Precise counting results for closed orbits of Anosov flows

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ABSTRACT : We study the problem of counting closed geodesics according to their lengths and under homological constraints on a compact surface of negative curvature. We show how to use Dolgopyat's recent results to obtain a full asymptotic expansion, in addition to the leading term given by Lalley.

We first state the properties of the stable and unstable leaves used by Chernov and Dolgopyat; then we introduce the usual transfer operators and we prove the result with the help of a dynamical ζ-function.

RESUME : Nous étudions un problème de dénombrement de géodésiques fermées, classées selon leur longueur et leur classe d'homologie, sur une surface compacte de courbure négative. Nous expliquons comment les travaux récents de Dolgopyat permettent de donner un développement asymptotique complet, en plus du terme principal déjà obtenu par Lalley.

Nous commençons par énoncer les propriétés des feuilletages stable et instable utilisées par Chernov et Dolgopyat, puis nous introduisons des opérateurs de transfert dont l'étude nous permettra de conclure, par l'intermédiaire de fonctions ζ dynamiques.

1 Introduction and statement of results

The first counting results for closed orbits of a hyperbolic flow date back to 1961, when Huber introduced techniques from analytic number theory to count closed geodesics on a compact manifold of constant negative curvature ([10]) . He used Selberg's trace formula, which gives a relation between a certain ζ-function and a Fredholm determinant associated to the Laplace operator, and he obtained a result analogous to the famous "prime number theorem" :

Theorem 1.1 *(Prime number theorem) When $x > 0$, define $\pi(T)$ to be the number of primes in $[0, T]$. Then $\pi(T) \sim \frac{T}{\log(T)}$ as $T \longrightarrow +\infty$.*

Huber's theorem reads as follows :

Theorem 1.2 *Let V be a compact n-dimensional manifold of constant curvature -1. When $T > 0$, define $\pi(T)$ to be the number of closed geodesics γ with length $l(\gamma)$ in $[0, T]$. Then $\pi(T) \sim \frac{e^{(n-1)T}}{(n-1)T}$ as $T \longrightarrow +\infty$.*

(Huber actually proved it for surfaces only).

In 1969 Margulis proved a similar result for manifolds of variable negative curvature ([18]); he used mostly the hyperbolic structure of the geodesic flow. The definition of "hyperbolicity" was stated the same year by Anosov in [2] and it is now called the "Anosov property" :

Definition 1.3 *A smooth flow (ϕ_t) on a compact manifold M is said to be "Anosov" if the tangent bundle TM can be decomposed into the direct sum of three subbundles : $TM = E_s \oplus E_0 \oplus E_u$, with the following properties : if we choose an arbitrary Riemannian metric $\| \, . \, \|$ on M, there exists $C, \gamma > 0$ such that for all $x \in M$,*

- *For all $v \in E_s(x)$, for all $t > 0$, $\| d\phi_t(v) \| < C \| v \| e^{-\gamma t}$*
- *For all $v \in E_u(x)$, for all $t > 0$, $\| d\phi_{-t}(v) \| < C \| v \| e^{-\gamma t}$*
- *$\dim E_0(x) = 1$ and E_0 is tangent to the flow*

(This property does not depend on the choice of the norm $\| \, . \, \|$)

The distributions E_s and E_u are respectively called the stable and unstable distributions of the flow.

In the same paper ([2]) Anosov showed that the geodesic flow on the tangent bundle of a manifold of negative curvature has this property. Geodesic flows still remain the most studied Anosov flows (for other examples see [2]).

In 1973 Bowen and Ratner constructed "Markov sections" for Anosov flows ([5],[24]). This allows these flows to be coded by suspensions over subshifts of finite type, so that Ruelle's theory on symbolic dynamics applies : one can define "dynamical ζ-functions" whose poles are now related to the spectrum of "transfer operators". This gives the same kind of estimates as Theorem 1.2 for general Anosov flows; the dimension $n-1$ has to be replaced by the topological entropy of the flow.

It is possible to introduce additional conditions on the closed orbits we want to count. Here are some examples of counting results that were obtained for closed geodesics in a given homology class, using either harmonic analysis and the Laplace operator in the case of constant negative curvature, or symbolic dynamics and transfer operators in the case of variable curvature :

Theorem 1.4 *(Katsuda-Sunada, [12]) Let (ϕ_t) be a smooth, transitive, weak-mixing Anosov flow on a compact manifold M. Let A be an abelian group and $[.] : H_1(M, \mathbb{Z}) \longrightarrow A$ a surjective homomorphism. When γ is a closed curve on M we denote $[\gamma]$ the image of its homology class under $[.]$. Let m be* *the measure of maximal entropy with respect to* (ϕ_t). *One assumes that the asymptotic winding cycle of* m *vanishes, i.e :* $\int_M < \omega, \frac{d\phi_t}{dt}(x) > dm(x) = 0$, *for any closed 1-form* ω.

Fix $\alpha \in A$. *When* $T > 0$, *define* $\pi(\alpha, T)$ *to be the number of closed orbits* γ *with period* $l(\gamma) \leq T$ *and such that* $[\gamma] = \alpha$. *Then*

$$\pi(\alpha, T) \sim \frac{1}{l(2\pi)^{d/2} Vol(A^*)} \frac{e^{hT}}{hT^{d/2+1}}$$

Here d *is the rank of* A, l *its order of torsion,* A^* *is the set of characters of* A, *and* h *is the topological entropy of the flow. The precise definition of the volume* $Vol(A^*)$ *is given in [12].*

(In [29] Sharp proved a more general version of this theorem, assuming that there is a certain Gibbs measure whose winding cycle vanishes - and not necessarily the measure of maximal entropy)

Theorem 1.5 *(Phillips-Sarnak, [20]). The notation are the same as in Theorem 1.4. Now* $M = T^1V$ *is the tangent bundle of a* n*-dimensional manifold* V *of constant negative curvature, and* (ϕ_t) *is the geodesic flow on* M. *Then there exists a sequence of real numbers* $c_0, c_1, \cdots$ *such that*

$$\pi(\alpha, T) = \frac{e^{(n-1)T}}{l(2\pi)^{d/2} Vol(A^*)(n-1)T^{d/2+1}} (1 + \frac{c_1}{T} + \cdots)$$

In this last example, a full asymptotic expansion can be given because the spectrum of the Laplace operator is well known; it was not the case for transfer operators until Dolgopyat proved a new upper bound on their norms in 1997 (in the case of 3-dimensional Anosov flows). This allowed Pollicott and Sharp to give a full asymptotic expansion in the case of surfaces of variable negative curvature, with or without homological constraints ([22], [23]).

In this paper we study a more general type of homological constraints, first introduced by Lalley in 1989 ([16]). We will first state the result in the case of geodesic flows on manifolds, with the same notations as in Theorem 1.5, except that we will now consider only *torsion free* integral homology classes on M : we take for A the torsion free component of the homology group $H_1(M, \mathbb{Z})$, and we still denote it $H_1(M, \mathbb{Z})$. This restriction is only intended to simplify our calculations and it is possible to drop it. In fact, in the case of compact oriented surfaces of negative curvature, $H_1(M, \mathbb{Z}) = H_1(V, \mathbb{Z})$ and it has no torsion. From now on, we will denote by $[\gamma]$ the torsion free homology class of a closed geodesic γ.

Let us fix $\xi \in H_1(M, \mathbb{R})$, $\alpha \in H_1(M, \mathbb{Z})$, and $\delta > 0$. Since $H_1(M, \mathbb{Z})$ is a lattice in $H_1(M, \mathbb{R})$, we can choose an "integral part" map $[.] : H_1(M, \mathbb{R}) \longrightarrow H_1(M, \mathbb{Z})$ (we use the same brackets to denote two different things but we hope it will not cause too much confusion). We define Γ to be the set of closed geodesics and we want to study the asymptotic behaviour in T of the quantity

$$\pi(\xi,\alpha,\delta;T) = Card\{\gamma\in\Gamma, T\le l(\gamma)\le T+\delta, [\gamma]=\alpha+[T\xi]\}$$

It is necessary to consider the integral part $[T\xi]$, and not $T\xi$, simply to ensure that $\pi(\xi,\alpha,\delta;T)$ does not equal zero for the trivial reason that $[\gamma]\in H_1(M,\mathbb{Z})$ whereas $T\xi\in H_1(M,\mathbb{R})\setminus H_1(M,\mathbb{Z})$.

Our main interest is the dependence on ξ and T of these quantities. We have introduced the quantity α mainly in order to obtain Sharp-Pollicott's result ([23]) as a special case ($\xi=0$); but it should be interesting also to study the dependence on α of the coefficients (see Remark 3 at the end of this section). The number δ is arbitrary and has not much incidence on the result.

We need to introduce the following functions H and P :

$$\begin{array}{rcl} P : H^1(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ [\omega] & \longmapsto & \sup\limits_{m\in\mathcal{M}} h(m) + <[\omega],[m]> \end{array}$$

where $\mathcal{M}$ is the set of ϕ-invariant probability measures on M, $h(m)$ is the metric entropy of $m\in\mathcal{M}$, and $[m]\in H_1(M,\mathbb{R})$ is the winding cycle of m (see [27]), defined by duality by the equality :

$$<[\omega],[m]> = \int_M <\omega, \frac{d\phi_t}{dt}(x)> dm(x)$$

when ω is a closed 1-form and $[\omega]$ is its cohomology class.

$$\begin{array}{rcl} H : H_1(M,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \sup\limits_{m\in\mathcal{M},[m]=\xi} h(m) \end{array}$$

The function H is supported by the compact convex $C=\{[m], m\in\mathcal{M}\}\subset H_1(M,\mathbb{R})$. It is explained in [4] that P is analytic and stricly convex in $H^1(M,\mathbb{R})$ and that H is analytic and strictly concave on the interior $\overset{\circ}{C}$. Also, they are related by Legendre duality : $P=(-H)^*$.

Lalley's theorem reads :

Theorem 1.6 *([16]) For all $\xi\in\overset{\circ}{C}$,*

$$\pi(\xi,\alpha,\delta;T)\sim c_0(\xi^T,\delta)\frac{e^{TH(\xi^T)-<u^T|\alpha>}}{T^{d/2+1}}$$

where we have defined

$$c_0(\xi^T, \delta) = \frac{1}{(2\pi)^{d/2} Vol(A^*) \sqrt{\mid det P''(u^T) \mid}} \left(\frac{e^{\delta P(u^T)} - 1}{P(u^T)} \right)$$

and $\xi^T := \frac{[T\xi]}{T}$ *and* $u^T := -\nabla H(\xi^T)$.
Notice that $\xi^T = \longrightarrow \xi$ *and* $u^T = \longrightarrow u^\xi := -\nabla H(\xi)$. *as* $T \longrightarrow +\infty$.

(Lalley proved it for ξ in a neighbourhood of 0, and Babillot and Ledrappier showed in [4] that this neighbourhood is precisely $\overset{\circ}{C}$)

We shall now apply Dolgopyat's new estimates to give the full asymptotic expansion :

Theorem 1.7 *In the case of the geodesic flow on a* surface *of negative curvature, there are analytic functions* $c_1, c_2, c_3, \cdots$ *on* $\overset{\circ}{C} \times H_1(M, \mathbb{R}) \times \mathbb{R}$ *such that, for all* $N \in \mathbb{N}$,

$$\pi(\xi, \alpha, \delta; T) = \frac{e^{TH(\xi^T) - <u^T|\alpha>}}{T^{d/2+1}} \left(c_0(\xi^T, \delta) + \sum_{k=1}^{N} \frac{c_k(\xi^T, \alpha, \delta)}{T^k} + O(T^{-(N+1)}) \right)$$

Moreover, the c_k*'s are polynomial with respect to* α.

Remarks :
(1) Our method gives, in theory, explicit expressions of the c_k's in terms of the functions P and H and their derivatives; however, it seems very difficult to give the meaning or even a simple expression of these coefficients. We can at least say that there are no fractional powers of $\frac{1}{T}$ in the expansion, which the tauberian theorem used by Pollicott and Sharp in [22] did not show.
(2) Since δ is fixed once and for all, and since we are not going to discuss the dependence on δ, we may as well drop all the δ's in the argument of the c_k's.
(3) It is not clear to the author whether or not the theorem is valid for higher dimensional manifolds. Almost all the references mentioned in this article include proofs for surfaces only. The author does not know if Dolgopyat's estimates are true in higher dimensions; even so, the curvature of the manifold should be $\frac{1}{4}$-pinched for the horocyclic foliations to be C^1.

We will actually work in the more general setting of Anosov flows :

Let M be a compact manifold on which a transitive, topologically mixing Anosov flow (ϕ_t) is given. Let us denote by Γ the set of its closed orbits. When $\gamma \in \Gamma$, let $l(\gamma)$ denote its period. Let $F = (F_1, \cdots, F_d)$ be an $\mathbb{R}^d$-valued Hlder continuous function on M. We shall assume that the closed subgroup G of $\mathbb{R}^{d+1}$ generated by the vectors $(l(\gamma), \int_\gamma F)$, $(\gamma \in \Gamma)$, is of the form $G = \mathbb{R} \times A$, where A is a lattice in $\mathbb{R}^d$. The choice of a fundamental domain of A allows us to define the "integral part" of a vector $[.] : \mathbb{R}^d \longrightarrow A$. We fix δ an arbitrary positive number. For $\alpha \in A$ and $\xi \in \mathbb{R}^d$, we study the quantity

$$\pi(\xi, \alpha, \delta; T) = Card\{\gamma \in \Gamma, T \leq l(\gamma) \leq T + \delta, \int_\gamma F = \alpha + [T\xi]\}$$

Let us denote by $\mathcal{M}$ the set of ϕ-invariant probability measures on M and C the compact convex set $C = \{\int F dm, m \in \mathcal{M}\} \subset \mathbb{R}^d$. For $\xi \in \mathbb{R}^d$, define as previously $H(\xi) = \sup\{h(m), m \in \mathcal{M}, \int F dm = \xi\}$. For $u \in \mathbb{R}^d$, define $P(u) = \sup_{m \in \mathcal{M}} h(m) + \int (\sum u_i F_i) dm$; it is the pressure of the function $u_1 F_1 + \cdots + u_d F_d$. As before, P and H are related by Legendre duality. Moreover P is analytic and strictly convex on $\mathbb{R}^d$, and H is continuous on C, analytic and strictly concave on $\overset{\circ}{C}$.

In this context, Lalley's result remains true; so does our theorem, but under some restrictions on the flow :

Theorem 1.8 *Suppose $dim\, M = 3$, the characteristic foliations of the flow are of class C^1 and uniformly jointly non-integrable (see section 2 for definitions). Then there are analytic functions $c_1, c_2, c_3, \cdots$ on $\overset{\circ}{C} \times \mathbb{R}^d \times \mathbb{R}$ such that, for all $N \in \mathbb{N}$,*

$$\pi(\xi, \alpha, \delta; T) = \frac{e^{TH(\xi^T) - <u^T|\alpha>}}{T^{d/2+1}} \left(c_0(\xi^T, \delta) + \sum_{k=1}^{N} \frac{c_k(\xi^T, \alpha, \delta)}{T^k} + O(T^{-(N+1)}) \right)$$

Moreover, the c_k's are polynomial with respect to α.

Remark : The author does not know if it is still true when $dim\, M > 3$.

We would like to stress the fact that it can be interesting to see the condition $\int_\gamma F = \alpha + [T\xi]$ as a condition on $\frac{1}{l(\gamma)} \int_\gamma F$ rather than a condition on $\int_\gamma F$: this condition implies that $\frac{1}{l(\gamma)} \int_\gamma F$ should be near ξ. In this perspective, it is interesting to relate our result to Kifer's large deviations result for the probability measures

$$\frac{1}{l(\gamma)} \int_\gamma \delta_t dt$$

Kifer's theorem implies that $\limsup \frac{1}{T} \log \pi(\xi, \alpha, \delta; T) \leq H(\xi)$ for all ξ. Our result can be seen as a "precise large deviations" result. We refer the reader to [14] and [4] for more details.

In particular, we can use Theorem 1.8 to count closed orbits under homological constraints : let d be the first Betti number of M and let $(\omega_1, \cdots, \omega_d)$ be a family of closed harmonic 1-forms whose cohomology classes form an integral basis of $H^1(M, \mathbb{R})$. Set $F_i(x) =< \omega_i | \frac{d\phi_t}{dt}(x) >$. Then A is the torsion free component of $H^1(M,\mathbb{Z})$ and $\int_\gamma F$ represents the (torsion free) homology class of γ. Since the geodesic flow on a surface satisfies the hypotheses of Theorem 1.8 ([6], Theorem 18.1), we obtain Theorem 1.7 as a special case.

Remarks :

(1) In the case of the geodesic flow on a manifold of constant negative curvature and when $\xi = 0$, it was already known (Theorem 1.5) that there were no terms in $\frac{1}{T^{n+\frac{1}{2}}}$. This is a consequence of the invariance of the geodesic flow under the map $TV \to TV$, $v \mapsto -v$. Now for a general Anosov flow, this property is replaced by our assumption that G decomposes as a product $G = \mathbb{R} \times A$.

(2) In fact we shall prove a stronger theorem : for each compact subset K of $\overset{\circ}{C}$, there exists a positive constant c such that, for all $\xi \in K$,

$$\mid \pi(\xi,\alpha,T) - \frac{e^{TH(\xi^T)-<u^T|\alpha>}}{T^{d/2+1}} \left(c_0(\xi^T) + \sum_{k=1}^{N} \frac{c_k(\xi^T,\alpha)}{T^k} \right) \mid \leq c\, \frac{e^{TH(\xi^T)-<u^T|\alpha>}}{T^{d/2+1+N+1}}$$

(That is to say, the O of Theorem 1.8 is locally uniform in ξ)

(3) The presence of the term $e^{-<u^T|\alpha>}$ shows that if $u^\xi \neq 0$ the closed geodesics homologically in the direction ξ are not uniformly distributed with respect to α. For the geodesic flow, $u^\xi \neq 0$ is equivalent to $\xi \neq 0$.

(4) When $\xi \notin C$, the quantity $\pi(\xi,\alpha,\delta;T)$ vanishes for all T's sufficiently large, since in that case there exists an $\eta > 0$ such that $\mid \int_\gamma F - l(\gamma)\xi \mid > \eta l(\gamma)$, for all γ. When $\xi \in \partial C$, little is known about the asymptotics of $\pi(\xi,\alpha,\delta;T)$. In the case of the geodesic flow on a manifold V, the convex set C is the unit ball for the stable norm defined by Federer, and elements of the boundary of C have a very interesting interpretation in terms of "geodesic laminations"([3]). We hope to be able to use the properties of laminations in order to find a polynomial upper bound on $\pi(\xi,\alpha,\delta;T)$ when V is a surface.

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2 The differential structure of Anosov flows

We will use Dolgopyat's theorem on transfer operators in order to prove our main theorem. Therefore we will first explain the definition of "Markov rectangles" he used in his proof. These rectangles are slightly different from Bowen's, since Dolgopyat needed to keep track of the smoothness (or lack of smoothness) of the stable and unstable distributions. In fact, Dolgopyat did not use the usual coding of Anosov flows by suspensions over subshifts of finite type; instead, following Chernov ([6]), he worked directly on the Markov sections and made use of their differentiable properties. Finally,we formulate the Uniform Non-Integrability hypothesis introduced by Chernov ([6]) and which is the key of Dolgopyat's proof.

2.1 Regularity and local product structure

The stable and unstable distributions E_s and E_u defined in section 1 are continuous, and their dimensions are constant : $dimE_s = k$, $dimE_u = l$ (under the assumptions of Theorem 1.8, $k = l = 1$) . Furthermore, they are always integrable and the integral leaves can be defined as :

$$W_{ss}(x) = \{y \in V/d(\phi_t x, \phi_t y) \underset{t\to+\infty}{\longrightarrow} 0\}$$

$$W_{su}(x) = \{y \in V/d(\phi_{-t} x, \phi_{-t} y) \underset{t\to+\infty}{\longrightarrow} 0\}$$

They are usually called "strong stable" and "strong unstable" manifolds. We shall also need the following "weak stable" and "weak unstable" foliations :

$$W_s(x) = \cup_{t\in\mathbb{R}}\phi_t W_{ss}(x)$$

$$W_u(x) = \cup_{t\in\mathbb{R}}\phi_t W_{su}(x)$$

which are the orbits of the former under the flow and which also are the integral leaves of the distributions $E_s \oplus E_0$ and $E_0 \oplus E_u$. It is also useful to introduce the local version of these leaves :

$$W_{*s}(x,\varepsilon) = \{y \in W_{*s}(x), d(\phi_t x, \phi_t y) \leq \varepsilon \, \forall t > 0\}$$

$$W_{*u}(x,\varepsilon) = \{y \in W_{*u}(x), d(\phi_{-t} x, \phi_{-t} y) \leq \varepsilon \, \forall t > 0\}$$

(For further details, see [2], [5]).

Lemma 2.1 *(Local product structure, [5]) If we fix a $\delta > 0$, there exists $\eta > 0$ such that : if x and y satisfy $d(x, y) < \eta$, then there is a unique $\mid t \mid\leq \delta$ such that $W_{ss}(x,\varepsilon) \cap W_u(\phi_t y,\varepsilon) \neq \emptyset$. Moreover this intersection consists of a single point, denoted by $[x, y]$.*

In general, the stable and unstable distributions are not C^1 but only Hlder-continuous. Additional regularity is an exceptional phenomenon ([9]). However Anosov proved the following result about the dependence of the leaves on the base point :

Theorem 2.2 *([2]) The integral leaves $W_s(x)$ and $W_u(x)$ depend continuously on the point x.*

2.2 Markov rectangles

Definition 2.3 *A subset R of M will be called a rectangle if it is of the form $R = [U(R), S(R)]$, where $U(R)$ (respectively $S(R)$ is a closed subset of a strong unstable (resp. stable) such that $U(R) = Cl(IntU(R))$, (resp. $S(R) = Cl(IntS(R))$) (closure and interior are taken with respect to the induced topology on the leaves).*

The reader should be aware that it is not exactly Bowen's definition of a rectangle : Bowen's rectangles were subsets of differentiable balls but they did not a priori contain strong stable or unstable leaves. Instead, Dolgopyat's rectangles contain strong stable leaves and a strong unstable leaf, so that they are not differentiable a priori. Bowen's rectangles are the projections of Dolgopyat's on small diffentiable discs, along the direction of the flow.

The set R is partitioned into (strong) stable leaves $W_s \cap R$, denoted by W_s^R, and into "unstable" leaves $W_u \cap R = W_u^R$. Only one of these "unstable" leaves, namely $U(R)$, is actually a strong unstable leaf of the flow; the others are unstable leaves deformed along the trajectories of the flow, but they will be unstable leaves for the first return time. The following result is a consequence of the work of Ratner and Bowen ([24], [5]) :

Theorem 2.4 *There exists a finite collection $(R_i)_{i=1,\cdots,k}$ of rectangles which are transverse to the flow and such that :*
- There exists $\alpha > 0$ (that can be chosen arbitrarily small), such that $M = \cup\phi_{[-\alpha,0]}R_i$
- $diam\, R_i < \alpha$
- The first return map $\bar{T} : \mathcal{R} = \cup R_i \longrightarrow \mathcal{R}$ satisfies $\bar{T}(W_s^{\mathcal{R}}(x)) \subset W_s^{\mathcal{R}}(\bar{T}x)$, $\bar{T}^{-1}(W_u^{\mathcal{R}}(x)) \subset W_u^{\mathcal{R}}(\bar{T}^{-1}x)$

For $x \in \mathcal{R}$, let $\tau(x)$ designate its first return time into $\mathcal{R}$: $\bar{T}(x) = \phi_{\tau(x)}(x)$. The manifold M can thus be modelled by $\mathcal{R}^\tau = \{(x,s) \in \mathcal{R} \times \mathbb{R}, 0 \leq s \leq \tau(x), (x,\tau(x)) \sim (\bar{T}(x), 0)\}$, and the flow is vertical translation with constant speed. We do not want to use subshifts of finite type because the smoothness of τ is crucial in Dolgopyat's proof. Thanks to our choice of rectangles, τ is constant along the stable leaves of the rectangles, and τ has the same regularity as the distribution E_s. Furthermore we will see that the mixing properties of the flow are related to some properties of τ.

2.3 Non joint integrability of E_u and E_s :

Anosov's alternative ([2]) gives the relationship between the integrability of $E_s \oplus E_u$ and the mixing rate of the flow.

Theorem 2.5 *If E_s and E_u are jointly integrable, and if one of the integral leaves of the distribution $E_s \oplus E_u$ is compact, then the flow is a suspension*

Figure 1: Non joint integrability.

Figure 2: Markov sections.

of a diffeomorphism of this leaf by a constant roof function. In particular the flow is not topologically mixing.

From now on we shall assume that the flow is topologically mixing. In other terms, the distribution $E_s \oplus E_u$ is not integrable, or τ cannot be expressed as $\tau = c + g - g \circ \bar{T}$ (c a constant, g a function on $\mathcal{R}$).

In [6] (paragraph 13), Chernov introduced a quantitative aspect for the non-integrability of $E_s \oplus E_u$: consider a point y in M. If we take a point $y' \in W_{su}(y)$ and a point $y'' \in W_{ss}(y)$, both close enough to y, then the leaves $W_{ss}(y')$ and $W_{su}(y'')$ will not intersect. Instead, there will be a "temporal distance" $t(y, y', y'') \neq 0$ between the two leaves, such that $\phi_{t(y,y',y'')} W_{su}(y'')$ intersects $W_{ss}(y')$ in exactly one point (see Fig. 1) . The Uniform Non-Integrability hypothesis reads as follows :

(UNI) There exists $c > 0$ and a small open ball $B \subset M$ such that, for all $y \in B$, for all $y' \in W_{su}(y) \cap B, y'' \in W_{ss}(y) \cap B$,

$$c^{-1} d_u(y, y\prime) d_s(y, y'') \leq \mid t(y, y\prime, y'') \mid \leq c \, d_u(y, y\prime) d_s(y, y'')$$

where d_s and d_u are the Riemannian distances on the characteristic foliations.

Since the stable and unstable foliations are invariant under the flow, (UNI) can also be expressed as follows in terms of the roof function τ (see Fig. 2) :

- There exists a rectangle R_{i_0} and an integer n such that $\bar{T}^{-n} R_{i_0}$ meets at least two rectangles R_{j_0} and R_{k_0};

- There exists an open ball in $U(R_{i_0})$ such that, if y, y' belong to this ball and if y_1, y'_1 (resp. y_2, y'_2) are the intersections of $\bar{T}^{-n} W_s^{\mathcal{R}}(y)$ and $\bar{T}^{-n} W_s^{\mathcal{R}}(y')$ with $U(R_{j_0})$ (resp. $U(R_{k_0})$), then

$$c^{-1} d_u(y, y') \leq \mid (\tau_n(y_2) - \tau_n(y_1)) - (\tau_n(y'_2) - \tau_n(y'_1)) \mid \leq c \, d_u(y, y')$$

for a certain constant $c > 0$.

Remark : This assumption is satisfied when dim $M = 3$, the flow is mixing and the characteristic distributions are of class C^1 (Chernov, [6], Theorem 18.1), in particular in the case of the geodesic flow on a surface of variable negative curvature.

Next we wish to make use of these phenomena to study the closed orbit distribution for an Anosov flow.

3 Fourier transforms and zeta-functions

3.1 Fourier analysis and the functions Z and Z'.

Suppose we are given a function $F = (F_1, \cdots, F_d)$ on M taking values in $\mathbb{R}^d$ ($d \geq 0$). We shall work under the following assumption :

(Assumption A) The closed subgroup G of $\mathbb{R}^{d+1}$ generated by the vectors $(l(\gamma), \int_\gamma F) = (\int_\gamma 1, \int_\gamma F)$, ($\gamma$ ranging over the set of closed orbits), has rank $d+1$.

The mixing assumption implies that the projection of G on the first coordinate axis is the whole $\mathbb{R}$. In fact, we will assume more :

(Assumption A') G can be written in the form $G = \mathbb{R} \times A$ in the canonical coordinates, A being a discrete subgroup of rank d in $\mathbb{R}^d$.

The set of characters A^* is compact; this will be crucial in sections 4 and 5 to obtain uniform estimates.

Since we will use the Fourier transform, we need to introduce the Haar measure on G. The space $\mathbb{R}^{d+1}$ is endowed with the usual euclidean structure, for which the canonical basis $(e_0, \cdots, e_d)$ is orthonormal. There exists a basis $(\epsilon_1, \cdots, \epsilon_d)$ of $\mathbb{R}^d$ in which A is defined by $A = \{w_1\epsilon_1 + \cdots + w_d\epsilon_d \ / \ \forall i = 1, \cdots, d, w_i \in \mathbb{Z}\}$. For elements of G we shall try to stick to the letters w_i for coordinates on $(\epsilon_1, \cdots, \epsilon_d)$ and t for the coordinate on e_0. Let $\mathcal{E}$ be the parallelogram built on $(\epsilon_1, \cdots, \epsilon_d)$. G is endowed with the Haar measure

$$d\mu = \frac{1}{(2\pi)^{d+1}} Vol(\mathcal{E}) dt \times \sharp$$

where $\sharp$ is the counting measure on the discrete component of G and $Vol(\mathcal{E})$ is the euclidean volume of $\mathcal{E}$.

The dual $G^* = \mathbb{R} \times A^*$ of G is a quotient of $\mathbb{R}^{d+1}$ by a discrete subgroup of rank d : $\{v \ /; \forall x \in G / < v|x > \in 2\pi\mathbb{Z}\}$. It comes with the quotient metric and the Lebesgue measure $dm(y, v) = Vol(\mathcal{E}) dy dm(v)$ where $dm(v) = dv_1 \cdots dv_d$. Here we identify G^* with a fundamental domain in $\mathbb{R}^{d+1}$ and we let $(y, v_1, \cdots, v_d)$ denote the coordinates on $(e_0, \epsilon_1, \cdots, \epsilon_d)$ of an element of G^*. We have $m(A^*) = \frac{(2\pi)^d}{Vol(\mathcal{E})}$ and $Vol(A^*) = \frac{1}{Vol(\mathcal{E})}$.

For $\xi \in \mathbb{R}^d$ we will denote by $[\xi]$ its "integral part", that is to say, the unique element of A such that $\xi \in [\xi] + \mathcal{E}$.

We want to study the asymptotic behaviour in T of the expressions

$$\pi(\xi,\alpha,\delta;T) = Card\{\gamma \in \Gamma, T \le l(\gamma) \le T+\delta, \int_\gamma F = \alpha + [T\xi]\}$$

and

$$\pi'(\xi,\alpha,\delta;T) = Card\{\gamma \in \Gamma', T \le l(\gamma) \le T+\delta, \int_\gamma F = \alpha + [T\xi]\}$$

for $\alpha \in A$, $\xi \in \mathbb{R}^d$ and an arbitrary positive δ. Γ' is the set of "prime" closed orbits, whereas when we write Γ we mean "non necessarily prime" orbits, i.e more exactly $\Gamma = \mathbb{N}^* \times \Gamma'$ with $l(n,\gamma) = nl(\gamma)$.

We shall first consider

$$\pi(\xi,\psi;T) := \sum_{\gamma\in\Gamma} \psi(l(\gamma) - T, \int_\gamma F - [T\xi])$$

and

$$\pi'(\xi,\psi;T) := \sum_{\gamma\in\Gamma'} \psi(l(\gamma) - T, \int_\gamma F - [T\xi])$$

when ψ is a function on G with compact support and with C^3-regularity in the first variable. Then we will be able to deal with step functions and even with Riemann integrable functions (with compact support).

Recall the formal definition of the Fourier-Laplace transform :

$$\begin{array}{rcl} \hat{\psi} : \mathbb{C} \times (\mathbb{R}^d \oplus iA^*) & \longrightarrow & \mathbb{C} \\ ((x+iy),(u+iv)) & \longmapsto & \displaystyle\int_G \psi(t,w) e^{-(x+iy)t - <u+iv|w>} d\mu(t,w) \end{array}$$

Fourier's inversion formula yields, at least formally, for all $u \in \mathbb{R}^d$ and $x \in \mathbb{R}$,

$$\begin{aligned} \pi(\xi,\psi;T) &= \sum_{\gamma\in\Gamma} \int_{G^*} e^{(x+iy)(l(\gamma)-T) + <u+iv| \int_\gamma F - [T\xi]>} \hat{\psi}(x+iy, u+iv) dm(y,v) \\ &= e^{xT - <u|[T\xi]>} \int_{G^*} Z(x+iy, u+iv) e^{iyT - i<v|[T\xi]>} \hat{\psi}(-(x+iy), u+iv) dm(y,v) \end{aligned}$$

where we have defined

$$Z(s,z) = \sum_{\gamma\in\Gamma} e^{<z| \int_\gamma F> - sl(\gamma)}$$

for $(s,z) \in \mathbb{C} \times (\mathbb{R}^d \oplus iA^*)$. This transformation makes sense when :
(1) ψ satisfies the Fourier inversion theorem. In particular, $\hat{\psi}$ should be

integrable with respect to (y, v).
(2) The sum defining Z is absolutely convergent. This happens when

$$x = \Re s > P(u) = P(\Re z) = \sup_{m \in \mathcal{M}} h_m + < u| \int F dm >$$

(see Proposition 4.2 below).

Similarly, $\pi'(\xi, \psi; T)$ can be studied with a formula involving

$$Z'(s, z) = \sum_{\gamma \in \Gamma'} e^{<z| \int_\gamma F > - sl(\gamma)}$$

instead of Z. For the rest of this paper, u varies in K_0, a compact neighborhood of a point $u^\xi \in \mathbb{R}^d$ which will be set at the beginning of paragraph 5.2.

4 Transfer operators

If we define $\tau_n(\omega) = \tau(\omega) + \tau \circ \bar{T}(\omega) + \cdots + \tau \circ \bar{T}^n(\omega)$ and $\bar{f}_n(\omega) = \bar{f}(\omega) + \bar{f} \circ \bar{T}(\omega) + \cdots + \bar{f} \circ \bar{T}^n(\omega)$ we can now express the function Z as

$$Z(s, z) = \sum_{n=1}^{\infty} \sum_{\omega \in \mathcal{R}, \bar{T}^n \omega = \omega} \frac{1}{\mathcal{N}(\omega)} e^{<z|\bar{f}_n(\omega)> - s\tau_n(\omega)}$$

and

$$Z'(s, z) = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\omega \in \mathcal{R}, \mathcal{N}(\omega) = n} e^{<z|\bar{f}_n(\omega)> - s\tau_n(\omega)}$$

where $\mathcal{N}(\omega)$ is the smallest period of ω under the map $\bar{T}$.

These quantities are classically interpreted as generalized determinants of transfer operators.

In order to introduce transfer operators, we need to consider only the expansive part of $\bar{T}$; that is, "restrict" the maps $\bar{T}$, τ, $\bar{f}$ to the unstable leaves, in a way such that the previous expressions remain unchanged. That is possible, but one has to be careful about the regularity of the functions involved :

4.1 Restriction to the unstable leaves

Amongst the functions defined on $\mathcal{R}$, we are interested in those constant on each stable leaf $W_s^{\mathcal{R}}$, for they can be seen as functions on $\mathcal{U} = \cup U_i$ ($U_i = U(R_i)$). Under the assumptions of Theorem 1.8, recall that the U_i's can be chosen to be diffeomorphic to (connected) segments of dimension 1 ([6], Thm 9.1, [17]).

Let $p : \mathcal{R} \longrightarrow \mathcal{U}$ be the projection along stable leaves. Now $T = p \circ \bar{T}$ is an expansive map from $\mathcal{U}$ into itself.

Because of our construction of rectangles, τ is constant on each stable leaf, so that we can consider it a function on $\mathcal{U}$.

It is possible to replace the function $\bar{f}$ by a function $f = \bar{f} + g \circ \bar{T} - g$ constant on each stable leaf of $\mathcal{R}$, so that the previous expression of Z remains true with f instead of $\bar{f}$ ([19], Proposition 1.2). But, in doing so, one usually loses some regularity. The function $\bar{f}$ (as well as τ), has the same regularity as the stable foliation : it is, in general, piecewise Hlder. But the function f will have a lot of points of discontinuity (the points in $\cup T^{-n}\partial\mathcal{U}$). Anyway, since the integrals of f along closed orbits are always in A, we can use Proposition 5.2 of [19] and assume that f is A-valued; then f is locally constant.

Owing to the expansiveness and to the Markov property, there is a natural one-to-one correspondence between the periodic orbits of $\bar{T}$ and those of T, given by the projection along stable leaves. We can now write :

$$Z(s, z) = \sum_{n=1}^{\infty} \sum_{\omega \in \mathcal{U}, T^n\omega = \omega} \frac{1}{\mathcal{N}(\omega)} e^{<z|f_n(\omega)> - s\tau_n(\omega)}$$

and there is a similar expression for Z'.

4.2 Definition of L and relation with Z

We will determine the domain of Z by studying the norm of the operators

$$\begin{aligned} &L(s, z) : C^1(\mathcal{U}) \longrightarrow C^1(\mathcal{U}) \\ &L(s, z)g(\omega) = \sum_{\omega' \in \mathcal{U}, T\omega' = \omega} e^{<z|f(\omega')> - s\tau(\omega')} g(\omega') \end{aligned}$$

The following lemma, due to Ruelle ([26], p187), gives a relation between the spectral properties of L and the domain of Z. As above u varies in a compact subset $K_0 \subset \mathbb{R}^d$.

Lemma 4.1 *Define $Z_n(s, z) = \sum_{\omega \in U, T^n\omega = \omega} e^{<z|f_n(\omega)> - s\tau_n(\omega)}$. For all $i \in \{1, \cdots, k\}$, choose a point $x_i \in Int(U_i)$. Denote by χ_{U_i} the characteristic function of U_i. Then for all compact subset $K \subset \mathbb{R}$, there exists a $\theta < 1$ such that, for all $s = x + iy$ ($x \in K$), for all $z = u + iv$ ($u \in K_0$), for all n large enough :*

$$\mid Z_n(s, z) - \sum_{i=1}^{k} L(s, z)^n \chi_{U_i}(x_i) \mid \leq (\theta \rho(L(x, u)))^n$$

where $\rho(L(x, u))$ is the spectral radius of $L(x, u)$

4.3 Spectral properties of L and consequences

The function P defined in paragraph 1 describes the spectral properties of L.

Proposition 4.2 *([19]) The function P is analytic and it has the property that, if $x = P(u)$, the operator $L(x, u)$ has 1 as a simple eigenvalue, and all other eigenvalues are < 1 in modulus. If $x > P(u)$ then the spectral radius of $L(s, z)$ ($\Re s = x, \Re z = u$) is strictly less than 1.*

As a result, the function

$$\begin{aligned}\mathcal{Z}(s, z) &:= \sum_{n=1}^{\infty} \frac{1}{n} \sum_{\omega \in U, T^n\omega=\omega} e^{<z|f_n(\omega)>-s\tau_n(\omega)} \\ &= \log \prod_{\gamma \in \Gamma'} (1 - e^{-sl(\gamma)+<z|\int_\gamma F>})^{-1}\end{aligned}$$

which is the logarithm of the generalized determinant of $1 - L(s, z)$, is absolutely convergent when $x > P(u)$ and has a singularity for $x = P(u)$, $y = 0, v = 0$.

This is also true for the functions Z and Z' because of the following lemma :

Lemma 4.3 *There exists $\varepsilon > 0$ such that the functions $\mathcal{Z} - Z$ and $\mathcal{Z} - Z'$ extend to a bounded analytic function on $\{u \in K_0, x \geq P(u) - \varepsilon\}$.*

Proof : Write

$$\mathcal{Z}(s, z) = \sum_{k \geq 1} \frac{1}{k} Z'(ks, kz)$$

and

$$Z(s, z) = \sum_{k \geq 1} Z'(ks, kz)$$

Then use the computation made in [19], pp $115-116$, and the compactness of $K_0 \times A^*$ to get the result.

The definition of the group G and of the dual G^* has the following consequence : suppose that the function $y + v_1 f_1 + \cdots + v_d f_d$ on M is cohomologous to a $2\pi\mathbb{Z}$-valued function. Then $(y, v) = (0, 0)$ in G^*. As a consequence ([19], Chapter 4) :

Proposition 4.4 *The operator $L(P(u) + iy, u + iv)$ does not have 1 as an eigenvalue unless $(y, v) = (0, 0) \in G^*$. Moreover P is strictly convex.*

As a result, $\mathcal{Z}, Z$ and Z' have an analytic continuation on the boundary $\{x = P(u)\}$ except for singularities on the set $\{x = P(u), (y, v) = (0, 0)\}$.

In what follows we assume that the condition (UNI) is satisfied and that the stable and unstable foliations are C^1. Dolgopyat proved the following estimate on the norm of the operator L :

Theorem 4.5 *([8]) Let K_0 be a compact subset of $\mathbb{R}^d$. There exist $B > 0$, $\varepsilon > 0$, $\alpha \in (0,1)$ and some constants B_1, B_2 such that : for every function $q \in C^1(U)$, for every $z = u + iv \in K_0 \times A^*$, for all $s = x + iy$ such that $\mid y \mid > B$ and $\mid x - P(u) \mid \leq \varepsilon$, we have*

$$\| L^{[B_1 \log |y|]}(s,z) q \|_\infty \leq B_2 \| q \|_{C^1} \frac{1}{\mid y \mid^\alpha}$$

Let us recall the general idea of the proof :

Remember the expression of L given at the beginning of paragraph 4.2. For all n, we have :

$$L^n(x+iy, u+iv) g(\omega) = \sum_{\omega' \in \mathcal{U}, T^n \omega' = \omega} e^{<u+iv|f_n(\omega')> - (x+iy)\tau_n(\omega')} g(\omega')$$

This quantity may be large if all the terms in the sum have similar arguments, otherwise it will be "small". Notice that the presence of v is harmless since it varies in a compact set. But we need to have a closer look at the argument $y\tau_n(\omega')$.

The second expression of (UNI) (paragraph 2.3) tells us that we can choose, for many ω's, two particular pre-images ω_1, ω_2 of ω, such that the quantities $\tau_n(\omega_2)$ and $\tau_n(\omega_1)$ are different enough. Hence, if y is large enough, the arguments $y\tau_n(\omega_2)$ and $y\tau_n(\omega_1)$ will be very different and $L^n g(\omega)$ will not be too large. Since it happens for many ω's, the function $L^n(x + iy, u + iv)g$ will be bounded in $\mathcal{L}^1$-norm, with respect to the equilibrium measure of $< u|f > -P(u)\tau$ (and the bigger y, the smaller $|| L^n(x+iy, u+iv)g ||_{\mathcal{L}^1}$).

Then it is still necessary to show that $\mathcal{L}^1$-contraction implies C^1-contraction.

It can be done using the facts that $\mid L^N(x+iy, u+iv)g \mid \leq C L^{N-n}(P(u), u) \mid L^n(x+iy, u+iv)g \mid$ and that $L^{N-n}(x,u) \mid L^n(x+iy, u+iv)g \mid$ is very near $|| L^n(x+iy, u+iv)g ||_{\mathcal{L}^1}$ for large N (it is a property of equilibrium measures).

Remark : We use the compactness of $K_0 \times A^*$ to state this version of Dolgopyat's theorem. The complete proof is exactly the same as in [8]; notice that we don't have to resort to any approximation since f is locally constant.

We can now deduce the following extension of the domain of Z. Using the preceding results and the compactness of A^* and K_0, we get :

Proposition 4.6 *There exist $B > 0$, $\varepsilon > 0$ and an open neighborhood V_0 of 0 in A^* such that :*
(1) $Z(s,z)$ is analytic in $\{u \in K_0, x > P(u) - 2\varepsilon, \mid y \mid > B\}$
(2) The function $u \mapsto P(u)$ can be continued analytically in a complex neighborhood of K_0, yielding a function $z \mapsto P(z)$ such that $Z(s,z) + \log(s - P(z))$

has an analytic continuation in the region $\{u \in K_0, v \in V_0, x > P(u) - 2\varepsilon, \mid y \mid < B\}$ *(we use the usual branch of the logarithm). Moreover we can choose* V_0 *such that* $\Re P(z) > P(u) - \varepsilon/2$ *and* $\mid \Im P(z) \mid < B$ *when* $v \in V_0$*.*
(3) There exists $\varepsilon \geq \varepsilon' > 0$ *such that* $Z(s, z)$ *is analytic in* $\{u \in K_0, v \notin \bar{V}_0, x > P(u) - 2\varepsilon'\}$
These results are also valid for Z'*.*

Proof : It is sufficient to prove the same for the function $\mathcal{Z}$. Singularities of $\mathcal{Z}(s, z)$ appear when $L(s, z)$ has 1 as an eigenvalue.

(a) Dolgopyat's spectral theorem implies that there exists $\varepsilon > 0$ and B such that $L(x + iy, u + iv)$ has spectral radius strictly less than 1 when $u \in K_0$ and $x > P(u) - 2\varepsilon$, $\mid y \mid > B$. Thus Z is analytic in this region.

(b) Consider the function

$$\mathcal{P}(x, u) := \sup_{\nu\ T\text{-invariant}} h_\nu + \int (< u|f > -x\tau) d\nu$$

which is defined on a neighborhood of $(P(u^\xi), u^\xi)$. $\mathcal{P}$ can be continued analytically in a complex neighborhood of $(P(u^\xi), u^\xi)$. Usual properties of pressure functions ([19], Chapter 6) show that $\mathcal{P}(P(u^\xi), u^\xi) = 0$. Remember that $L(s, z)$ is quasi-compact and that its essential spectrum consists of the whole disc $\{\mid w \mid \leq \lambda e^{\mathcal{P}(x,u)}\}$, with $\lambda < 1$ ([19], Chapter 10). For (s, z) in a neighbourhood of $(P(u^\xi), u^\xi)$, $L(s, z)$ has a simple isolated eigenvalue of maximum modulus : $e^{\mathcal{P}(s,z)}$. Babillot and Ledrappier showed in [4] that the function $\mathcal{Z}(s, z) + \log(1 - e^{\mathcal{P}(s,z)})$, defined a priori when $x > P(u)$, extends analytically to a neighborhood of $(P(u^\xi), u^\xi)$. Finally,

$$\frac{\partial \mathcal{P}}{\partial s}(P(u^\xi), u^\xi) = -\int \tau d\nu_{<u^\xi|f>-P(u^\xi)\tau} \neq 0$$

where $\nu_{<u^\xi|f>-P(u^\xi)\tau}$ is the Gibbs state of the potential $< u^\xi|f > -P(u^\xi)\tau$. Weierstrass' theorem ([11], Th. 7.5.1) allows us to write

$$1 - e^{\mathcal{P}(s,z)} = c(s, z)(s - P(z))$$

where c is an analytic function such that $c(P(u^\xi), u^\xi) \neq 0$ and P is an analytic continuation of the real P function to a neighborhood of u^ξ. This yields (2).

(c) Finally, (3) can be derived from (1) combined to Proposition 4.4.

We conclude this section by showing an upper bound on the function $\mathcal{Z}$. As usual, we consider $s = x + iy$ and $z = u + iv$. If x is close to $P(u)$ then $\rho(L(x, u))$ is close to 1 so that $\theta\rho(L(x, u)) < \theta' < 1$ in Lemma 4.1.

$$
\begin{aligned}
| \mathcal{Z}(s,z) | &\leq \sum_{n=1}^{+\infty} \frac{1}{n} \mid Z_n(s,z) \mid \\
&\leq \sum_n \frac{1}{n} \left(\sum_i \mid L^n(s,z)\chi_{U_i}(x_i) \mid \right) + \sum_n \theta'^n \\
&\leq \sum_k \frac{1}{kB_1 \log \mid y \mid} \sum_{n=k[B_1 \log|y|]}^{(k+1)[B_1 \log|y|]-1} \| L^{n-k[B_1 \log|y|]} \| \| L^{k[B_1 \log|y|]} \chi_{U_i} \| + C \\
&\leq \frac{C}{\log \mid y \mid} \sum_{k=0}^{+\infty} \frac{1}{k \mid y \mid^{k\alpha}} \left(\sum_{l=0}^{[B_1 \log|y|]-1} e^{l\mathcal{P}(x,u)} \right) + C \\
&\leq C \max\Big(\log \mid y \mid, \mid y \mid^{B_1 \mathcal{P}(x,u)} \Big) + C
\end{aligned}
$$

The second inequality comes from Lemma 4.1 and the fourth one from Theorem 4.5.

The same upper bound is valid for Z and Z'.

5 Asymptotic expansion

5.1 Integration with respect to s : the Cauchy formula.

We want an asymptotic expansion of $e^{-TP(u)+<u|[T\xi]>}\pi(\xi,\psi;T)$ into powers of $T^{-1/2}$. In what follows, the letters C_i, $i=1,2,\cdots$ will denote constants which are not only uniform in z but also independent of ψ. They can depend on N (the order of the expansion). They can be chosen to depend continuously on ξ; this motivates Remark (2) following Theorem 1.8.

We take ψ of class C^3 in the first variable, with compact support, and $x > P(u)$, so that we may write :

$$
\pi(\xi,\psi;T) = \int_{\mathbb{R}\times A^*} Z(x+iy,u+iv) e^{(x+iy)T-<u+iv|[T\xi]>} \hat{\psi}(-(x+iy),u+iv)dydm(v)
$$

Then, using Fubini's theorem,

$$
\begin{aligned}
\pi(\xi,\psi;T) = &\int_{V_0} dm(v) e^{-<u+iv|[T\xi]>} \left(\int_{\mathbb{R}} Z(x+iy,u+iv) e^{(x+iy)T} \hat{\psi}(-(x+iy),u+iv)dy \right) \\
&+ \int_{A^*-V_0} dm(v) e^{-<u+iv|[T\xi]>} \left(\int_{\mathbb{R}} Z(x+iy,u+iv) e^{(x+iy)T} \hat{\psi}(-(x+iy),u+iv)dy \right)
\end{aligned}
$$

When $v \notin V_0$, Cauchy's theorem and our upper bounds of (4.2.2) yield

$$
\begin{aligned}
&\mid \int_{\mathbb{R}} Z(x+iy, u+iv) e^{(x+iy)T} \hat{\psi}(-(x+iy), u+iv) dy \mid \\
&= \mid \int_{\mathbb{R}} Z(P(u)-\varepsilon+iy, u+iv) e^{(P(u)-\varepsilon'+iy)T} \hat{\psi}(-(P(u)-\varepsilon'+iy), u+iv) dy \mid \\
&\leq C_1 \mid\mid \partial_t^3 \psi \mid\mid_{L^1} \, e^{T(P(u)-\varepsilon')}
\end{aligned}
$$

Here we choose ε such that $B_1 \mathcal{P}(P(u)-\varepsilon', u) < 1$ and we use the fact that $\mid \hat{\psi}(-(P(u)-\varepsilon'+iy), u+iv) \mid \leq C_2 \frac{||\partial_t^3 \psi||_{L^1}}{|y|^3}$ for $u \in K_0$, $v \in A^*$.

When $v \in V_0$, one has, using Cauchy's formula and Proposition 4.6 (2), (and always denoting $z = u+iv$)

$$
\begin{aligned}
&\int_{\mathbb{R}} Z(x+iy, u+iv) e^{(x+iy)T} \hat{\psi}(-(x+iy), u+iv) dy \\
&= i \int_{\mathcal{C}(x)} \log(s-P(z)) \hat{\psi}(-s, z) e^{sT} ds \\
&- i \int_{\{\Re s = P(u)-\varepsilon, |\Im s| > 2B\}} Z(s,z) e^{sT} \hat{\psi}(-s,z) ds \\
&- i \int_{\{\Re s = P(u)-\varepsilon, |\Im s| < 2B\}} \left(Z(s,z) + \log(s-P(z))\right) e^{sT} \hat{\psi}(-s,z) ds
\end{aligned}
$$

In the right hand side term we use as the path of integration

$$
\begin{aligned}
\mathcal{C}(x) = \quad & \{\Im s = -2B, P(u)-\varepsilon \leq \Re s \leq x\} \\
\cup \quad & \{\Re s = x, -2B \leq \Im s \leq 2B\} \\
\cup \quad & \{\Im s = 2B, x \geq \Re s \geq P(u)-\varepsilon\}
\end{aligned}
$$

oriented counterclockwise. We obtain the following inequality :

$$
\begin{aligned}
&\mid \int_{\mathbb{R}} Z(x+iy, u+iv) e^{(x+iy)T} \hat{\psi}(-(x+iy), u+iv) dy - i \int_{\mathcal{C}(x)} \log(s-P(z)) \hat{\psi}(-s,z) e^{sT} ds \mid \\
&\leq C_3 \mid\mid \partial_t^3 \psi \mid\mid_{L^1} \, e^{T(P(u)-\varepsilon)}
\end{aligned}
$$

After integration by parts with respect to y, we get

$$
\begin{aligned}
\mid i \int_{\mathcal{C}(x)} \log(s-P(z)) \hat{\psi}(s,z) e^{sT} ds \quad & + \quad \frac{i}{T} \int_{\mathcal{C}(x)} \frac{\hat{\psi}(-s,z)}{s-P(z)} e^{sT} ds \\
& - \quad \frac{i}{T} \int_{\mathcal{C}(x)} \log(s-P(z)) \frac{\partial \hat{\psi}}{\partial s}(-s,z) e^{sT} ds \mid \\
& \leq \quad C_4 \mid\mid \psi \mid\mid_{L^1} \, e^{T(P(u)-\varepsilon)}
\end{aligned}
$$

The residue formula implies that

$$| \int_{\mathcal{C}(x)} \frac{\hat{\psi}(s,z)}{s-P(z)} e^{sT} ds - 2i\pi\hat{\psi}(-P(z),z)e^{P(z)T} | \leq C_5 \parallel \psi \parallel_{L^1} e^{T(P(u)-\varepsilon)}$$

Let M be an integer which we will fix in Section 5.2. We iterate the same operation $M+1$ times :

$$\begin{aligned}
&| \int_{\mathcal{C}(x)} \log(s-P(z))\hat{\psi}(s,z)e^{sT} ds - 2\pi \sum_{k=0}^{M} \frac{1}{T^{k+1}} \frac{\partial^k \hat{\psi}}{\partial s^k}(-P(z),z)e^{P(z)T} \\
&- \frac{i}{T^{M+1}} \int_{\mathcal{C}(x)} \log(s-P(z)) \frac{\partial^{M+1}\hat{\psi}}{\partial s^{M+1}}(s,Z)e^{sT} ds \,| \\
&\leq C_6 \left(|| \, \psi \, ||_{L^1} + \cdots + || \, t^M \psi \, ||_{L^1} \right) e^{T(P(u)-\varepsilon)}
\end{aligned}$$

Finally, we obtain that, when ψ is a function with compact support and of class C^3 in the first variable, one has for all $u \in K_0$ and for all $x > P(u)$,

$$\begin{aligned}
| \, \pi(\xi,\psi;T) \quad &- \quad 2\pi \sum_{k=0}^{M} \frac{1}{T^{k+1}} \int_{V_0} dm(v) e^{-<u+iv|[T\xi]>} \frac{\partial^k \hat{\psi}}{\partial s^k}(-P(z),z)e^{P(z)T} \\
&- \quad \frac{i}{T^{M+1}} \int_{V_0} \int_{\mathcal{C}(x)} \log(s-P(z)) \frac{\partial^{M+1}\hat{H}}{\partial s^{M+1}}(s,z)e^{sT-<z|[T\xi]>} ds dm(v) \,| \\
&\leq C_7 \left(|| \, \partial_t^3 \psi \, ||_{L^1} + || \, \psi \, ||_{L^1} + \cdots + || \, t^M \psi \, ||_{L^1} \right) e^{T(P(u)-\varepsilon)}
\end{aligned}$$

Then, by the dominated convergence theorem, one sees that, uniformly in $u \in K_0$,

$$\begin{aligned}
&\lim_{x \to P(u)} \int_{V_0} \int_{\mathcal{C}(x)} \log(s-P(z)) \frac{\partial^{M+1}\hat{\psi}}{\partial s^{M+1}}(s,z)e^{sT-<z|[T\xi]>} ds dm(v) \\
&= \int_{V_0} \int_{\mathcal{C}(P(u))} \log(s-P(z)) \frac{\partial^{M+1}\hat{\psi}}{\partial s^{M+1}}(s,z)e^{sT-<z|[T\xi]>} ds dm(v)
\end{aligned}$$

so that we ultimately proved the following lemma :

Lemma 5.1 *When ψ is of the class C^3 in the first variable t, with compact support,*

$$\begin{aligned}
| \, \pi(\xi,\psi;T) \quad &-2\pi \sum_{k=0}^{M} \frac{1}{T^{k+1}} \int_{\Re z=u, v=\Im z \in V_0} dm(v) e^{-<z|[T\xi]>} \frac{\partial^k \hat{\psi}}{\partial s^k}(-P(z),z)e^{P(z)T} \, | \\
&\leq C_8 \frac{|| \, t^{M+1}\psi \, ||_{L^1}}{T^{M+1}} e^{TP(u)-<u|[T\xi]>} \\
&+ C_9 \left(|| \, \partial_t^3 \psi \, ||_{L^1} + || \, \psi \, ||_{L^1} + \cdots + || \, t^M \psi \, ||_{L^1} \right) e^{T(P(u)-\varepsilon)-<u|[T\xi]>}
\end{aligned}$$

5.2 Integration with respect to z : the saddle-point method.

In this last section we want to use the saddle-point method to expand the term

$$\int_{V_0} \frac{\partial^k \hat{\psi}}{\partial s^k}(-P(u+iv), u+iv) e^{TP(u+iv)-<u+iv|[T\xi]>} dm(v)$$

The saddle-point method ([7]) tells us to choose u such that $\nabla P(u) = [T\xi]/T$.

Proposition 5.2 *∇P is a homeomorphism from $\mathbb{R}^d$ onto the interior of the convex compact set $C = \{\int F d\mu, \mu \in \mathcal{M}\}$.*

Proof : As explained in [4], it is a consequence of Th. 26.5 of [25], since P is analytic and a has positive definite Hessian matrix.

Consequently we take $\xi \in \overset{\circ}{C}$.

In that case, when T is large enough, we can define $\xi^T = \frac{[T\xi]}{T} \in \overset{\circ}{C}$ and $u = u^T = (\nabla P)^{-1}(\xi^T)$. We now choose for K_0 a compact neighborhood of $u^\xi = (\nabla P)^{-1}(\xi)$.

Define $H(\xi) = \inf_{u \in \mathbb{R}^d} P(u) - < u|\xi >$, for $\xi \in \mathbb{R}^d$. One also has $H(\xi) = \sup\{h(m), m \in \mathcal{M}, \int F dm = \xi\}$.

Theorem 5.3 *Assume the characteristic foliations are of class C^1 and uniformly jointly non-integrable. Then, when ψ is a Riemann-integrable function with compact support on G, $\pi(\xi, \psi; T)$ has the following asymptotic expansion up to any order N :*

$$\pi(\xi, \psi; T) = \frac{e^{TH(\xi^T)}}{T^{d/2+1}} \left(\frac{(2\pi)^{1+d/2}}{\sqrt{| \, det P''(u^T) \, |}} \hat{\psi}(-P(u^T), u^T) + \sum_{k=1}^{N} \frac{c_k(u^T, \alpha)}{T^k} + O(T^{-(N+1)}) \right)$$

The c_k's are analytic functions on $\mathbb{R}^d \times A$, polynomial with respect to α; their expressions involve the functions P, H and ψ as well as their derivatives and their Fourier transforms.

Notice that $\xi^T \longrightarrow \xi$ and $u^T \longrightarrow u^\xi$ as $T \longrightarrow +\infty$.

Proof : (a) Let us first consider the case when ψ is of class C^3 in the first variable. Denoting by $\bar{\psi}_j(u+iv)$ the function $\partial_j \hat{\psi}(-P(u+iv), u+iv)$, one can write, for $M > N$ and $1 \leq j \leq M$,

$$\int_{||v|| \leq \rho} \bar{\psi}_j(u^T + iv) e^{TP(u^T+iv)-<u^T+iv|[T\xi]>} dm(v)$$

$$= \frac{e^{-<u^T|[T\xi]>}}{T^{d/2}} \int_{||v|| \leq \sqrt{T}\rho} \bar{\psi}_j(u^T + i\frac{v}{\sqrt{T}}) e^{TP(u^T+i\frac{v}{\sqrt{T}})-<u^T+i\frac{v}{\sqrt{T}}|[T\xi]>} dm(v)$$

$$= \frac{e^{TP(u^T)-<u^T|[T\xi]>}}{T^{d/2}} \int_{||v|| \leq \sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)+TR_{2N+3}(u^T, \frac{iv}{\sqrt{T}})} \bar{\psi}_j(u^T + i\frac{v}{\sqrt{T}}) dm(v)$$

with

$$P(u^T+i\frac{v}{\sqrt{T}}) = P(u^T)+i < \nabla P(u^T)|\frac{v}{\sqrt{T}} > -\frac{1}{2T}P''(u^T)(v,v)+R_{2N+3}(u^T,\frac{iv}{\sqrt{T}})$$

Here $R_{2N+3}(u^T,\frac{iv}{\sqrt{T}})$ is given by the Taylor formula :

$$R_n(u,v) = \frac{1}{3!}P^{(3)}(u).v^3 + \cdots + \frac{1}{n!}P^{(n)}(u).v^n + \int_0^1 \frac{(1-t)^n}{n!}P^{(n+1)}(u+tv).v^{n+1}dt$$

so that

$$\begin{aligned} TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) &= \frac{i^3}{6T^{1/2}}P^{(3)}(u^T).v^3 + \cdots + \frac{i^{2N+3}}{(2N+3)!T^{N+1/2}}P^{(2N+3)}(u^T).v^{2N+3} \\ &+\frac{i^{2N+4}}{T^{N+1}}\int_0^1 \frac{(1-t)^{2N+3}}{(2N+3)!}P^{(2N+4)}(u^T+it\frac{v}{\sqrt{T}}).v^{2N+4}dt \end{aligned}$$

Next we introduce the polynomials

$$Q_{2N+3}(u^T).(iv,T^{-1/2}) = \frac{i^3}{6T^{1/2}}P^{(3)}(u^T).v^3+\cdots+\frac{i^{2N+3}}{(2N+3)!T^{N+1/2}}P^{(2N+3)}(u^T).v^{2N+3}$$

Taylor's formula shows the existence of a constant K, dependent of N but independent of ρ, such that :
(1)$| TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) |\leq KT^{-1/2} \| v \|^3$
(2)$| TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) |\leq K\rho \| v \|^2$
(3)$\| TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) - Q_{2N+3}(u^T).(iv,T^{-1/2}) \|\leq K\frac{\|v\|^{2N+4}}{T^{N+1}}$
(4)$\| TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) - Q_{2N+3}(u^T).(iv,T^{-1/2}) \|\leq K\rho^{2N+2} \| v \|^2$.
if $\| v \|\leq \sqrt{T}\rho$

Then

$$\begin{aligned} \left| e^{TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}})} - \sum_{k=0}^{2N+1}\frac{(TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}))^k}{k!} \right| &\leq e^{|TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}})|}\frac{| TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}) |^{2N+2}}{(2N+2)!} \\ &\leq e^{K\rho\|v\|^2}T^{-(N+1)} \| v \|^{6N+6} \frac{K^{2N+2}}{(2N+2)!} \end{aligned}$$

so that

$$| \int_{\|v\|\leq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)} \left(e^{TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}})} - \sum_{k=0}^{2N+1} \frac{(TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}))^k}{k!} \right) \bar{\psi}_j(u^T + i\frac{v}{\sqrt{T}})dm(v) |$$

$$\leq \| t^j\psi \|_{L^1} \frac{K^{2N+2}}{(2N+2)!} T^{-(N+1)} \int_{||v||\leq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)} e^{K\rho\|v\|^2} \| v \|^{6N+6} dm(v)$$

$$\leq C_{10} \, || \, t^j\psi \, ||_{L^1} \, T^{-(N+1)}$$

provided ρ is small enough for the integral

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}P''(u^T)(v,v)} e^{K\rho||v||^2} \, || \, v \, ||^{6N+6} \, dm(v)$$

to be finite.

Developing the expression $\sum_{k=0}^{2N+1} \frac{(TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}))^k}{k!}$ yields

$$\sum_{k=0}^{2N+1} \frac{(TR_{2N+3}(u^T,\frac{iv}{\sqrt{T}}))^k}{k!} = \sum_{k=0}^{2N+1} T^{-k/2}W_k(u^T).iv + Y_N(u^T,v,T^{-1/2})$$

where $W_k(u^T).iv$ is a polynomial function of iv depending analytically on u^T, and $|| \, Y_N(u^T,v,T^{-1/2}) \, || \leq \frac{||v||^q}{T^{N+1}}$ for some integer q (depending on N).

One has $TP(u^T) - < u^T|[T\xi] >= TH(\xi^T)$, since $P = (-H)^*$. Thus we get

$$| \int_{||v||\leq\rho} \bar{\psi}_j(u^T + iv) e^{TP(u^T+iv) - <u^T+iv|[T\xi]>} dm(v)$$

$$e^{TH(\xi^T)} \int_{||v||\leq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)} (\sum_{k=0}^{2N+1} T^{-k/2}W_k(u^T).iv) \bar{\psi}_j(u^T + i\frac{v}{\sqrt{T}})dm(v) |$$

$$\leq C_{11} e^{TH(\xi^T)} \, || \, t^j\psi \, ||_{L^1} \, T^{-(N+1)}$$

and if we expand $\bar{\psi}_j$ in a neighborhood of u^T,

$$| \int_{||v||\leq\rho} \bar{\psi}_j(u^T + iv) e^{TP(u^T+iv) - <u^T+iv|[T\xi]>} dm(v)$$

$$-e^{TH(\xi^T)} \int_{||v||\leq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)} (\sum_{k=0}^{2N+1} T^{-k/2}W_k(u^T).iv)(\sum_{l=0}^{2N+1} i^l T^{-l/2} \frac{\bar{\psi}_j^{(l)}(u^T).v^l}{l!})dm(v) |$$

$$\leq C_{12} \, e^{TH(\xi^T)} \sup_{n_1+|n_2|\leq 2N+2} || \, t^{j+n_1} w^{n_2} \psi \, ||_{L^1} \, T^{-(N+1)}$$

To obtain (1), all we have to do is notice that

$$\int_{\|v\|\geq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)}|W_k(u^T).iv|.|\bar{\psi_j}^{(l)}(u^T).v^l|dm(v)$$
$$\leq C_{13} \sup_{n_1+|n_2|\leq l+1} \| t^{j+n_1}w^{n_2}\psi \|_{L^1} \int_{\|v\|\geq\sqrt{T}\rho} e^{-\frac{1}{2}P''(u^T)(v,v)} \| v \|^{\alpha_{k,l}} dm(v)$$
$$\leq C_{14} \sup_{n_1+|n_2|\leq 2N+2} \| t^{j+n_1}w^{n_2}\psi \|_{L^1} e^{-C_{15}T}$$

Finally,

$$| \int_{||v||\leq\rho} \bar{\psi}_j(u^T+iv)e^{TP(u^T+iv)-<u^T+iv|[T\xi]>}dm(v)$$
$$-\frac{e^{TH(\xi^T)}}{T^{d/2}}\int_{v\in\mathbb{R}^d} e^{-\frac{1}{2}P''(u^T)(v,v)}(\sum_{k=0}^{2N+1}T^{-k/2}W_k(u^T).iv)(\sum_{l=0}^{2N+1}i^lT^{-l/2}\frac{\bar{\psi}^{(l)}(u^T).v^l}{l!})dm(v) |$$
$$\leq C_{16}\, e^{TH(\xi^T)} \sup_{n_1+|n_2|\leq 2N+2} || t^{j+n_1}w^{n_2}\psi ||_{L^1} T^{-(N+1)}$$

If we compare to lemma 4.4, it turns out that the remaining terms are negligible in front of the latter expression. Gather all terms of same order; a more careful computation shows that the coefficients of odd powers of $T^{-1/2}$ vanish, because they are expressed as the integrals of odd functions over $\mathbb{R}^d$. (1) is now proved. More precisely, the term $O(T^{-(N+1)})$ can be bounded by

$$C_{17}\left(\sum_{n_1+|n_2|\leq 4N} || t^{n_1}w^{n_2}\psi ||_{L^1}\right)T^{-(N+1)} + C_{18} || \partial_t^3\psi ||_{L^1} e^{-\varepsilon T}$$

We will need this now to resort to an approximation argument in the next paragraph.

Notation : We will write $\theta_N(\psi)$ for $\sum_{n_1+n_2\leq 4N} || t^{n_1}w^{n_2}\psi ||_{L^1}$.

(b) We now come to the case of a function ψ which is piecewise constant. For all T take ψ_T^- and ψ_T^+ of class C^3, with compact supports, such that :
(1)$\psi_T^- \leq Q \leq \psi_T^+$
(2)$\theta_N(\psi_T^+ - \psi_T^-) \leq e^{-\beta T}\theta_N(\psi)$
(3)$|| \partial_t^3\psi_T^* ||_{L^1} \leq e^{3\beta T} || \psi ||_{L^1}$
(4)$|| \psi_T^* ||_\infty \leq 2 || \psi ||_\infty$ This can be done by a convolution argument with respect to t.

Then write

$$\pi(\xi,\psi;T) \leq \pi(\xi,\psi_T^+,T)$$

so that

$$
\begin{aligned}
\pi(\xi,\psi;T) \quad - \quad & 2\pi\sum_{k=0}^{N}\frac{1}{T^{k+1}}\int_{V_0} dm(v)e^{-<u+iv|[T\xi]>}\frac{\partial^k\hat{\psi_T^+}}{\partial s^k}(-P(z),z)e^{P(z)T} \\
& \leq C_8\frac{||\ t^{N+1}\psi_T^+\ ||_{L^1}}{T^{N+1}}e^{TH(\xi^T)} \\
& +C_9\left(||\ \partial_t^3\psi_T^+\ ||_{L^1}+||\ \psi_T^+\ ||_{L^1}+\cdots+||\ t^N\psi_T^+\ ||_{L^1}\right)e^{TH(\xi^T)}
\end{aligned}
$$

The same technique as in (a) can be used to show that

$$
\begin{aligned}
& |\int_{V_0} dm(v)e^{-<u+iv|[T\xi]>}\left(\frac{\partial^k\hat{\psi_T^+}}{\partial s^k}-\frac{\partial^k\hat{\psi}}{\partial s^k}\right)(-P(z),z)e^{P(z)T}\ | \\
& \qquad \sim C_{19}\frac{e^{TH(\xi^T)}}{T^{d/2+1}}\ |\left(\frac{\partial^k\hat{\psi_T^+}}{\partial s^k}-\frac{\partial^k\hat{\psi}}{\partial s^k}\right)(-P(u^T),u^T)\ | \\
& \qquad \leq C_{20}\theta_N(\psi)\frac{e^{TH(\xi^T)-\beta T}}{T^{d/2+1}}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\pi(\xi,\psi;T) \quad - \quad & 2\pi\sum_{k=0}^{N}\frac{1}{T^{k+1}}\int_{V_0} dm(v)e^{-<u+iv|[T\xi]>}\frac{\partial^k\hat{\psi}}{\partial s^k}(-P(z),z)e^{P(z)T} \\
& \leq C_{20}\theta_N(\psi)\frac{e^{TH(\xi^T)-\beta T}}{T^{d/2+1}} \\
& +C_8\frac{\theta_N(\psi)}{T^{N+1}}e^{TP(u)-<u|T\xi>} \\
& +C_9\ ||\ \psi\ ||_{L^1}\ e^{T(P(u)-\varepsilon+3\beta)-<u|[T\xi]>}
\end{aligned}
$$

Doing the same with ψ_T^-, we obtain that

$$
\begin{aligned}
|\ \pi(\xi,\psi;T) \quad - \quad & 2\pi\sum_{k=0}^{N}\frac{1}{T^{k+1}}\int_{V_0} dm(v)e^{-<u+iv|[T\xi]>}\frac{\partial^k\hat{\psi}}{\partial s^k}(-P(z),z)e^{P(z)T}\ | \\
& \leq C_{20}\theta_N(\psi)\frac{e^{TH(\xi^T)-\beta T}}{T^{d/2+1}} \\
& +C_8\frac{\theta_N(\psi)}{T^{N+1}}e^{TP(u)-<u|[T\xi>} \\
& +C_9\ ||\ \psi\ ||_{L^1}\ e^{T(P(u)-\varepsilon+3\beta)-<u|[T\xi]>}
\end{aligned}
$$

Then choose $\beta<\varepsilon/3$ and obtain the same expansion as in (a).

(c) The result for Riemann-integrable functions is obtained the same way.

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