olevskii [2] proved the following

Theorem 2. *In one dimension if both the support Λ of a crystalline measure μ and the support S of its Fourier transform are uniformly discrete sets, then μ is a generalized Dirac comb.*

The problem is still open in dimension $n \geq 2$.

2 Almost periodic measures

Let μ be a crystalline measure. We have

$$\sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda(x) = \sum_{y \in S} b(y) \exp(2\pi i x \cdot y) \tag{5}$$

This raises the following issue: In what sense is the right hand side of (5) almost periodic?

Let us begin by recalling the definition of almost periodic functions in the sense of H. Bohr. A complex valued continuous function f defined on $\mathbb{R}^n$ is almost periodic in the sens of Bohr if for every positive ϵ one can find a a finite subset $F = F_\epsilon \subset \mathbb{R}^n$ and a trigonometric sum $g(x) = \sum_{y \in F} a(y) \exp(2\pi i x \cdot y)$ such that $\sup_{x \in \mathbb{R}^n} |f(x) - g(x)| \le \epsilon$.

Definition 3. *A tempered distribution $S \in \mathcal{S}'(\mathbb{R}^n)$ is an almost periodic distribution if for every testing function $\phi \in \mathcal{S}(\mathbb{R}^n)$ the convolution product $S \star \phi$ is an almost periodic function in the sense of Bohr.*

This definition was given by L. Schwartz in [7].

Definition 4. *A Borel measure μ on $\mathbb{R}^n$ is an almost periodic measure if for every compactly supported continuous function g the convolution product $\mu \star g$ is an almost periodic function in the sense of Bohr.*

A generalized Dirac comb is an almost periodic measure.

Lemma 3. *Every crystalline measure μ is an almost periodic distribution.*

Indeed if $\phi \in \mathcal{S}(\mathbb{R}^n)$ has a compactly supported Fourier transform $g = \mu \star \phi$ is a finite trigonometric sum. Such testing functions ϕ are dense in the Schwartz class which implies Lemma 1.

Surprisingly a crystalline measure is not always an almost periodic measure (Theorems 3 and 4 below). If Λ is a model set [4] the measure $\sigma = \sum_{\lambda\in\Lambda} \delta_\lambda$ is not an almost periodic measure (it is not even an almost periodic distribution). It is a *generalized almost periodic measure* [5]. It means that for every $\epsilon > 0$, one can find two almost periodic measures μ_ϵ and ν_ϵ such that $\mu_\epsilon \leq \sigma \leq \nu_\epsilon$ and $\mathcal{M}(\nu_\epsilon - \mu_\epsilon) \leq \epsilon$ where $\mathcal{M}(\mu) = \lim_{T\to\infty} \frac{\mu([-T,T])}{2T}$.

Definition 5. *Let us denote by $\mathcal{M}$ the Banach space of all Borel measures μ such that*

(a) μ is an almost periodic measure

(*b*) *the distributional Fourier transform of μ is also an almost periodic measure.*

The norm in the Banach space $\mathcal{M}$ is

$$\|\mu\|_{\mathcal{M}} = \sup_{x\in\mathbb{R}^n} |\mu|(x+U) + \sup_{\xi\in\mathbb{R}^n} |\hat{\mu}|(\xi+U) \tag{6}$$

where U is the unit ball. Every $\mu \in \mathcal{M}$ is a purely atomic measure. The support Λ of μ is a numerable set and the support S of $\hat{\mu}$ is also a numerable set. In general these supports are not closed. The study of $\mathcal{M}$ is still in its infancy. The measure constructed by Lev and Olevskii in [3] is an almost periodic measure. It is not the case for the crystalline measure defined by Theorem 3 (next section).

3 Guinand's approach

We first prove Theorem 1 in one dimension.

By Legendre theorem an integer $n \geq 0$ can be written as a sum of three squares (0^2 being admitted) if and only if n is not of the form $4^j(8k+7)$. For instance 0, 1, 2, 3, 4, 5, 6 are sums of three squares but 7 is not. Let $r_3(n)$ be the number of decompositions of the integer $n \geq 1$ into a sum of three squares (with $r_3(n) = 0$ if n is not a sum of three squares). More precisely $r_3(n)$ is the number of points $k \in \mathbb{Z}^3$ such that $|k|^2 = n$. We have $r_3(4n) = r_3(n)$, $r_3(0) = 1$, $r_3(1) = 6, \ldots$

Guinand began his seminal work [1] with a lemma

Lemma 4. *For all $x > 0$ we have*

$$1 + \sum_1^\infty r_3(n) \exp(-\pi n x) = x^{-3/2} +$$

$$x^{-3/2} \sum_1^\infty r_3(n) \exp(-\pi n / x) \tag{7}$$

The functional equation satisfied by the Jacobi theta function is:

$$\sum_{\infty}^{\infty} \exp(-\pi k^2 x) = x^{-1/2} \sum_{\infty}^{\infty} \exp(-\pi k^2/x) \tag{8}$$

and this identity raised to the cubic yields (7).

Guinand considered the family of odd functions $f_x(t) = t\exp(-\pi x t^2)$ of the argument $t \in \mathbb{R}$ indexed by the parameter $x > 0$. Then

$$\hat{f}_x(y) = -ix^{-3/2} y \exp(-\pi y^2/x)$$

and (5) can be written

$$\frac{df_x}{dt}(0) + \sum_{1}^{\infty} r_3(n) n^{-1/2} f_x(\sqrt{n}) =$$

$$i\frac{d\hat{f}_x}{dt}(0) + i\sum_{1}^{\infty} r_3(n) n^{-1/2} \hat{f}_x(\sqrt{n}) \tag{9}$$

Guinand introduced the odd distribution

$$\sigma = -2\frac{d}{dt}\delta_0 + \sum_1^\infty r_3(n) n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) \tag{10}$$

We have $\sum_0^N r_3(n) n^{-1/2} \sim 2\pi N$ which implies that σ is a tempered distribution. Guinand proved the following

Lemma 5. *The distributional Fourier transform of σ is $-i\sigma$.*

Indeed (10) can be rewritten as $\langle \sigma, f_x \rangle = i\langle \sigma, \hat{f}_x \rangle$ or $\langle \sigma, f_x \rangle = i\langle \hat{\sigma}, f_x \rangle$. But the collection of odd functions f_x, $x > 0$, is total in the subspace of odd functions of the Schwartz class. This implies Lemma 3.

A variant on Guinand's distribution σ is the measure $\tilde{\sigma} = -4\pi t + \sum_1^\infty r_3(n) n^{-1/2}(\delta_{\sqrt{n}} - \delta_{-\sqrt{n}})$. Since $\mathcal{F}(\frac{d}{dt}\delta_0 - 2\pi t) = -i(\frac{d}{dt}\delta_0 - 2\pi t)$ we also have

$$\mathcal{F}(\tilde{\sigma}) = -i\tilde{\sigma}.$$

We now move one small step beyond Guinand's work and prove Theorem 1. Let $\alpha \in (0,1)$ and set

$$\tau_\alpha(t) = (\alpha^2 + \frac{1}{\alpha})\,\sigma(t) - \alpha\sigma(\alpha t) - \sigma(t/\alpha) \tag{11}$$

Then $\frac{d}{dt}\delta_0$ disappears in this linear combination. On the Fourier transform side

$$\hat{\tau}_\alpha(y) = (\alpha^2 + \frac{1}{\alpha})\,\hat{\sigma}(y) - \hat{\sigma}(y/\alpha) - \alpha\hat{\sigma}(\alpha y) = -i\tau_\alpha$$

We had $\sigma([1,N]) \simeq N^2$ as $N \to \infty$ while $\tau_\alpha([1,N]) \simeq N$ which is due to the subtle cancellation introduced in the linear combination in (12).

Fix $\alpha = 1/2$ in the preceding construction, let $\tau = \tau_{1/2}$ and define $\chi(n) = -1/2$ if $n \geq 1$, $n \notin 4\,\mathbb{N}$, $\chi(n) = 4$ if $n \in 4\,\mathbb{N}$, $n \notin 16\,\mathbb{N}$ and $\chi(n) = 0$ if $n \in 16\,\mathbb{N}$. Then we have

Theorem 6. *The Fourier transform of the measure*

$$\tau = \sum_{1}^{\infty} \chi(n) r_3(n) n^{-1/2}(\delta_{\sqrt{n}/2} - \delta_{-\sqrt{n}/2}) \tag{12}$$

is $-i\tau$.

The square roots of square-free integers are linearly independent over $\mathbb{Q}$. Therefore the $\mathbb{Z}-$module generated by the support of τ is not finitely generated. The measure τ is not an almost periodic measure. It is however an almost periodic distribution.

Here is our second example. Observe that for every function f the Fourier transform of $\cos(\pi x)[f(x-1/2)-f(x+1/2)]$ is $i\cos(\pi\xi)[\hat{f}(\xi-1/2)-\hat{f}(\xi+1/2)]$. This simple observation leads to a variant on the measure τ of Theorem 3. Let σ be the Guinand distribution and consider the measure $\rho=\cos(\pi x)[\sigma(x-1/2)-\sigma(x+1/2)]$. The derivative of the Dirac mass $\frac{d\delta_0}{dx}$ is moved at $1/2$ and $-1/2$ and then transformed into Dirac masses after being multiplied by $\cos(\pi x)$. On the Fourier transform side the derivative of the Dirac mass $\frac{d\delta_0}{dy}$ is transformed into a Dirac mass after multiplication by $\sin(\pi y)$ and then the resulting measure is translated by $\pm 1/2$. Finally the Fourier transform of ρ is ρ. We have $\rho=$

$$2\pi\delta_{1/2}+2\pi\delta_{-1/2}+\sum_1^\infty \sin(\pi\sqrt{n})r_3(n)n^{-1/2}(\delta_{(\sqrt{n}+1/2)}+\delta_{(\sqrt{n}-1/2)}+\delta_{(-\sqrt{n}+1/2)}+\delta_{(-\sqrt{n}-1/2)}).$$

One is tempted to replace $1/2$ by 0 in the definition of ρ. But $\tilde{\rho}=\pi\delta_0+\sum_1^\infty \sin(\pi\sqrt{n})r_3(n)n^{-1/2}(\delta_{\sqrt{n}}+\delta_{-\sqrt{n}})$ is not a crystalline measure: $\frac{d}{dx}\delta_{\pm 1/2}$ shows up in the Fourier transform of $\tilde{\rho}$.

It is now easy to prove Theorem 1 in any dimension

n. It suffices to consider the tensor product $\mu = \mu_1 \otimes \mu_2 \ldots \otimes \mu_n$ between n copies of the measure of Theorem 3.

Notice that

$$\sum_1^\infty r_3(n) n^{-1/2} (\delta_{\sqrt{n}} - \delta_{-\sqrt{n}}) = \sum_{k \in \mathbb{Z}^3,\, k \neq 0} \frac{1}{|k|} (\delta_{|k|} - \delta_{-|k|})$$

which paves the road to our third example (next section).

4 The third example

As it was shown in the preceding section it suffices to construct a one dimensional example.

Theorem 7. *Let* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin \mathbb{Z}^3$ *and* $\beta = (\beta_1, \beta_2, \beta_3) \notin \mathbb{Z}^3$*. Then the distributional Fourier transform of the measure*

$$\sigma_{(\alpha,\beta)} = \sum_{k \in \mathbb{Z}^3} \frac{\exp(2\pi i k \cdot \beta)}{|k+\alpha|} (\delta_{|k+\alpha|} - \delta_{-|k+\alpha|})$$

is

$$\mathcal{F}(\sigma_{(\alpha,\beta)}) = -i \exp(-2\pi i \alpha \cdot \beta) \overline{\sigma_{(\beta,\alpha)}}$$

What happens if $\beta = 0$? The Fourier transform of the measure

$$\sigma_{(\alpha,0)} = \sum_{k \in \mathbb{Z}^3} \frac{1}{|k+\alpha|} (\delta_{|k+\alpha|} - \delta_{-|k+\alpha|})$$

is not a measure. When $\beta = (\beta_1, \beta_2, \beta_3) \notin \mathbb{Z}^3$ cancellations are introduced in $\sigma_{(\alpha,\beta)}$ by the phase factor $\exp(2\pi i k \cdot \beta)$ and are playing a seminal role.

If $1, \beta_1, \beta_2, \beta_3$ are linearly independant over then for $k \neq l$, $k, l \in \mathbb{Z}^3$, $|k+\beta| \neq |l+\beta|$. It implies $\int_x^{x+1} d|\sigma_{(\alpha,\beta)}|(t) \simeq x$, $x \to \infty$, and this estimate is optimal. Therefore $\sigma_{(\alpha,\beta)}$ is a tempered measure which is not an almost periodic measure. Our fourth construction yields an almost periodic measure.

We have $\sigma_{(\alpha,\beta)} = \sigma_{(-\alpha,-\beta)}$. If $\alpha_1 = \beta_1 = 1/2, \alpha_2 = \alpha_3 = \beta_2 = \beta_3 = 0$, then $\sigma_{(\alpha,\beta)} = 0$.

There exist infinitely many $\alpha \in \mathbb{R}^3$ such that the set of all $|k+\alpha|$, $k \in \mathbb{Z}^3$, is linearly independent over $\mathbb{Q}$. The following lemma shows that an odd crystalline measure whose spectrum is linearly independent over $\mathbb{Q}$ cannot be an almost periodic measure. Let $n = 1$, $\mu = \sum_{\lambda \in \Lambda} a(\lambda)\delta_\lambda$ and $\hat{\mu} = \sum_{y \in S} b(y)\delta_y$. If μ is odd so is its Fourier transform $\hat{\mu}$. Then $S = -S$ and $b(-s) = -b(s)$, $s \in S$.

Lemma 8. *Let us assume that* $S \cap (0, \infty)$ *is* $\mathbb{Q}-$*linearly independent. Then if* μ *is an odd crystalline measure, it cannot be an almost periodic measure.*

We argue by contradiction and suppose that for every compactly supported function ϕ the convolution product $f(x) = \mu \star \phi = \sum_{\lambda \in \Lambda} a(\lambda)\phi(x - \lambda)$ is an almost periodic function in the sense of Bohr. We are assuming now that ϕ is even, real valued, and does not belong to the Wiener algebra $A(\mathbb{R})$. The Wiener algebra [6] is the algebra consisting of Fourier transforms of functions in $L^1(\mathbb{R})$. Using (5) we have $f(x) = \sum_{s \in S} b(s)\hat{\phi}(s) \exp(2\pi i s x)$. Let $z_s \in \mathbb{C}$ any sequence of $\pm i$ such that $z_{-s} = \overline{z}_s$, $s \in S$. Then there exists a sequence x_k tending at infinity such that $\exp(2\pi i x_k s) \to z_s$, $s \in S$, as k tends to infinity. This implies $f(x_k) \to \sum_{s \in S} z_s b(s)\hat{\phi}(s)$ and $|\sum_{s \in S,\, s>0} b(s)\hat{\phi}(s)(z_s - \overline{z}_s)| \leq \|f\|_\infty$. By an appropriate choice of $z_s = \pm i$, $s \in S$, we obtain $\sum_{s \in S,\, s>0} |b(s)\hat{\phi}(s)| \leq 2\|f\|_\infty$. Therefore $f(x)$ locally belongs to the Wiener algebra. If $a(\lambda_0) \neq 0$ and if the support of the continuous function ϕ is contained in $[-\eta, \eta]$ where η is small enough then $f(x)$ coincides with $a(\lambda_0)\phi(x - \lambda_0)$ on $[\lambda_0 - \eta, \lambda_0 + \eta]$. This implies that ϕ belongs to the Wiener algebra. We reached a contradiction.

It is interesting to let α and β tend to 0. Then the limit of $\sigma_{(\alpha,\beta)}$ is the Guinand distribution

$$-2\frac{d}{dt}\delta_0 + \sum_{k\in\mathbb{Z}^3;\, k\neq 0} \frac{1}{|k|}(\delta_{|k|} - \delta_{-|k|}).$$

Therefore Theorem 4 gives another proof of Lemma 3: the Fourier transform of the Guinand distribution σ_0 is $-i\sigma_0$.

Here is a two dimensional example. Let $A \in SO(4,\mathbb{R})$ and let A' be the 3×4 matrix obtained by deleting the last row a from A. Let $\alpha = (\alpha_1, \alpha_2, \alpha_3) \notin \mathbb{Z}^3$.

Theorem 9. *The distributional Fourier transform of the (two dimensional) measure*

$$\sigma_\alpha = \sum_{k\in\mathbb{Z}^4} \frac{\exp(2\pi i A'(k)\cdot\alpha)}{|A'(k)+\alpha|}(\delta_{(|A'(k)+\alpha|, a\cdot k)} - \delta_{(-|A'(k)+\alpha|, a\cdot k)})$$

is

$$\mathcal{F}(\sigma_\alpha) = -\frac{i}{2}\exp(-2\pi i|\alpha|^2)\overline{\sigma_\alpha}$$

Here also the Fourier transform of

$$\sum_{k\in\mathbb{Z}^4} \frac{1}{|A'(k)+\alpha|}(\delta_{(|A'(k)+\alpha|,a\cdot k)} - \delta_{(-|A'(k)+\alpha|,a\cdot k)})$$

is not a measure and the phase factors are playing a seminal role.

Theorem 4 is a particular case of a more general statement :

Theorem 10. *Let μ be a crystalline measure on $\mathbb{R}^3$. We then have $\mu = \sum_{\lambda\in\Lambda} a(\lambda)\delta_\lambda$ and $\hat{\mu} = \sum_{y\in S} b(y)\delta_y$. Let us assume that $0 \notin \Lambda$ and $0 \notin S$. Let us consider the one dimensional measure*

$$\sigma_\Lambda = \sum_{\lambda\in\Lambda} \frac{a(\lambda)}{|\lambda|}(\delta_{|\lambda|} - \delta_{-|\lambda|}) \tag{13}$$

Then σ_Λ is a crystalline measure and the distributional Fourier transform of σ_Λ is $-i\sigma_S$ where $\sigma_S = \sum_{y\in S} \frac{b(y)}{|y|}(\delta_{|y|} - \delta_{-|y|})$.

We prove Theorem 6.

The measures σ_Λ and σ_S are odd. In order to check the identity

$$\langle\hat{\phi}, \sigma_\Lambda\rangle = \langle\phi, \sigma_S\rangle \tag{14}$$

for every testing function ϕ it suffices to do it for every odd ϕ. Let us write $\varphi = \hat{\phi}$. Then φ is also a odd function and the left hand side of (14) is

$$s(\phi) = 2\sum_{\lambda\in\Lambda} a(\lambda)\frac{\varphi(|\lambda|)}{|\lambda|} \tag{15}$$

We introduce the radial function $\Phi(x) = \frac{\varphi(|x|)}{|x|}$ which belongs to $\mathcal{S}(\mathbb{R}^3)$. Then

$$s(\varphi) = 2\sum_{\lambda\in\Lambda} a(\lambda)\Phi(\lambda) \tag{16}$$

We have for every testing function f

$$\sum_{\lambda\in\Lambda} a(\lambda)\hat{f}(\lambda) = \sum_{y\in S} b(y)f(y) \tag{17}$$

This is now applied to the radial function $F(x) = -i\frac{\phi(|x|)}{|x|}$.

Lemma 11. *The Fourier transform of F is $\hat{f}(x) = \Phi(x)$.*

Indeed the $3-D$ Fourier transform $\mathcal{F}(f)$ of a radial function $f \in L^1(\mathbb{R}^3)$ is

$$\mathcal{F}(f)(y) = 4\pi \int_0^\infty f(r) \frac{\sin(2\pi|y|r)}{2\pi|y|r}\, r^2\, dr \tag{18}$$

Therefore if ϕ is an odd function in $\mathcal{S}(\mathbb{R})$ and if $\varphi = \hat{\phi}$ is its $1-D$ Fourier transform we have

$$\mathcal{F}(\frac{\varphi(|\cdot|)}{|\cdot|}) = -i\frac{\phi(|\cdot|)}{|\cdot|}$$

or

$$\mathcal{F}(\frac{\phi(|\cdot|)}{|\cdot|}) = i\frac{\varphi(|\cdot|)}{|\cdot|} \tag{19}$$

which proves Lemma 5.

Finally (15), (16) and (19) yield $s(\varphi) = 2i\sum_{y\in S} b(y)\frac{\phi(|y|)}{|y|}$. This is $\langle \tau, \phi \rangle$ which ends the proof.

Theorem 4 is a corollary of Theorem 6 if we oberve that the Fourier transform of $\sum_{k\in\mathbb{Z}^3} \exp(2\pi i k\cdot\beta)\delta_{k+\alpha}$ is $\exp(-2\pi i\beta\cdot\alpha)\sum_{k\in\mathbb{Z}^3} \exp(-2\pi i k\cdot\alpha)\delta_{k+\beta}$.