QUANTUM ERGODICITY AND BEYOND. *WITH A GALLERY OF PICTURES.*

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The vibrations of a guitar string, electromagnetic waves, seismic waves, or states of quantum systems: all these physical phenomena are described by mathematical equations belonging to the family of *wave equations*. The results we describe here deal with *linear* wave equations in a *closed* cavity. Apparently simple equations, such as the d'Alembert equation $\frac{\partial^2\psi}{\partial t^2} = \Delta\psi$ or the Schrödinger equation $\frac{\partial\psi}{\partial t} = i\Delta\psi$ are still at the heart of active research. One major challenge is to understand the "scar" phenomenon, namely, the possibility for stationary waves (or waves evolved over a long time) to stay localized in the vicinity of periodic billiard trajectories, or periodic geodesics. The interpretation of scars is controversial: simple heuristic arguments suggest that it is difficult for a wave to follow an *unstable* billiard orbit for a long time, while [55, 26, 24] give a reasoning in favor of the enhancement along unstable orbits. Numerical computations such as shown Figures 0.1 and 0.2 show that some stationary waves do have a slightly higher intensity around unstable periodic orbits in billiards.

The series of papers [2, 12, 11, 14, 71, 35, 46, 4, 6] eradicates the possibility of *strong scars* in the case of negatively curved manifolds (leaving open the possibility of *partial scars*, as explained later). This answered a long-standing open question, and also constituted a progress towards the Quantum Unique Ergodicity conjecture of Rudnick and Sarnak [74, 73]. This conjecture, formulated for negatively curved manifolds, predicts that the

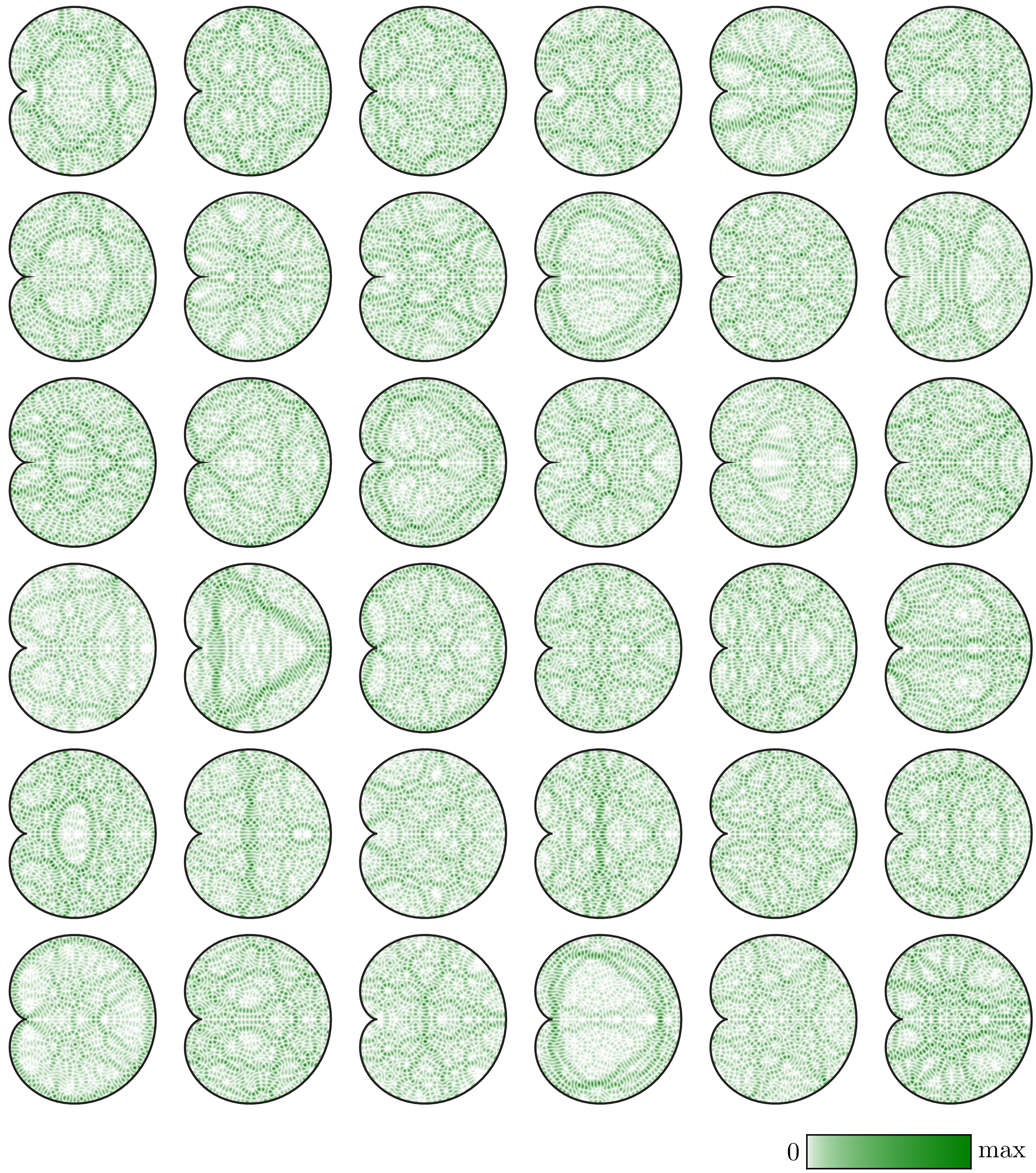

FIGURE 0.1. Plot of $|\psi_n(x, y)|^2$ for the cardioid billiard with odd symmetry, for consecutive states starting from $n = 700$.

stationary solutions of the Schrödinger equation should occupy phase space uniformly, in the small wavelength limit. Quantum Unique Ergodicity would imply the absence of scars,

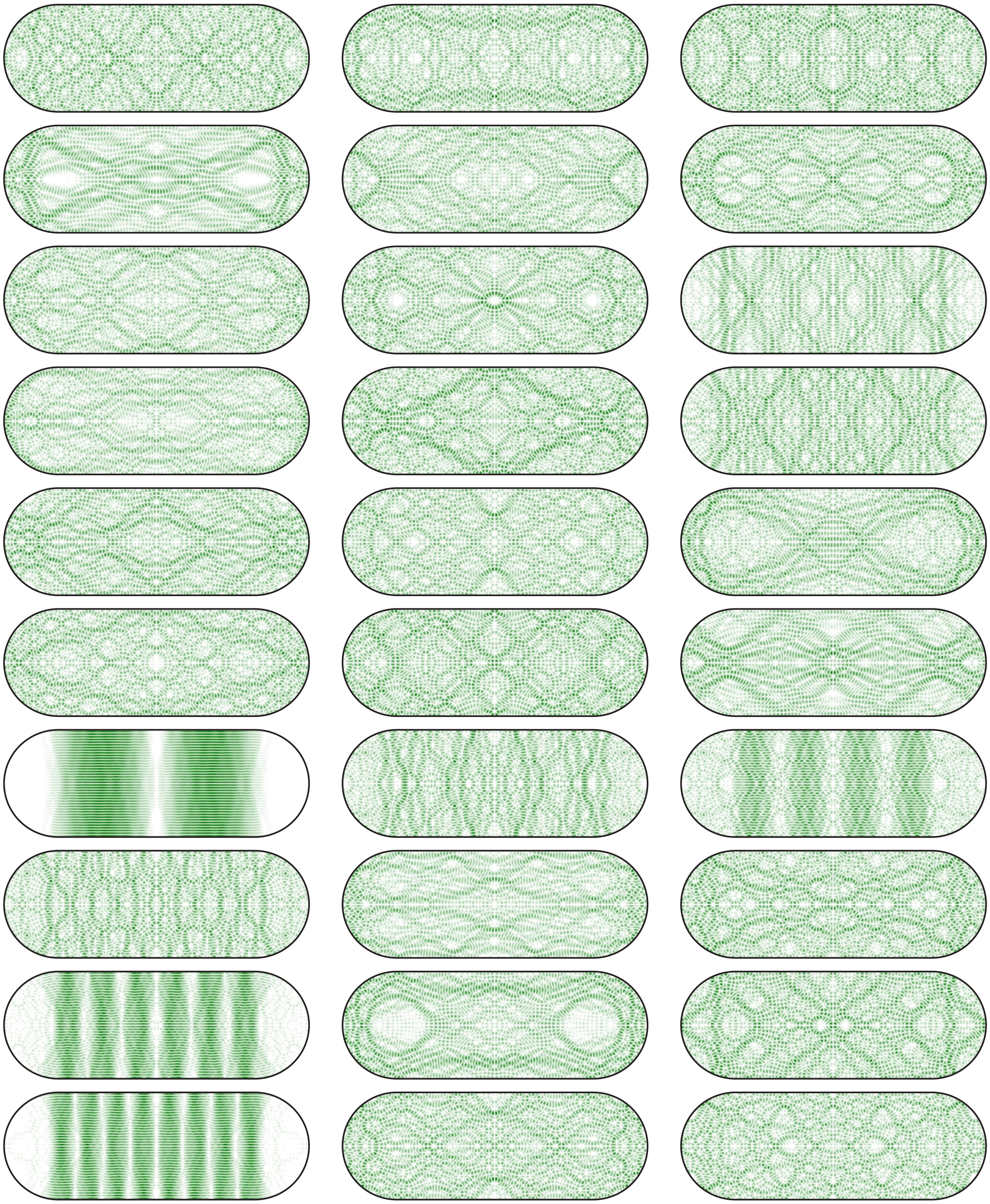

FIGURE 0.2. Plot of $|\psi_n(x,y)|^2$ for the stadium billiard with odd-odd symmetry, for consecutive states starting from $n = 758$.

but has not been proven rigourously except in a specific "arithmetic" setting [62, 79], quite remote from the physical models. On the contrary, the ideas introduced in [2, 12] could in principle apply to very general geometries.

One of the novelties in this work is the use of a concept from ergodic theory, the Kolmogorov-Sinai entropy, to study the qualitative behaviour of solutions of linear wave equations. Ergodic theory was born at the beginning of the twentieth century, with the foundational work of Poincaré and Birkhoff, and has been developed over the years to provide tools to study the large time behaviour of solutions of Ordinary Differential Equations, from a statistical point of view. Ergodic theory usually deals with finite-dimensional, non-linear systems. It is well suited for the study of classical mechanics. On the other hand, wave mechanics is described by Partial Differential Equations which require to work with infinite dimensional function spaces. The notions of "chaos", "entropy"... that have been developed in ergodic theory do not apply naturally to these non-commutative dynamical systems. In our work we used the semiclassical approximation as a bridge between quantum dynamics and classical ergodic theory. This circle of ideas is usually referred to as "quantum chaos", where "quantum" just means "wave mechanics", and chaos refers to the classical theory of chaos in ergodic theory.

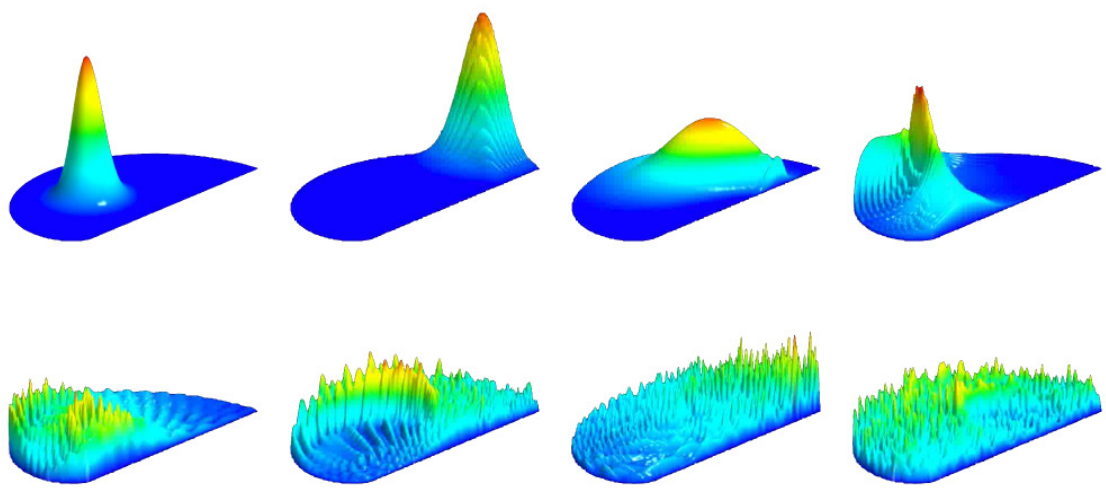

FIGURE 0.3. Time evolution of a wave-packet in the desymmetrized cardioid billiard with Dirichlet boundary conditions.

When trying to understand the large time behaviour of waves, it is clear that the answer depends of the ambient geometry. Figure 0.3 shows snapshots of the propagation of an initial "gaussian wave packet" in a half cardioid, and gives an idea of the complexity of the phenomenon. The movie suggests that an equilibrium regime is reached, in which the wave fills uniformly all the space at its disposal. Of course, the simulation only runs over a finite time, and one may imagine that the wave packet later comes back to its original state, if one waits long enough. Although this is difficult to believe, there is today no mathematical result ruling out this possibility. In particular, in certain situations, like integrable systems, the occurence of revivals has been observed in numerical investigations and are also rigorously proven for specific examples. The origin of such revivals are specific relations among the eigenvalues. Something like this is not expected for chaotic systems leading to the common belief that there are no such revivals (but also see [82]).

In a closed domain, there are countably many stationary waves, that is waves that oscillate in a stationary manner. These correspond to eigenfunctions of the laplacian,

$$\Delta\psi_n = -k_n^2\psi_n,$$

i.e. the Helmholtz equation, with the physically relevant boundary conditions. Figures 0.1 and 0.2 display samples of consecutive laplacian eigenfunctions in a stadium-shaped domain and in a cardioid with Dirichlet boundary conditions, i.e. $\psi_n(x, y) = 0$. Numerically the eigenvalues have been computed using the boundary element method (see e.g. [21] and references therein) in the case of the stadium billiard and by means of the conformal mapping technique for the cardioid billiard [72]. From the eigenvalues, together with the corresponding normal derivative function along the billiard boundary, one can reconstruct the eigenfunction at any point inside the billiard.

Since we deal with linear wave equations, the propagation of any wave can be written as a superposition of stationary waves. For the Schrödinger equation, solutions are of the form

$$\psi(t, x) = \sum_n a_n e^{-itk_n^2}\psi_n(x)$$

where the coefficients a_n can be expressed in terms of the initial conditions. For the numerical computation leading to Figure 0.3 the first 2000 eigenstates of the cardioid billiard have been used.

Thus, in principle everything boils down to understanding the eigenfunctions ψ_n and the eigenfrequencies k_n of the laplacian. However, this statement is mostly of theoretical interest: one of the difficulties we meet is the fact that the eigenfunctions and eigenvalues are not given by explicit expressions. It is difficult to lay hands on individual eigenfunctions otherwise than by numerical methods, that give visual pictures without explaining: for instance, the interesting geometric patterns of Figures 0.1 and 0.2 are only very partially understood! Here, we will mostly discuss the phenomenon called "scarring", namely enhancement of an eigenstate in the vicinity of a periodic trajectory, see Figures 0.4 and 0.5. The "bouncing ball modes" of the stadium also want an explanation; and the *nodal lines* (places where the eigenstates vanish) also form a very beautiful and mysterious lace.

Working with explicit models is sometimes possible, but one rapidly runs into delicate problems of number theory. As an example, let us consider the solutions of

$$\Delta\psi = -k^2\psi$$

in the cube $[0, \pi]^d$, with Dirichlet boundary conditions. Of course, k^2 must be an integer, and a general solution can be expanded into Fourier series as

$$\psi(x) = \sum_{(k_1,\ldots,k_d)\in\mathbb{N}^d\setminus(0,\ldots,0), k_1^2+\ldots+k_d^2=k^2} c_{k_1,\ldots,k_d}\sin(k_1x_1 + \ldots + k_dx_d).$$

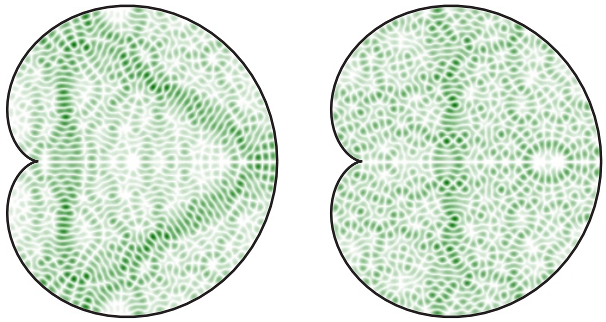

FIGURE 0.4. Selected states in the cardioid showing higher intensity ("scarring") around unstable periodic orbits.

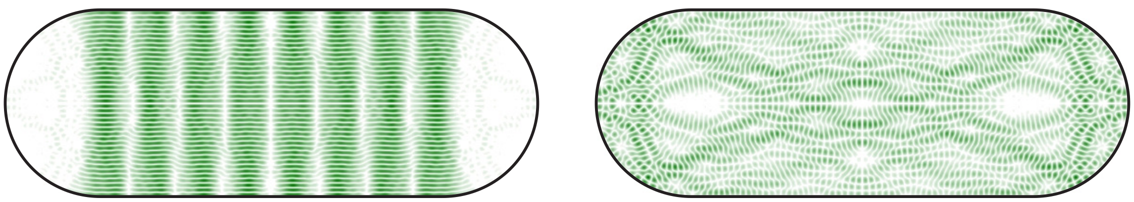

FIGURE 0.5. Selected states in the stadium showing a bouncing–ball mode and an eigenstate with slightly higher intensity ("scarring") around an unstable periodic orbit.

The sum involves all the possible ways of writing k^2 as a sum of d squares : describing and counting these are a notoriously difficult problem in number theory. Even in a cube, the study of the Schrödinger equation is a rich subject, at the frontier between harmonic analysis, number theory, and the analysis of partial differential equations [28, 27, 29, 59, 58, 31, 32, 33, 9, 1, 7]. For instance, proving the impossibility of scars in a (2-dimensional) square is easy, but in cubes of higher dimensions this requires some elaborate Fourier analysis and/or discussion of the geometric properties of integer points lying on a sphere [27, 59, 9, 1].

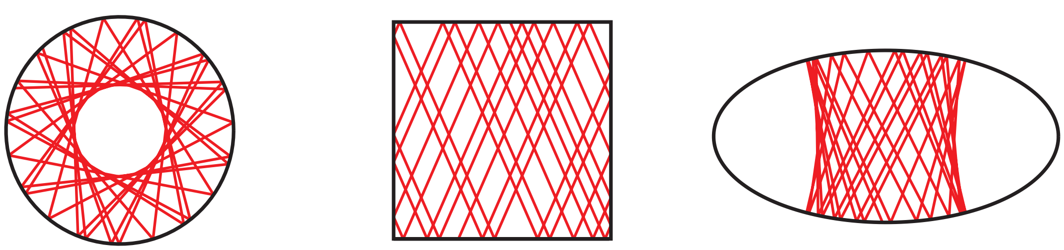

FIGURE 0.6. The billiards in a circle, square and ellipse examples for integrable dynamics. The resulting orbits show regular dynamics.

Coming back to a general setting, an interesting approach to understand the propagation of waves is to use the *semiclassical approximation*. In this approximation the wavefronts

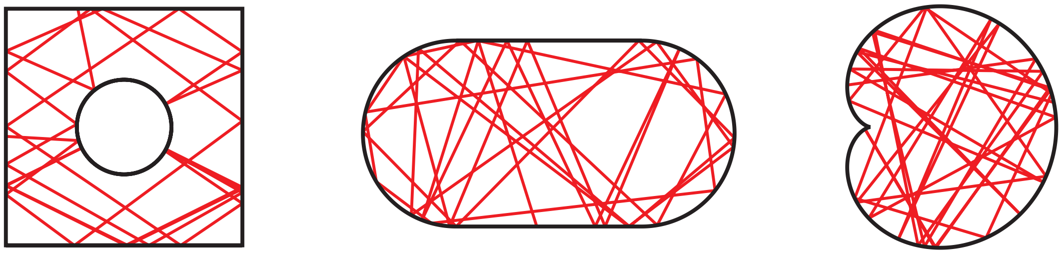

FIGURE 0.7. The billiards in a square with a circular scatterer (so-called Sinai-billiard), in a stadium shaped boundary and in the cardioid are examples for chaotic dynamics, more precisely, they are proven to be ergodic, mixing, K–systems and Bernoulli. The resulting orbits show irregular dynamics.

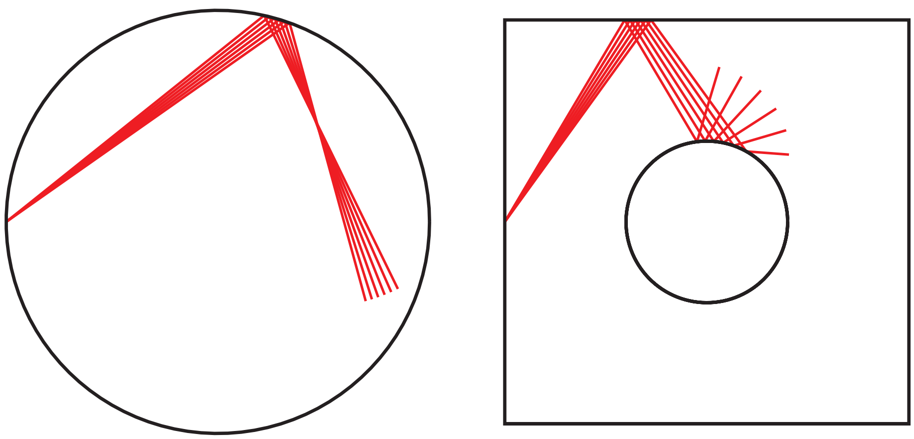

FIGURE 0.8. Trajectories started in the same point but with different initial angles stay close to each other in the case of integrable dynamics while for the chaotic systems a strong defocussing occurs. This is the origin of the chaotic properties and illustrates the exponential sensitivity on the initial conditions in such systems.

propagate like in a billiard: in straight lines as long as the movement is free, and bouncing on obstacles in the usual way (angle of incidence = angle of reflection). Figures 0.6 and 0.7 show individual billiard trajectories for several well known billiards. Figure 0.8 shows the emission of a beam of trajectories in a small angular interval. According to the semiclassical approximation, the wave emitted by one point source will disperse in the same way. However, as the name indicates, this is only an imperfect description of what actually happens. The dispersion of rays explains only the initial dispersion of the wave-packet. The smaller the wavelength, the smaller the error: the semiclassical approximation is well-adapted to describe the propagation of waves with small wavelength and large frequency. This approximation gives for instance the link between Huygens' wave theory of light and the geometric optics of Descartes. The mathematical technique that allows to formalize this is a sophisticated version of Fourier analysis, called *microlocal analysis* or *semiclassical analysis*, developed to a point of extreme refinement by Lars Hörmander and his school [54].

The semiclassical approximation gives a fair account of the first four snapshots of the movie: we indeed observe that the center of the wave packet moves along a straight line and effectively bounce on the boundary like a ball. Later, the wave-packet starts dispersing, interferences occur, and we are out of the range of usefulness of the semiclassical approximation. This happens at a time called the Ehrenfest time, which (for chaotic systems) typically grows like $\log k_n$ when the frequency k_n goes to infinity [45, 40, 85, 25, 34]. Beyond that time, the complicated interference effects are described by sums of oscillatory terms known as "exponential sums". These are in general impossible to cope with, except for certain explicit models where number theory can help.

Research has focussed on *completely integrable* billiards on the one hand, Figure 0.6, and *chaotic* billiards on the other hand, Figure 0.7. For the integrable case, Figure 0.9 shows in green, a few stationary waves in the disc; in red, the trajectory of a billiard ball with approximately the same angular momentum as the eigenfunction directly on its right. In a disc, the trajectory of the ball is organized in a very simple way: the angle of reflection on the boundary is always the same, and the trajectory is tangent to an inner disc called the caustic. As beside the energy, the angular momentum is an independent conserved quantity, this is a *completely integrable system*, the most ordered class of systems in the theory of dynamical systems. The images of stationary states offer a certain visual similarity. This is of course expected, as in the semiclassical limit the quantum states should resemble more and more the corresponding classical dynamics. [1]

In a chaotic system, on the other hand, the ball thrown at random (typically) follows a chaotic trajectory and eventually visits uniformly all the space at disposal (Figure 0.7). Chaotic billiards also have countably many periodic trajectories, that occupy a dense set of measure zero in phase space (Figure 0.10). The global orbit structure is thus extremely rich and complex. Here again, this is reflected in the variety of patterns in Figures 0.1 and 0.2.

Up to now our discussion has been based on intuition and the observation of pictures, without any rigourous mathematical formulation. Let us focus on the chaotic case, and state a few theorems whose proofs are based on the semiclassical approximation and require to overcome the difficulties mentioned above.

The "quantum ergodicity theorem" (Shnirelman, Zelditch, Colin de Verdière [78, 88, 41]) deals with the case of "ergodic" systems and has been extended to the case of billiards [48, 86]. Classical ergodicity means that a ball thrown in a random direction will visit phase space uniformly: for almost all initial conditions, the average time spent in a given region is proportional to the volume of that region. The Sinai billiard, stadium billiard

[1]More explicitely, according to the semiclassical eigenfunction hypothesis [70, 23, 83] one expects that quantum states concentrate on those regions in phase space, which a typical orbit explores in the long time limit. For integrable systems these invariant regions are the regular tori, while for chaotic systems one expects a uniform distribution on the energy shell. Note: In the latter case this follows [19] from the quantum ergodicity theorem, with the restriction to a subsequence of density 1.

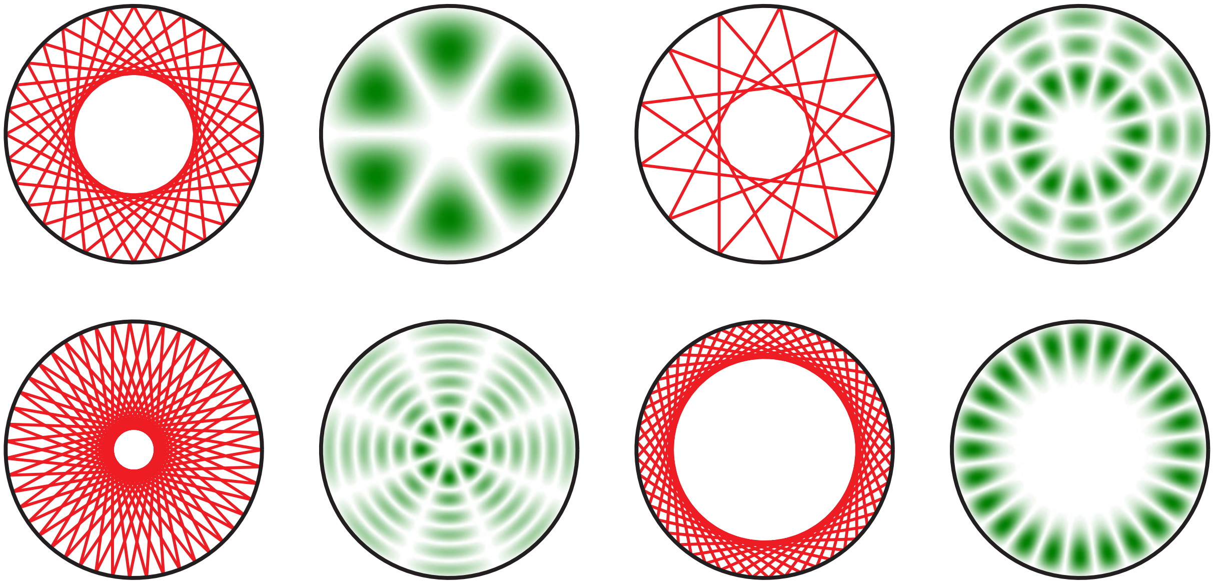

FIGURE 0.9. Plot of $|\psi_{nm}(x,y)|^2$ for selected states in the circle billiard.

and cardioid billiard are known to be ergodic [77, 36, 84, 64]. The quantum ergodicity theorem proves a correspondence between the ergodicity property of a billiard table and the behaviour of waves: it establishes that a typical stationary wave will occupy phase space uniformly.

For a precise formulation of the theorem, we let (X, g) be a compact Riemannian manifold (the billiard table). We denote by T^*X the cotangent bundle over X, and $S^*X \subset T^*X$ the unit cotangent bundle. An element of S^*X is denoted by (x, θ), where $x \in X$ and $\theta \in S^*_x X$ is a vector of norm 1; x represents "position", whereas θ represents "velocity", or "momentum". The phase space S^*X is endowed with the Liouville measure, $\mathrm{Vol}(dx) d\sigma_x(\theta)$, where Vol is the Riemannian volume element and $d\sigma_x(\theta)$ is the uniform measure on the sphere $S^*_x X$.

Theorem 0.1. [78, 88, 41] *Assume that the geodesic motion (billiard motion) on S^*X is* **ergodic** *for the Liouville measure*[2].

Let $(\psi_n)_{n\in\mathbb{N}}$ be an orthonormal basis of the Hilbert space $L^2(X)$, formed of eigenfunctions of the Laplacian ($\Delta\psi_n = -k_n^2\psi_n$, with $k_n \le k_{n+1}$).

[2]This theorem was first proved for boundaryless Riemannian manifolds, then extended to manifolds with boundary in [48, 86]. This covers the case of Euclidean billiards. For arbitrary manifolds the motion in straight lines is replaced by motion along geodesics. A version for general Hamiltonian systems appears in [53] `AB: the initial part of this footnote is already discussed a bit before in the text --` *has been extended to .....*

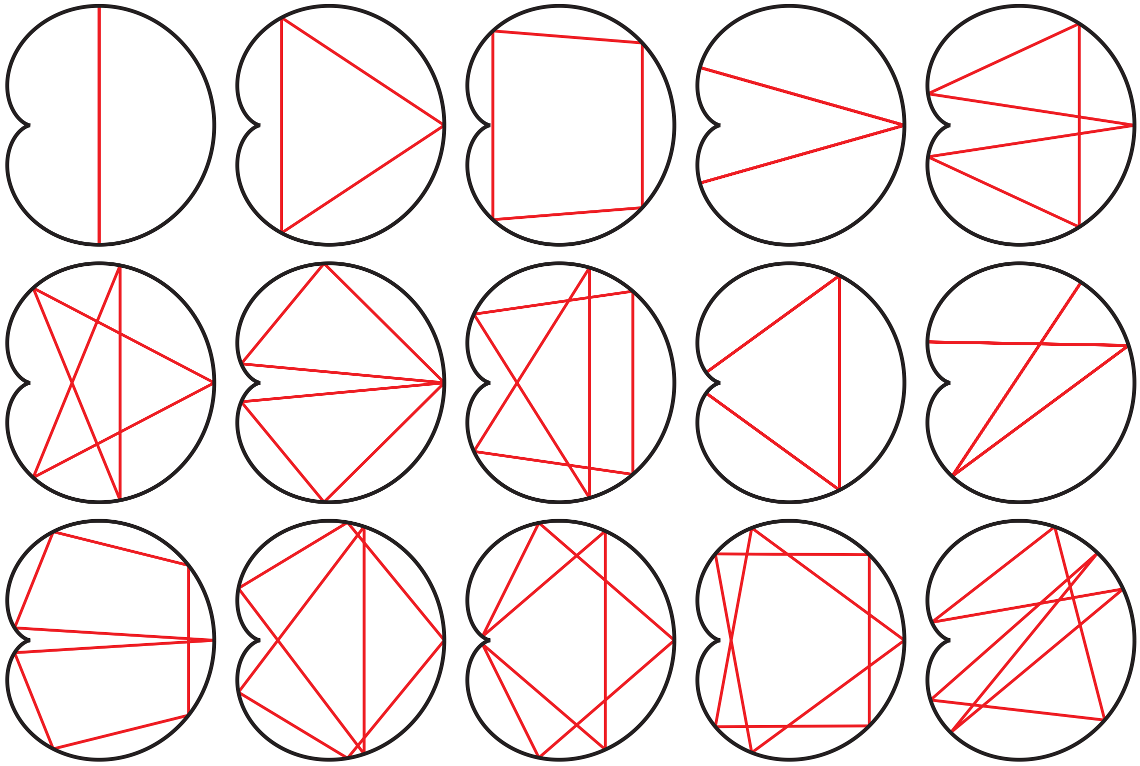

FIGURE 0.10. Periodic orbits in the cardioid billiard. For this system a proven symbolic dynamics exists [18, 17, 44] such that periodic orbits can be determined in a systematic way.

Let A be a pseudodifferential operator of order 0. Then, there exists a subset $\mathcal{S} \subset \mathbb{N}$ of density[3] *1, such that*

$$\langle \psi_n, A\psi_n \rangle \underset{\substack{n\to\infty,\\ n\in\mathcal{S}}}{\longrightarrow} \int_{S^*X} \sigma(A)(x,\theta)\,\mathrm{Vol}(dx)d\sigma_x(\theta) \tag{0.1}$$

where $\sigma(A)$ stands for the principal symbol of A.

The principal symbol $\sigma(A)$ is a function on S^*X. In quantum mechanical terms, it is the "classical observable" corresponding to the "quantum observable" A. The quantum ergodicity theorem can be interpreted as saying that almost all eigenfunctions become uniformly distributed over the phase space S^*X, both in the "position" variable x and in the "momentum" variable θ. An interesting special case is when $a(x)$ is a function on X and A is the operator of multiplication by a; in that case, equation (0.1) simply reads

$$\int_X a(x)|\psi_n(x)|^2\,\mathrm{Vol}(dx) \underset{\substack{n\to\infty,\\ n\in\mathcal{S}}}{\longrightarrow} \int_X a(x)\,\mathrm{Vol}(dx).$$

[3] $\mathcal{S} \subset \mathbb{N}$ being of density 1 means that $\frac{\sharp\{n\in\mathcal{S}, n\leq N\}}{N} \underset{N\to+\infty}{\longrightarrow} 1$, such that the statement holds for almost all elements, i.e. the complement of $\mathcal{S}$ is negligible.

Thus in position space almost all eigenstates converge (in the weak sense) to a constant. For an introduction to the quantum ergodicity theorem for physicists see [20].

The quantum ergodicity theorem is in agreement with the idea that the semiclassical approximation allows to describe qualitatively the propagation of waves. [4] It proves that most eigenfunctions in Figures 0.1 and 0.2 occupy space uniformly. However, the stadium-shaped billiard is extremely mysterious: numerical simulations [55, 69, 65] show the existence of infinitely many "bouncing ball modes", like the one shown in Figure 0.5. These modes bounce back and forth between the two parallel walls, without visiting the bends. In agreement with the quantum ergodicity theorem, eigenfunctions of this type are extremely rare, their number is negligible in the collection of eigenfunctions. Nevertheless, they seem to exist. In [81, 19, 63] their number has been investigated numerically and theoretically, giving arguments in favor of $N_{\rm bb}(E) \sim E^{3/4}$. The existence of bouncing-ball modes was only proved mathematically around 2008 by Andrew Hassell [51]. To be more specific, consider the one-dimensional family of stadia, obtained by varying the width of the inner rectangle, while the height is fixed. What Hassell proved is that, for "almost all" stadia, there is a sequence of eigenfunctions that concentrate partially on the bouncing-ball trajectories. The proof is not constructive, and thus does not allow to predict for which stadia and which frequencies these "bouncing ball modes" will be observed.

The work of Anantharaman and Anantharaman–Nonnenmacher deals with the eigenfunctions shown in Figures 0.4 and 0.5, which are enhanced in a neighbourhood of an unstable periodic billiard trajectory. This phenomenon named *scarring* [55] generally means, for physicists, a visual enhancement. Mathematicians are more cautious: they want a precise definition of the word *enhanced* ; and they require the existence of infinitely many such eigenfunctions to speak about "scarring". With their restricted definition, mathematicians tend to think that scarring on an unstable periodic trajectory cannot occur. Anantharaman and Nonnenmacher give a diplomatic answer: "scarring" on an unstable orbit, if it ever occurs, can only be partial. This means that part of the energy of oscillation must be localized away from the unstable orbit.

To state rigourous results, we will restrict our study to the case of *negatively curved* manifolds, closed and without boundary[5]. In addition to ergodicity, the classical dynamics (namely the motion of a particle along geodesics) is extremely chaotic: strong mixing properties, exponential instability,... Although the dynamics is deterministic, it has the property of being a *Bernoulli system*, meaning that the long time behaviour of the particle is as random as if it were flipping coins to decide where it is going. All this is already true for the above billiards [39], but the geodesic motion on negatively curved manifolds is, in addition, a *uniformly hyperbolic* dynamical system : this is a tame form of chaos, that has been studied very early and is now well understood [15, 56, 57, 52].

[4] AB: I am not sure whether I fully see the connection between the propagation and the stationary states ;-).

[5] Some of these results have also been proved for the "cat-map" on a torus [46, 35]. In principle the same results could also be proven rigourously for certain types of billiards, but this would be extremely technical work and has not been done yet. The adaptation to *non-positively curved* surfaces is already a technical challenge [71].

For such systems, it is generally believed that the convergence (0.1) takes place for the whole sequence of eigenfunctions (i.e. $\mathcal{S} = \mathbb{N}$, in other words, there is no exceptional subsequence). This is the Quantum Unique Ergodicity conjecture of Rudnick and Sarnak [74, 73]. Numerical computations already indicated [16] that there is no such scarring for an explicit example of a system on a surface of constant negative curvature. Up to now, the conjecture has only being proven for the so-called "arithmetic" surfaces (for instance, the modular surface obtained as the quotient of the hyperbolic plane $\mathbb{H}^2$ by $SL(2,\mathbb{Z})$), and for simultaneous eigenfunctions of the laplacian and the Hecke operators [62, 30, 79]. Since these arithmetic surfaces are somehow more explicit than the other hyperbolic surfaces, this is again one case where number theory can come to the rescue – in a very elegant manner. Below, we deal with general negatively curved manifolds, for which even the phenomenon of "scarring" is not fully understood. But let us give a mathematical definition of scarring.

Let $B \subset \mathbb{N}$ be an infinite subset such that, for any pseudodifferential operator A of order 0,

$$\langle \psi_n, A\psi_n \rangle \underset{\substack{n\to\infty,\\ n\in B}}{\longrightarrow} \int_{S^*X} \sigma(A)(x,\theta)d\mu(x,\theta) \tag{0.2}$$

for some probability measure μ. We will say that the sequence of eigenfunctions $(\psi_n)_{n\in B}$ exhibits *strong scarring* if μ is a measure that is *entirely* supported on periodic classical trajectories. We will say that the sequence of eigenfunctions $(\psi_n)_{n\in B}$ exhibits *partial scarring* on a periodic classical trajectory γ if $\mu(\gamma) > 0$. In the case of an ergodic geodesic flow (in particular, on negatively curved manifolds), the quantum ergodicity theorem implies that such a set B must have zero density.

What is proven in various contexts in [2, 12, 11, 3, 14, 71, 35, 46, 4, 6] is that, if such a convergence as (0.2) occurs, then the measure μ must have a high "dynamical complexity" (measured thanks to the notion of Kolmogorov-Sinai entropy). This rules out the possibility of strong scarring – but leaves open the possibility of partial scarring. If partial scarring occurs, our result implies that $\mu(\gamma)$ cannot be too large. This means that the "scars" that one is tempted to see in Figures 0.4 and 0.5 are never very sharp. We stress the fact that partial scarring has been observed in certain examples of chaotic systems [60, 47]; thus we do not completely exclude the possibility of partial scarring on negatively curved manifolds.

To finish, let us go back to our simplest example, the cube. As alluded to earlier, in a euclidean cube, scarring is impossible, due to the following result. Let us take any sequence $(u_n)_{n\in\mathbb{N}}$ such that $\int_{[0,\pi]^d} |u_n(x)|^2 dx = 1$. After extracting a subsequence, one can always assume that there is a probability measure μ on $[0,\pi]^d$ such that

$$\int_{t=0}^{1}\int_{[0,\pi]^d} a(x)|e^{it\Delta}u_n(x)|^2 dxdt \underset{n\to+\infty}{\longrightarrow} \int_{[0,\pi]^d} a(x)d\mu(x)$$

for any continuous observable $a(x)$. It is shown in [] that μ must have a density: $d\mu(x) = \rho(x)dx$ (thus excluding the possibility of scars) and that for any open set $\Omega \subset [0,\pi]^d$ there exists $c_\Omega > 0$, independent on the initial conditions (u_n), such that $\rho(\Omega) > c_\Omega$ (which means that the solution of the Schödinger equation must spend some uniform amount of time within Ω). Because of spectral degeneracies it is easy to build examples where $\rho(x)dx$ is not the uniform measure. However, obtaining more precise information on the density ρ is a subtle issue.

In recent work, techniques from ergodic theory have also been used to study the spectrum of the damped wave equation on negatively curved manifolds [5] or in the square [8] and to characterize the controllability of the Schrödinger equation in various geometries [9, 13, 10]. These ideas also helped to understand the resonance spectrum of *open* systems [67, 68, 76, 37]. We stress the fact that the mathematical work described here deals with $|\psi_n(x)|^2$ as being a density: in expressions such as (0.2) we look at the average of this density over sets of finite, fixed sizes. With this approach, many features seen in numerics remain mysterious: nodal lines, fluctuations of eigenfunctions around their "typical" statistical bevaviour, estimates of L^p norms and description of the "peaks" of eigenfunctions, value distribution, restriction of eigenfunctions to submanifolds,... Another challenge is to understand generic systems in which one has a coexistence of regular and chaotic motion. Even classically, this numerically observed coexistence has only been established rigorously in few cases. Quantum mechanically one expects that in the semiclassical limit eigenstates either concentrate on regular tori or within the chaotic regions. However, the convergence to this limit can be extremely slow and many important physical effects are relevant away from the semiclassical limit: e.g. dynamical tunnelling [42, 61] partial barriers [66], deviations from universal spectral statistics [22], ...

To learn more: We refer the reader to the lecture notes [3] for details about the work of Anantharaman–Nonnenmacher. The survey papers [87] and [75] and books in quantum chaos [49, 80, 50] offer a more detailed and exhaustive presentation of the various aspects of the subject.

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