COUNTING GEODESICS WHICH ARE OPTIMAL IN HOMOLOGY

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The profusion of recent papers investigating the properties of optimal orbits of dynamical systems, generalizing or contradicting some results specific to Lagrangian systems, leaves no doubt as to the interest of such a study in a general setting. However, it seems that a unified motivation remains to be found.

Optimal orbits and optimal measures appear most naturally in the Aubry-Mather theory for Lagrangian systems : they are orbits, or measures, minimizing globally the integral of the lagrangian. In this context they are classically called action-minimizing orbits (and measures). More generally, given a dynamical system and a potential f on the phase space, one may ask for a description of the orbits, or of the invariant measures, minimizing the integral of f. Such a question appears for instance in [YH] (from where we took the expression "optimal orbits") where it is linked to the question of "controlling chaos"; in [CLT], as a variant of Mañe's work on Lagrangian systems. In [J] and in [Bo] specific examples are treated for their intrinsic interests.

In the papers mentioned above, the dynamical systems under study are expanding or hyperbolic. It is a situation where the behaviour of arbitrary orbits is very well known : the system is expansive, satisfies specification – in fact, lots of properties which allow to manipulate the orbits as if they were trajectories of a random walk. A good measure theoretical approach of such systems is provided by thermodynamical formalism, which gives a description of the distribution of arbitrary orbits in terms of Gibbs measures. An interesting observation is that optimal measures can be obtained as limits of Gibbs measures when a certain parameter (playing the role of the inverse of the temperature) tends to infinity. Therefore one can ask for analogs to some results about arbitrary orbits : counting results for closed optimal orbits, spatial distribution ? The paper [YH] deals for instance with the existence of closed orbits sufficiently close to optimal. However, it seems very difficult to obtain more refined results in this direction. As will appear later, properties of optimal orbits depend a lot on the geometric features of the system; optimal orbits can behave chaotically for some systems and in an almost-periodic way for other systems (as is most often the case).

Here we study the situation of the geodesic flow on a surface of negative curvature. We are interested in closed geodesics which are optimal, or close to optimal, in homology : that is to say, they minimize the length in their homology classes. The aim is to obtain some counting results as well as to raise some questions about the transition from the "chaotic behaviour" of arbitrary orbits (described by Gibbs measures) to the almost-periodic behaviour of optimal orbits.

The starting point of this work was a paper by Lalley ([Lal2]) in which he describes closed orbits of Anosov flows, with prescribed Birkhoff average of a given

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potential f. For non-optimal orbits (for f) he can prove some results which stand in the continuity of the famous result of Margulis ([M]) : exponential multiplicative asymptotic for the number of orbits, and equidistribution with respect to a Gibbs measure. His result applies in particular when we want, as in the present paper, to count closed geodesics of a negatively curved surface with a prescribed *average winding cycle*. The proof relies on coding the flow, then on the spectral analysis of a Markovian transition operator. However, this technique fails if we want to count *optimal* orbits – here this means that the winding cycles of the geodesics have stable norms close to 1. As mentioned before there is no solution to such questions in a general abstract setting ; results seem to depend on the specific topological properties of the manifold as well as of the flow. Here we will use very strongly the topological properties of optimal geodesics of surfaces, in particular their almost-periodic behaviour. We will show that in certain optimal homological directions (the rational ones), the growth of the number of closed geodesics is at most polynomial (Theorem 1.1.3); we will also show that in other directions the growth is still subexponential, but possibly no longer polynomial (Theorem 1.1.5).

Our counting technique is very different from that used in the non-optimal case. However in the course of our investigation we will observe some interesting similarities. The Lagrangian variational principle can be obtained as a limit of the thermodynamic variational principle, when temperature tends to zero. Instead of characterizing equilibrium measures by the Gibbs property, one can characterize minimizing measures by their support : the Mather set. We will also see the role played by a semi-group, the Lax-Oleinik semi-group (Section 4). This semi-group is not Markovian, in fact non-linear; it is the semi-group giving the viscosity solutions of the Hamilton-Jacobi equation. The convergence in infinite time of the Lax-Oleinik semi-group will be used to describe the topological distribution of optimal orbits, exactly as the convergence of Markovian operators influences the measure-theoretical distribution of arbitrary orbits. We will see that optimal orbits will spend most time close to the so-called Aubry set; the latter is the set of accumulation points of closed quasi-minimizing geodesics, and admits a nice description in terms of the fixed points of the Lax-Oleinik semi-groups. On the coincidence, or not, of Aubry set with Mather set will depend some features of the growth of optimal orbits.

To conclude, we should note that in this particular setting, the question can by raised one level higher if we remember that geodesics can be regarded as limits of brownian trajectories when some normalisation parameters tend to infinity. For such a system, one can consider three levels of optimization (or, on the contrary, of randomness) : brownian motion is the case of completely free trajectories on the surface; geodesics are trajectories which minimize the action locally; "optimal" geodesics minimize the action globally. The relation between brownian motion and globally minimizing trajectories will be partially dealt with in an independent appendix; there we recall in particular how to obtain Lax-Oleinik semi-groups from twisted heat semi-groups by the method of vanishing viscosity. This is striking here, because twisted heat semi-groups are usually used to count closed geodesics with homological constraints in constant negative curvature.

1. Growth rates for quasi-minimizing closed geodesics

Let V be a compact orientable Riemannian manifold endowed with a metric of negative sectional curvature. Let $H_1(V,\mathbb{R})$ denote its first homology space with real coefficients, and $H_1(V,\mathbb{Z})$ its first homology group. The space $H_1(V,\mathbb{R})$ is a finite dimensional real vector space, and $H_1(V,\mathbb{Z})$ can be seen as the lattice of integer points in $H_1(V,\mathbb{R})$. We fix once and for all an "integral part" mapping $[\ .\ ]$ from $H_1(V,\mathbb{R})$ to $H_1(V,\mathbb{Z})$. More explicitly, we choose an arbitrary fundamental domain $\mathcal{E}$ of the lattice $H_1(V,\mathbb{Z}) \subset H_1(V,\mathbb{R})$, and we define $[\xi]$ as the unique element of $H_1(V,\mathbb{Z})$ such that $\xi \in \mathcal{E} + [\xi]$.

We will denote Γ the set of oriented closed geodesics γ of V – we recall that it is in one-to-one correspondence with $\pi_1(V)$, the set of free homotopy classes of loops in V. There is a surjective map

$$\begin{array}{ccc} \Gamma & \longrightarrow & H_1(V,\mathbb{Z}) \\ \gamma & \mapsto & [\gamma] \end{array}$$

which assigns to a closed geodesic γ its homology class $[\gamma]$ (we use the same notation $[.]$ for different objects, hoping it will not cause too much confusion).

Given $\alpha \in H_1(V,\mathbb{Z})$, $\xi \in H_1(V,\mathbb{R})$ and $\delta > 0$, Lalley described the asymptotic behaviour, for $T \longrightarrow \infty$, of the set

$$\Gamma(\xi,\alpha,\delta;T) = \{\gamma \in \Gamma, T-\delta \leq l(\gamma) \leq T, [\gamma] = \alpha + [T\xi]\}$$

where $l(\gamma)$ denotes the length of $\gamma \in \Gamma$. Under some assumptions on the vector ξ, he found the asymptotic behaviour of the quantity

$$\pi(\xi,\alpha,\delta;T) = Card\,\Gamma(\xi,\alpha,\delta;T)$$

as well as obtained an asymptotic equidistribution theorem for the geodesics in $\Gamma(\xi,\alpha,\delta;T)$.

These geodesics have *average winding cycle* (or *average homology cycle*) $\frac{[\gamma]}{l(\gamma)}$ fixed and equal to ξ, up to a bounded error – it is necessary to allow this error because $[\gamma]$ belongs to $H_1(V,\mathbb{Z})$ whereas $l(\gamma)\xi$ may not. The vector ξ can be thought of as of a constraint exerted on the geodesics, which forces them to align themselves in a certain direction in the homology space.

Such questions are usually treated by identifying the set Γ of oriented closed geodesics with the set of closed orbits of the geodesic flow, thus bringing counting problems into the field of dynamical systems. The geodesic flow (ϕ_t) acts on the unit tangent bundle SV of the manifold, and it has the Anosov property when V has negative curvature. Lalley used the thermodynamical formalism for such flows, and introduced the functions H (*contracted entropy*) and P (*contracted pressure*) on the first homology and cohomology spaces. Let us denote $\mathcal{M}_1(SV)$ the set of (ϕ_t)-invariant probability measures on SV, endowed with the weak topology. The function P is defined by :

$$\begin{array}{rcl} P : H^1(V,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ [\omega] & \longmapsto & \sup\limits_{\nu \in \mathcal{M}_1(SV)} h(\nu) + \langle [\omega],[\nu]\rangle \end{array}$$

In this definition, $h(\nu)$ is the metric entropy of $\nu \in \mathcal{M}_1(SV)$, and $[\nu] \in H_1(V,\mathbb{R})$ is the asymptotic cycle of ν (Schwartzman [Sc]), defined by duality by the equality :

$$\langle [\omega],[\nu]\rangle = \int_{SV} \langle \omega, X\rangle d\nu(X)$$

when ω is a closed 1-form and $[\omega]$ is its cohomology class (the same definition would make sense for any positive (ϕ_t)-invariant measure with compact support on the tangent bundle TV).

The entropy on $H_1(V,\mathbb{R})$, defined as follows, is the Legendre transform of P :

$$\begin{array}{rcl} H : H_1(V,\mathbb{R}) & \longrightarrow & \mathbb{R} \\ \xi & \longmapsto & \sup_{\nu\in\mathcal{M}_1(SV),[\nu]=\xi} h(\nu) \end{array}$$

The function H is supported on the compact convex set $C = \{[\nu], \nu \in \mathcal{M}_1(SV)\} \subset H_1(V,\mathbb{R})$; notice that C is symmetric with respect to the origin. One can prove that the function P is analytic and strictly convex on $H^1(V,\mathbb{R})$, and that H is continuous on C, analytic and strictly concave on the interior $\overset{\circ}{C}$. We refer the reader to [BL] for proofs, and to [Mas] and [Mas1] for some geometrical properties of C. A nice description of a similar set C was obtained by Bousch ([Bo]) for a different dynamical system.

The first part of the following theorem says that, if the constraint ξ has a sufficiently small norm, the quantity $\pi(\xi,\alpha,\delta;T)$ will grow exponentially in T. However, for $\xi \neq 0$, the growth exponent $H(\xi)$ will be strictly less than the topological entropy, which was the growth rate obtained by Margulis for geodesics without constraints ([M]). The second part of the theorem deals with the question of the asymptotic distribution, in measure theoretic sense, of closed geodesics. For all $\gamma \in \Gamma$, we will denote by m_γ the Lebesgue measure carried by γ (seen as a curve in SV), parametrized by arc-length.

Theorem 1.0.1. *(Lalley* [Lal1], [Lal2]*, Babillot-Ledrappier* [BL]*)*

If $\xi \in \overset{\circ}{C}$, then :

(1) One has the following asymptotics, when T goes to infinity :

$$\pi(\xi,\alpha,\delta;T) \sim c_0(\xi^T,\delta)\frac{e^{TH(\xi^T)+\langle\nabla H(\xi^T),\alpha\rangle}}{T^{d/2+1}}$$

where

- $\xi^T = \frac{[T\xi]}{T} \underset{T\longrightarrow\infty}{\longrightarrow} \xi$
- $\nabla H(\xi^T)$ *is the differential of H, seen as an element of the dual $H^1(V,\mathbb{R})$.*
- $d = dim\, H_1(V,\mathbb{R})$
- $$c_0(\xi^T,\delta) = \frac{\sqrt{|\, det\, H''(\xi^T)\,|}}{(2\pi)^{d/2}Vol(H_1^*)}\left(\frac{1-e^{-\delta P(-\nabla H(\xi^T))}}{P(-\nabla H(\xi^T))}\right)$$

  where H_1^* is the set of characters of the lattice $H_1(V,\mathbb{Z})$.

(2) (Equidistribution) *One has the following convergence in $\mathcal{M}_1(SV)$ endowed with the weak topology :*

$$\frac{1}{\pi(\xi,\alpha,\delta;T)}\sum_{\gamma\in\Gamma(\xi,\alpha,\delta;T)}\frac{m_\gamma}{l(\gamma)} \underset{T\longrightarrow\infty}{\longrightarrow} m^\xi$$

where m^ξ is the unique element of $\mathcal{M}_1(SV)$ satisfying $[m^\xi] = \xi$ and $H(\xi) = h(m^\xi)$. It is a Gibbs measure, and supp $m^\xi = SV$. In particular, $\cup_T\Gamma(\xi,\alpha,\delta;T)$ is dense in SV.

Remark 1.0.2. The usual proof of this result involves the study of dynamical zeta-functions; their domain of convergence is related to the spectrum of Markovian transition operators :
– in constant curvature, the "twisted" Laplace operators (see the Appendix).
– in variable curvature, a twisted transfer operator L_ω : there exists a subshift of finite type (Σ, σ), and a positive function τ on Σ, such that the special flow on the suspension over X by the function τ is semi-conjugate to the geodesic flow. The action of L_ω acts on the space of functions on Σ looks like :

$$L_\omega f(x) = \sum_{\sigma y = x} e^{\int_0^{\tau(y)} \omega(y,s)ds - P([\omega])\tau(y)} f(y)$$

The measure m^ξ appearing in Theorem 1.0.1*(2)* is also the unique equilibrium measure achieving the supremum $\sup_{\nu \in \mathcal{M}_1(SV)} h(\nu) + \langle [\omega], [\nu] \rangle$, where $[\omega] = -\nabla H(\xi)$; it can be considered a function of ω, and denoted m_ω. Transversally to the flow, the measure m^ξ seen as a measure on Σ, is the product of the fixed points of the operator L_ω and of its dual L^*_ω.

In general, not much is known when $\xi \in \partial C$ (of course, it is easy to see that $\pi(\xi, \alpha, \delta; T) = 0$ for large enough T, when $\xi \notin C$). We will observe in paragraph 2.2 that measures with asymptotic cycle on ∂C are minimizing in the sense of Mather : they minimize the length, as well as the energy, in their homology class. Problems of two kinds arise for $\xi \in \partial C$. From the analytic point of view, the behaviour at infinity of the function P is not known, which prevents us from trying to adapt the classical technique of zeta-functions. From a more dynamical point of view, the restriction of the geodesic flow to the set $\overline{\cup_{[\nu]=\xi} supp\, \nu}$ has, generally speaking, no Markovian structure when $\xi \in \partial C$, and there is no meaningful definition of a Gibbs measure m^ξ. As a consequence, usual counting techniques fail.

This paper is devoted to describing the asymptotic behaviour of the set $\Gamma(\xi, \alpha, \delta; T)$ when $\xi \in \partial C$. Results analogous to Theorem 1.0.1 would be to find the asymptotic behaviour of $\pi(\xi, \alpha, \delta; T)$, as well as to find a measure which plays the role of the measure m^ξ. These questions remain open, but we give here some partial answers to the former question. We restrict ourselves to the case of a surface, where much information can be derived from topological considerations.

In the case when $\xi \in \partial C$ we will see that it is equivalent, and slightly simpler as far as notations are concerned, to drop the δ's and to study the sets $\Gamma(\xi, \alpha; T) = \{\gamma \in \Gamma, l(\gamma) \leq T, [\gamma] = [T\xi] + \alpha\}$ and their cardinality $\pi(\xi, \alpha; T)$.

It will sometimes be more convenient, as well as interesting, to look at closed geodesics whose average homology cycle is not necessarily prescribed to be near a single vector ξ, but rather, near a whole *face* of ∂C. In other words, one can count elements of the form $(l(\gamma), [\gamma])$ lying near some hyperplane of $\mathbb{R} \times H_1(V, \mathbb{R})$, instead of counting elements in a neighbourhood of a line.

A face of ∂C (or of C) is defined as the intersection of ∂C with one of its supporting hyperplanes. All faces are of the form $\{\xi \in C, \langle [\omega], \xi \rangle = 1\}$ where ω is a closed 1-form on V whose cohomology class $[\omega]$ satisfies $\langle [\omega], \xi' \rangle \leq 1$, for all $\xi' \in C$. We will always use this representation of faces, and we will denote F_ω the face associated to ω. For such an ω and for $\varepsilon, \delta, A > 0$, we introduce the sets

$$\Gamma(\omega, \varepsilon, A, \delta; T) = \{\gamma \in \Gamma, T - \delta \leq l(\gamma) \leq T, \langle [\omega], [\gamma] \rangle \geq (1 - \varepsilon)T - A\}$$

as well as their cardinality

$$\pi(\omega,\varepsilon,A,\delta;T) = Card\,\Gamma(\omega,\varepsilon,A,\delta;T)$$

Sometimes it will be possible to drop the dependence on δ :

$$\Gamma(\omega,\varepsilon,A;T) = \{\gamma \in \Gamma, l(\gamma) \leq T, \langle [\omega],[\gamma]\rangle \geq (1-\varepsilon)T - A\}$$

and

$$\pi(\omega,\varepsilon,A;T) = Card\,\Gamma(\omega,\varepsilon,A;T)$$

When we take $\varepsilon = 0$ we will simply write $\Gamma(\omega,A;T)$ and $\pi(\omega,A;T)$.

In this context Lalley's results also yield :

Proposition 1.0.3. *If $\varepsilon \in (0,1)$, then there exist a constant $c(\omega,\varepsilon,A,\delta)$ (which can be computed explicitly) such that*

$$\pi(\omega,\varepsilon,A,\delta;T) \sim c(\omega,\varepsilon,A,\delta)\frac{e^{TH(\omega,\varepsilon)}}{T^{3/2}}$$

with $H(\omega,\varepsilon) = \sup\{h(\nu), \langle[\omega],[\nu]\rangle = 1-\varepsilon\} > 0$.

Studying geodesics whose average homology cycle belongs to F_ω amounts to taking $\varepsilon = 0$ and to studying the set $\Gamma(\omega,A,\delta;T)$.

Definition 1.0.4. A closed geodesic $\gamma \in \Gamma$ is said to be (ω,A)-*quasi-minimizing* if it has the following property : if c is a family of closed curves on V, such that $\langle[\omega],[c]\rangle = \langle[\omega],[\gamma]\rangle$, then $l(c) \geq l(\gamma) - A$. The geodesic γ is said to be A-*quasi-minimizing*, if $l(c) \geq l(\gamma) - A$ for all multi-curves c with $[c] = [\gamma]$. For $A = 0$ we simply talk about ω-minimizing and minimizing geodesics.

We remark that geodesics in $\Gamma(\omega,A,\delta;T)$ are (ω,A)-minimizing.

When studying $\Gamma(\xi,\alpha,\delta;T)$ and $\Gamma(\omega,\varepsilon,A,\delta;T)$, profitable information can be derived from a Large Deviation result of Kifer ([Kif]), which has the following consequence :

Lemma 1.0.5. *(1) For all $\xi \in H_1(V,\mathbb{R})$,*

$$\limsup \frac{1}{T}\log \pi(\xi,\alpha,\delta;T) \leq H(\xi)$$

and for all ω as above and $\varepsilon \in [0,1)$,

$$\limsup \frac{1}{T}\log \pi(\omega,\varepsilon,A,\delta;T) \leq H(\omega,\varepsilon)$$

(2) If there exists a unique measure m^ξ (resp. $m_{\omega,\varepsilon}$) such that $[m^\xi] = \xi$ and $H(\xi) = h(m^\xi)$ (resp. $\langle[\omega],[m_{\omega,\varepsilon}]\rangle = 1-\varepsilon$ and $H(\omega,\varepsilon) = h(m_{\omega,\varepsilon})$) then

$$\frac{1}{\pi(\xi,\alpha,\delta;T)}\sum_{\gamma\in\Gamma(\xi,\alpha,\delta;T)}\frac{m_\gamma}{l(\gamma)} \underset{T\longrightarrow\infty}{\longrightarrow} m^\xi$$

(resp. $\frac{1}{\pi(\omega,\varepsilon,A,\delta;T)}\sum_{\gamma\in\Gamma(\omega,\varepsilon,A,\delta;T)}\frac{m_\gamma}{l(\gamma)} \underset{T\longrightarrow\infty}{\longrightarrow} m_{\omega,\varepsilon}$)

As we will see later, the quantities $H(\xi)$ and $H(\omega,\varepsilon)$ vanish in the boundary cases $\xi \in \partial C$ and $\varepsilon = 0$, so that $\pi(\xi,\alpha,\delta;T)$ and $\pi(\omega,A,\delta;T)$ grow sub-exponentially. In these cases, the assumptions of (2) are not satisfied, and it is not known whether there exist measures playing the role of m^ξ and $m_{\omega,\varepsilon}$.

1.1. **Rational constraints.** The description of the sets $\Gamma(\omega, A, \delta; T)$ will depend on the "rationality" of the face F_ω.

Definition 1.1.1. A point $\xi \in H_1(V, \mathbb{R})$ is called *rational* if there exists $r \in \mathbb{R} \setminus 0$ such that $r\xi \in H_1(V, \mathbb{Z})$.

Definition 1.1.2. A face of C is called *rational* if it contains a rational point in its relative interior.

We first prove the following :

Theorem 1.1.3. *(1) If ξ belongs to a rational face, then for all $\alpha \in H_1(V, \mathbb{Z})$, there exists a polynomial $Q_{\xi,\alpha}$ such that, for all T,*

$$\pi(\xi, \alpha; T) \leq Q_{\xi,\alpha}(T)$$

(2) The projection on V of the set $\cup_T \Gamma(\xi, \alpha; T)$ is nowhere dense in V; its closure has Lebesgue measure 0. Its Hausdorff dimension is 1, if it is not empty.

Let F_ω be a rational face containing ξ. The proof relies on a description of the *Mather set* $\tilde{M}_\omega$, defined as $\tilde{M}_\omega = \overline{\cup_{[m] \in F_\omega} supp\, m}$. In the rational case it is a union of disjoint closed simple geodesics. Such geodesics are isolated, for the Hausdorff distance, from other complete geodesics (in particular, small neighbourhoods of the set $\tilde{M}_\omega$ have zero entropy with respect to the geodesic flow).

Let Int be the algebraic intersection form on $H_1(V, \mathbb{R})$. Proving Theorem 1.1.3, we will in fact prove :

Proposition 1.1.4. *If F_ω is a rational face, then, for all $A, K \geq 0$, there exists a polynomial $Q_{\omega,A,K}$ such that, for all T,*

$$Card\, \{\gamma \in \Gamma(\omega, A; T), \mid Int([\gamma], \xi) \mid \leq K \text{ for all } \xi \in F_\omega\} \leq Q_{\omega,A,K}(T)$$

In general this is not true if we drop the condition on the intersection number, as we prove in Theorem 1.1.5. We use Fathi's remark that the phrase commonly used "Aubry-Mather set" represents two different objects, the Mather set $\tilde{M}_\omega$ defined above, and another set which we call the Aubry set $\tilde{A}_\omega$. It contains the Mather set, but can also contain orbits which are heteroclinic to the Mather set. We show that the existence of such orbits has strong implications on the growth rate for the number of (ω, A)-quasi-minimizing geodesics :

Theorem 1.1.5. *(1) If $\tilde{A}_\omega = \tilde{M}_\omega$, and if F_ω is a rational face, then, for all $A \geq 0$, $\pi(\omega, A; T)$ grows polynomially in T.*

(2) If the Aubry set $\tilde{A}_\omega$ and the Mather set $\tilde{M}_\omega$ are distinct, then for all $A > 0$ there exists a sequence $T_n \longrightarrow \infty$ such that $\pi(\omega, A; T_n)$ grows faster than any polynomial in T_n.

A priori, the Aubry set can be strictly larger than the Mather set. There are explicit examples of Hamiltonian systems for which Aubry and Mather sets do not coincide; however, it is not known what will the general picture look like for geodesic flows in negative curvature.

1.2. **Irrational constraints.** The case of rational faces stands in sharp contrast to the irrational case. In that case, many of the questions treated above remain open.

In Section 4 we use Fathi's weak KAM theorem to show that some steps leading to Proposition 1.1.4 and Theorem 1.1.3 still hold. But this time, arbitrarily small neighbourhoods of the Mather set have positive entropy :

Theorem 1.2.1. *Let F_ω be a non rational face of ∂C. Then for every neighbourhood $\tilde{W}$ of the Mather set $\tilde{M}_\omega$, and for every $\varepsilon \in (0,1)$, there exists a number $h(\tilde{W},\varepsilon) > 0$ such that, for all $A > 0$, for large enough T,*

$$Card\left(\Gamma(\omega,\varepsilon,A;T)\cap\{\gamma\in\Gamma,\gamma\subset\tilde{W}\}\right)\geq e^{h(\tilde{W},\varepsilon)T}$$

Thus, there exist exponentially many geodesics in $\Gamma(\omega,\varepsilon,A;T)$ lying in an arbitrary neighbourhood of the Mather set. This prevents us to conclude as in Theorem 1.1.3, and the question of estimating $\pi(\xi,\alpha;T)$ for arbitrary ξ remains open.

Remark 1.2.2. In the light of Theorem 1.2.1 it seems reasonable to conjecture that, if we let ε tend to 0, the entropy $H(\omega,\varepsilon)$ will decay faster to 0 when ω defines a rational face than when it defines a non-rational face. Also, Theorem 1.1.5 allows us to think that $H(\omega,\varepsilon)$ will decay faster when $\tilde{A}_\omega = \tilde{M}_\omega$ than when $\tilde{A}_\omega \neq \tilde{M}_\omega$.

Remark 1.2.3. The question of finding sharp estimates – that is, a satisfactory lower bound in 1.1.3 and 1.1.4 and an upper bound in 1.1.5 – remains open. It seems that topological considerations will not be sufficient, but that some analysis of the metric properties of V should come into play. Moreover, if we remember Theorem 1.0.1, it is obvious that the sharp growth rate as well as all the constants involved will be related to the behaviour at infinity of the pressure and of the entropy (explicit calculations can be carried out for reasonably simple Markov shifts).

In section 2 we collect all the necessary information on length and homology of geodesics and Mather theory, and we clarify the statements made above. In section 3 we prove Theorem 1.1.3. Eventually in section 4 we prove Theorems 1.1.5 and 1.2.1; this section also presents an attempt to relate the issue to the study of fixed points of the Lax-Oleinik operators.

2. Preliminaries on action minimizing laminations

2.1. **Basic facts about length and angles of intersection.** The following lemmas, which are very simple observations, are the key ingredient to the proof of the main result in Mather's description of minimizing measures (Theorem 2.2.5). They also play a major part in the proof of Theorem 1.1.3.

Lemma 2.1.1. *For small $a > 0$, there exists $\eta > 0$ such that, for every $\theta \in (0,\pi]$: if $g_1, g_2 : [0,a] \longrightarrow V$ are geodesic arcs, parametrized by arc-length, with $g_1(a) = g_2(0)$ and the angle between the tangent vectors at $g_1(a)$ and at $g_2(0)$ is greater than θ, then there exist a C^1 curve $\gamma_1 : [0,2a] \longrightarrow V$ such that $g_1(0) = \gamma_1(0)$, $g_2(a) = \gamma_1(2a)$, and*

$$l(g_1)+l(g_2)-l(\gamma_1)\geq\eta\theta^2$$

Proof. : This is a simple consequence of the local geometric properties of the manifold : in local charts the metric is equivalent to euclidean metric, for which the estimate is simple to check. □

We use the same notation *dist* for distance on V and distance on SV, measured with respect to the Riemannian metric. We will also need the following variant of the previous lemma (compare with the lemma on p186 of [Mat]) :

Lemma 2.1.2. *There exist $a, \rho, \eta > 0$ such that, for every $\delta > 0$ small enough : if $g_1, g_2 : [-a,a] \longrightarrow V$ are geodesic arcs, parametrized by arc length, with*

$dist(g_1(0), g_2(0)) \leq \delta$, and $dist(\dot{g}_1(0), \dot{g}_2(0) \geq \rho\delta$, then there exist C^1 curves $\gamma_1, \gamma_2 : [-a, a] \longrightarrow V$ such that $g_1(-a) = \gamma_1(-a)$, $g_1(a) = \gamma_2(a)$,$g_2(-a) = \gamma_2(-a)$, $g_2(a) = \gamma_1(a)$, and

$$l(g_1) + l(g_2) - (l(\gamma_1) + l(\gamma_2)) \geq \eta\delta^2$$

2.2. **Stable norm and Mather's theorem.** Some of the results of this section are true in any dimension, but we restrict our attention to surfaces : Propositions 2.2.2(1) and 2.4.2 only hold on surfaces; Mather's theorem holds in all dimension, but only in dimension 2 do the Mather sets admit a topological description suitable for counting orbits. Fathi's theorem also holds in any dimension; so in fact dimension does not play any role in the variational (Lagrangian) aspect of the problem, but it plays a crucial role in the counting technique, which relies on topology.

Definition 2.2.1. ([GLP]) The stable norm on $H_1(V, \mathbb{R})$ is defined by

$$\| \xi \|_s = \inf\{\sum | r_i | l(\gamma_i), \text{ where } r_i \in \mathbb{R}, \gamma_i \text{ are closed curves such that } \sum r_i[\gamma_i] = \xi\}$$

We denote $\mathcal{M}(TV)$ and $\mathcal{M}(SV)$ the set of positive measures invariant under the geodesic flow, respectively on the tangent bundle TV and on the unit tangent bundle SV. $\mathcal{M}_1(TV)$ and $\mathcal{M}_1(SV)$ are the corresponding subsets of probability measures.

Proposition 2.2.2. *(1) ([GLP]) If $\xi \in H_1(V, \mathbb{Z})$ then*

$$\| \xi \|_s = \inf\{\sum | p_i | l(\gamma_i), \textit{ where } p_i \in \mathbb{Z}, \gamma_i \textit{ are closed curves such that } \sum p_i[\gamma_i] = \xi\}$$

(2) ([Mas]) For all $\xi \in H_1(V, \mathbb{Z})$ one has

$$\| \xi \|_s = \inf\{m(SV), m \in \mathcal{M}(SV), [m] = \xi\}$$

and

$$\| \xi \|_s^2 = \inf\{\int_{TV} \| v \|^2 \, dm(v), m \in \mathcal{M}_1(TV), [m] = \xi\}$$

One can check that in all cases the inf is in actually a min; besides, the measures achieving the minimum in the last case are supported on $\| \xi \|_s .SV$. The last expression does not seem a very natural expression for the stable norm; the aim is to draw a relation with Mather theory for action minimizing measures of Lagrangian systems ([Mat]). Here the Lagrangian is simply $v \mapsto \frac{\|v\|^2}{2}$, and the *action minimizing measures* are the measures achieving the minimum in (2). By an homothety we can carry them to probability measures supported on SV. They are a generalization of the minimizing geodesics defined in 1.0.4 : a geodesic is minimizing if the uniform probability measure it carries is minimizing.

As a consequence of Proposition 2.2.2, the convex set C defined in paragraph 1 is the unit ball for the stable norm ([Mas]). Also, supporting hyperplanes of C are all of the form $\langle[\omega], \xi\rangle = 1$, where ω is a closed 1-form on V with dual stable norm $\| [\omega] \|_s = 1$. As before, we denote F_ω the corresponding face of C.

Definition 2.2.3. The set of ω-minimizing measures $\mathcal{M}_\omega$ is defined as the set of measures $m \in \mathcal{M}_1(SV)$ such that $[m] \in F_\omega$. The Mather set $\tilde{M}_\omega \subset SV$ is defined as

$$\tilde{M}_\omega = \overline{\cup_{m \in \mathcal{M}_\omega} supp\, m}$$

Remark 2.2.4. In thermodynamic formalism, the equilibrium measure m_ω which achieves the supremum $P([\omega]) = \sup_\nu h(\nu) + \langle[\omega],[\nu]\rangle$ is characterized by a Gibbsian property. Here, ω-minimizing measures are characterized by their support, namely the set $\tilde{M}_\omega$. Notice that any limit point of the family $(m_{\lambda\omega})$, as λ tends to infinity, is an ω-minimizing measure. In fact,

$$\frac{P(\lambda[\omega])}{\lambda} \underset{\lambda\longrightarrow\infty}{\longrightarrow} \parallel \omega \parallel_s$$

because $\lambda \parallel [\omega] \parallel_s \leq P(\lambda[\omega]) \leq h + \lambda \parallel [\omega] \parallel_s$, where h is the topological entropy of the geodesic flow.

Lemmas 2.1.1 and 2.1.2 show that geodesics with self-intersections are not minimizing. A major consequence is the following theorem, which holds in any dimension :

Theorem 2.2.5. *(Mather* [Mat]*) For all ω, the projection $p : SV \longrightarrow V$, restricted to $\tilde{M}_\omega$, is injective. Its inverse (considered as a mapping from $M_\omega = p(\tilde{M}_\omega)$ to $\tilde{M}_\omega$) is lipschitz. In particular, p is an homeomorphism from $\tilde{M}_\omega$ to M_ω.*

2.3. **The topology of geodesic laminations.**

Definition 2.3.1. Let λ be a compact subset of V.
(1) The set λ is a *geodesic lamination* if it is a disjoint union of immersed geodesics. In other words, there exists a compact subset $\bar{\lambda}$ of the projective tangent bundle $P(TV)$, invariant by the geodesic flow, such that the projection $\bar{p} : P(TV) \longrightarrow V$ restricted to $\bar{\lambda}$ is a homeomorphism onto λ.
(2) The set λ is a *measured lamination* if $\bar{\lambda}$ carries a flow-invariant measure ν. The projection of ν can be identified with a transverse measure for λ, and the projection of the support of ν is called the support of the transverse measure.
(3) A lamination λ is *orientable* if there exists a closed subset $\tilde{\lambda} \subset SV$ lying over $\bar{\lambda}$ such that the projection $p : SV \longrightarrow V$ restricted to $\tilde{\lambda}$ is a homeomorphism onto λ. (the choice of such a lift of λ is called an orientation of the lamination).

Remark 2.3.2. A geodesic lamination has empty interior : if it contained an open subset of V, then it would be equal to V itself, because geodesics diverge exponentially fast in the universal cover of V. But then V would carry a nowhere vanishing vector field, a contradiction with the fact that V has negative Euler curvature. One can also show that the intersection of V with any transversal is totally discontinuous ([Ot]). So, the lift $\bar{\lambda}$ is completely determined by λ, and one does not generally make the distinction between λ and $\bar{\lambda}$, or $\tilde{\lambda}$ if an orientation is given.

The two following propositions result from Theorem 2.2.5.

Proposition 2.3.3. *For every $\omega \in H^1(V,\mathbb{R})$ such that $\parallel \omega \parallel_s = 1$, $\mathcal{M}_\omega$ is an oriented geodesic lamination, and it is the support of a tranverse measure.*

Proposition 2.3.4. *(*[Lev2], [Mas1]*) The linear span of a face of C is a totally isotropic subspace of $H_1(V,\mathbb{R})$ for the intersection form Int.*

Recall that Int is a non-degenerate skew-symmetric form on $H_1(V,\mathbb{R})$. If $h, h' \in H_1(V,\mathbb{Z})$ are two integral homology classes, $Int(h,h')$ is the algebraic intersection number of any two closed (oriented) curves representing h and h'. Proposition 2.3.4 is a consequence of the fact that two geodesics in M_ω do not intersect transversally.

We now describe the topological properties of laminations, and check that measured laminations have zero entropy, so that H vanishes on ∂C as claimed in paragraph 1.

An oriented geodesic lamination which is the support of a measure is the union of a finite number of (compact) connected components. Some of these components are closed geodesics. The other components are called *minimal exceptional* components, they have the property that every half leaf of the component is dense in it. There are at most $3g-3$ closed components and g minimal exceptional components (we refer the reader to [Lev1] or [Lev2] for proofs).

We recall the transverse structure of minimal exceptional components. We won't use this description in its entirety, but we hope it will provide the reader with a comfortable picture of laminations.

Let c be such a component and I a small embedded segment transverse to the lamination (and such that $I \cap c' = \emptyset$ if c' is another connected component). Then $I \cap c$ is compact, has empty interior, and it is totally discontinuous : in other words it is a Cantor set. Consequently we can even assume that the endpoints of I do not belong to c. Recall that c is oriented, so that, if I is small enough, all leaves of c intersect I in the same direction.

Notations : If two points x, y lie on I, we denote $[x, y]$ the oriented subsegment of I going from x to y. If x, y are two points lying on the same leaf of c, we denote $/x, y/$ the corresponding oriented portion of leaf.

Let $T : I \cap c \longrightarrow I \cap c$ be the first return map along leaves of c. The transformation T has the structure of an "interval exchange". More precisely, there are two partitions of I into intervals of disjoint interiors, $P = (I^1, \cdots, I^n)$ (resp. $P' = (I'^1, \cdots, I'^n)$) such that $c \cap I^j \subset \mathring{I}^j$ and $T_c(I^j) \subset \mathring{I'}^j$; besides, two geodesic arcs of c of the form $/y, T_c(y)/$ and $/z, T_c(z)/$, with $y, z \in I^j$, are homotopic with endpoints varying respectively in I^j, I'^j, and $T_c : I^j \cap c \longrightarrow I'^j \cap c$ preserves the natural order on the intervals.

Because of this property, one has

Lemma 2.3.5. *Every measured lamination has zero entropy with respect to the geodesic flow.*

Proof. In fact, the first return time T has zero topological entropy. This comes from the remark that, for each N, at most $N.Card\, P$ atoms of the partition $\vee_{k=0}^{N} T^{-k} P$ are non-empty.

Consequently, T has null metric entropy with respect to any invariant measure on I. Any transverse measure supported on c gives a T-invariant measure on I, and Abramov's formula ([Ab]) implies that it has zero entropy. $\square$

Starting from such a partition P, one can construct a good looking neighbourhood $V_{I,P}(c)$ of c in SV the following way : consider all (oriented) geodesic arcs $\gamma : [0, t] \longrightarrow V$, such that there exists $j \in [1, n]$ with $\gamma(0) \in \mathring{I}^j$, $\gamma(t) \in \mathring{I'}^j$, and γ is homotopic, with endpoints varying in I^j, I'^j, to a geodesic arc in c of the form $/y, T_c(y)/$. The neighbourhood $V_{I,P}(c)$ is then defined as the set of vectors $v \in SV$ of the form $\dot{\gamma}(s)$ for some $\gamma : [0, t] \longrightarrow V$ as above, for some $s \in [0, t)$. Such a neighbourhood is called a "train-track" neighbourhhood : a curve in this neighbourhood follows the "rails" of c defined by the partition P, until it intersects

I. When it intersects I the track splits into several branches and the geodesic can be switched in one of the different directions.

2.4. **Rational asymptotic cycles.**

Proposition 2.4.1. *(*[Lev1], [Mas1]*) Let ν be a transverse measure of a geodesic lamination. If $[\nu]$ is rational, then the support of ν is a union of closed geodesics.*

Recall that a face F_ω is called rational if it contains a rational point in its relative interior. On surfaces, rational faces admit a nice description in terms of the topology of the associated Mather set :

Proposition 2.4.2. *(*[Mas2]*)The face F_ω is rational if and only if the Mather set M_ω is the union of a finite number of disjoint closed geodesics – or equivalently, the extremal points of F_ω are of the form $\frac{[\gamma]}{l(\gamma)}$, where γ is a closed minimizing geodesic.*

3. Proof of Theorem 1.1.3

From now on, ξ, α are fixed; ω is a closed 1-form such that $\| [\omega] \|_s = 1$ and $\langle [\omega], \xi \rangle = 1$, and such that F_ω is rational.

Section 3.1 describes the elements of $\Gamma(\xi, \alpha; T)$ when M_ω contains only closed geodesics. In Section 3.2 we finish the proof using combinatorial counting arguments.

We initially defined geodesics as (parametrized) curves in SV. But since we will be working only with geodesics and laminations, we will not always distinguish between the curves in SV, their projections on V, and their geometric images as subsets of V. Our geodesics will always be parametrized by arc-length. We will sometimes use the word "multicurve", meaning thereby a family of closed parametrized curves.

3.1. **Closed minimizing geodesics.**

Let c be a closed oriented geodesic; we fix a point $P \in c$ and a parametrization $c : [0, l(c)] \longrightarrow V$ such that $c(0) = c(l(c)) = P$. Let I be a small geodesic segment centered at P, orthogonal to c, of length $2r$ with r small. As at the end of paragraph 2.3, we define a neighbourhood $V_I(c) \subset SV$ of the oriented geodesic c. Consider all oriented geodesic arcs $\gamma : [0, t] \longrightarrow V$, with endpoints on I, and homotopic (also with endpoints varying in I) to $c : [0, l(c)] \longrightarrow V$. Then $V_I(c)$ is the set of vectors of the form $\dot\gamma(s)$ for such an arc $\gamma : [0, t] \longrightarrow V$, and for some $s \in [0, t)$. The set $V_I(c)$ is a neighbourhood (neither open nor closed) of c in SV.

Remark that the distance from any point of I to c is less than r. If $\tilde c$ is a lift of c to the universal covering $\tilde V$, the function "distance to $\tilde c$" is convex on $\tilde V$. As a consequence, the points of $p(V_I(c))$ are at distance from c less than r. In particular, for any $r' > r$, $p(V_I(c))$ is contained in $W = \{x \in V, d(x, c) < r'\}$; if r' is small enough, W is a collar neighbourhood of c in V, that is, homeomorphic to an annulus. From now on, let W be fixed, while I will be chosen sufficiently small.

Proposition 3.1.1. *Let c be a closed geodesic in M_ω and let I be as before. Then there exists K_0 such that, for all T, for all $\gamma \in \Gamma(\xi, \alpha; T)$, the number of connected components of $\gamma \cap V_I(c)$ (in SV) is smaller than K_0.*

Proof. We begin with the two following observations :

Observation 1 : By definition, the stable norm of the vectors $T\xi - [T\xi]$ are bounded in $H_1(V, \mathbb{R})$, say by a constant C independent of T. As a consequence, there exists $A = \mid \langle [\omega], \alpha \rangle \mid + C \geq 0$ such that, for all T, for all $\gamma \in \Gamma(\xi, \alpha; T)$, $\langle [\omega], [\gamma] \rangle \geq l(\gamma) - A$ (simply remember that $\langle [\omega], \xi \rangle = 1$ and write $\langle [\omega], [\gamma] \rangle = \langle [\omega], [T\xi] \rangle \geq \langle [\omega], T\xi \rangle - A = T - A \geq l(\gamma) - A$). In particular, if γ' is a multicurve such that $[\gamma'] = [\gamma]$, then $l(\gamma') \geq \| [\gamma'] \|_s \geq \langle [\omega], [\gamma'] \rangle \geq l(\gamma) - A$; this means that γ is A-quasi-minimizing. Notice also that the length $l(\gamma)$ itself has to be greater than $T - A$, for a certain A.

Observation 2 : For all $\lambda \geq 0$, for all $\gamma \in \Gamma(\xi; T)$

$$\langle [\omega], [\gamma] + \lambda [c] \rangle \geq l(\gamma) + \lambda l(c) - A$$

This is just a consequence of the fact that $\langle [\omega], [c] \rangle = l(c)$ and $\langle [\omega], \xi \rangle = 1$.

The rest of the proof will not depend much on α except for the constant A; in the rest of paragraph 3, we will omit all the α's in our notations.

We treat the case when γ is not homotopic to a multiple of c; if it is, then the number of connected components of $\gamma \cap V_I(c)$ is exactly 1.

If γ is a closed geodesic, all the connected components of $\gamma \cap V_I(c) \subset SV$ are contained in distinct components of $\gamma \cap (p^{-1}W)$. To see why, let γ_1 and γ_2 be two connected components of $\gamma \cap V_I(c)$. Because of the definition of $V_I(c)$, they are of the form $\gamma_j = [x_j, y_j)$ with $x_j, y_j \in I$. The closed curve γ decomposes as $\gamma = [x_1, y_1) \cup [y_1, x_2) \cup [x_2, y_2) \cup [y_2, x_1)$. Suppose $[y_1, x_2) \subset p^{-1}W$, then $dist(z, c) \leq r$ for all $z \in [y_1, x_2)$, always by convexity of the distance function in the universal covering; this has the consequence that $[y_1, x_2) \subset V_I(c)$ which is not possible if $\gamma_1 \neq \gamma_2$. The same argument shows that $[y_2, x_1) \not\subset p^{-1}W$.

There are two types of components of $\gamma \cap p^{-1}W$: those which do not intersect c (type 1), and those which intersect c (type 2). If I is chosen sufficiently small, a component of type 1 which contains a component of $V_I(c)$ automatically has a self-intersection in W; a component of type 2 has no self-intersection.

To see this, consider a lift $\tilde{c}$ of c to the universal cover $\tilde{V}$, and denote $\tilde{W}$ the lift of W containing $\tilde{c}$; that is to say, $\tilde{W} = \{x \in \tilde{V}, \tilde{d}(x, \tilde{c}) < r'\}$. Consider a connected component γ_1 of $\gamma \cap p^{-1}W$, and a lift $\tilde{\gamma}$ of γ such that $\tilde{\gamma} \cap \tilde{W}$ projects to γ_1. If $\tilde{\gamma}$ does not intersect $\tilde{c}$ in $\tilde{V}$, then γ_1 does not intersect c in V. If in addition γ_1 contains a connected component of $V_I(c)$, then $\tilde{\gamma}$ is close to $\tilde{c}$ in the space of geodesics of $\tilde{V}$. Let us consider the action of c on $\tilde{V}$; then $\tilde{\gamma}$ intersects $c\tilde{\gamma}$ in $\tilde{W}$, if the interval I is chosen sufficiently small. This means that γ_1 has a point of self-intersection in W.

If on the contrary $\tilde{\gamma}$ intersects $\tilde{c}$, then $\tilde{\gamma}$ is disjoint from all its translates under the action of c on $\tilde{V}$; this implies that γ_1 has no self-intersection in W.

Lemma 3.1.2. *There exists K_1 such that, for all T, for all $\gamma \in \Gamma(\xi; T)$, the number of components of $\gamma \cap V_I(c)$ contained in a component of type 1, is less than K_1.*

Proof. Let γ_1 be a component of $\gamma \cap p^{-1}W$ which contains a component of $\gamma \cap V_I(c)$. We claim that there exists $\varepsilon > 0$, depending only on the choice of W, such that, if we uncross all the self–intersections of this component, straighten the resulting multicurve to get a multigeodesic, we get a multigeodesic γ' such that $[\gamma'] = [\gamma]$ and $l(\gamma') < l(\gamma) - \varepsilon$.

To see this, one can first uncross all but one self-intersection of γ_1, straighten the resulting union of curves without moving the two endpoints of γ to get geodesics in W, in such a way that the result is a union of copies of c and of a geodesic arc with endpoints on ∂W *with exactly one self-intersection.* Negative curvature implies as previously that the geodesic arcs after modification are still contained in W. The

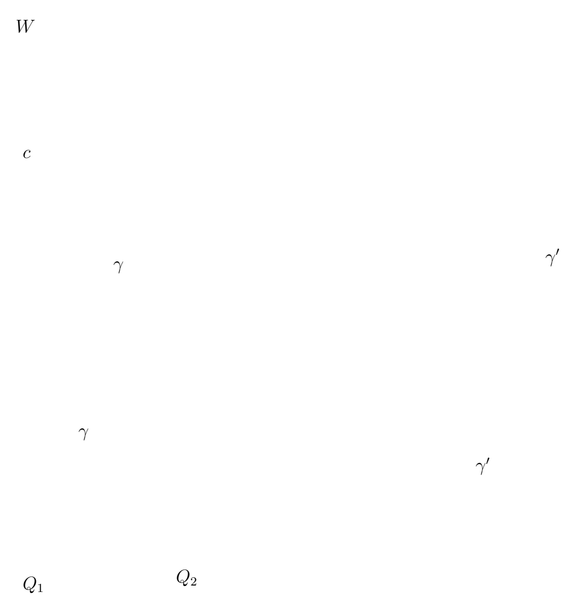

FIGURE 1. In a neighbourhood of a closed geodesic.

procedure, which is shown in Figure 3.1(a), does not alter the homology class but decreases the length.

Now the angle of the remaining intersection is bounded below by a certain $\theta > 0$, depending only on W. Indeed, the Closing Lemma implies that, for each neighbourhood Y of c in SV, there exists θ with the following property : if $\gamma' : [O, s] \longrightarrow V$ is a geodesic such that $\gamma'(0) = \gamma'(s)$, the angle between the tangent vectors at 0 and s is less than θ, and γ' is freely homotopic to c, then $\gamma' \subset Y$. On the other hand, if Y is chosen small enough and if $\gamma' \subset Y$, if γ' is not homotopic to a multiple of c, then the complete geodesic containing γ' will have a second point of transverse self-intersection in W. This shows the existence of a lower bound for the angle of self-intersection of geodesic arcs having exactly one self-intersection in W.

Noticing that the length of any closed curve freely homotopic to c is greater than $l(c) > 2a$, we can now apply Lemma 2.1.1, uncross the last intersection, and obtain a multicurve of same homology class than γ and of length less than $l(\gamma) - \varepsilon$, with $\varepsilon = \eta\theta^2$. Observation (1) now implies that this can happen at most $K_1 = \frac{A}{\varepsilon}$ times. $\square$

In what follows the letter i denotes the geometric intersection number.

Lemma 3.1.3. *There exists K_2 such that, for all T, for all $\gamma \in \Gamma(\xi; T)$, the intersection number $i(\gamma, c)$ is less than K_2.*

Proof. For all $\theta \in (0, \pi)$, we can write

$$i(c, \gamma) = i_{\geq\theta}(c, \gamma) + i_{<\theta}(c, \gamma)$$

where $i_{\geq\theta}(c,\gamma)$ (respectively $i_{<\theta}(c,\gamma)$) is the number of points of intersection between c and γ where the angle of intersection is greater than θ (resp. smaller than θ). The value of θ will be specified a bit later.

Notation : If c and γ intersect at a point Q, we will denote by $\measuredangle_Q(c,\gamma)$ the oriented angle between c and γ at Q.

Each time γ intersects c with an angle greater than θ, we can apply Lemma 2.1.1 to the multicurve composed of γ and a copy of c. Hence,

$$\langle\, [\omega], [\gamma] + i_{\geq\theta}(c,\gamma)[c]\,\rangle \leq l(\gamma) + i_{\geq\theta}(c,\gamma)l(c) - i_{\geq\theta}(c,\gamma)\eta\theta^2$$

This, combined with Observation 2, gives $i_{\geq\theta}(c,\gamma) \leq \frac{A}{\eta\theta^2}$.

Now we choose $\theta > 0$ small enough so that the following holds : if two oriented geodesic arcs g_1, g_2 contained in W and with endpoints on ∂W, intersect c respectively at points Q_1, Q_2, with $\measuredangle_{Q_1}(c,g_1) \in (0,\theta]$ and $\measuredangle_{Q_2}(c,g_2) \in [-\theta,0)$, then g_1 and g_2 have at least two points of intersection inside W.

Suppose that $\gamma \in \Gamma(\xi;T)$ intersects c in two points Q_1, Q_2, with $\measuredangle_{Q_1}(c,\gamma) \in (0,\theta]$ and $\measuredangle_{Q_2}(c,\gamma) \in [-\theta,0)$. We are in the situation illustrated on Figure 3.1(b), and we can uncross one self-intersection, as shown on the figure, to fall into the previous case of Figure 3.1(a). As we explained before, this will allow us to find a multi-curve γ' with $[\gamma'] = [\gamma]$ and $l(\gamma') \leq l(\gamma) - \varepsilon$. Because of Observation (1), this cannot happen more than $\frac{A}{\varepsilon}$ times.

What we have shown is that γ always intersects c with the same orientation, except for at most $i_{\geq\theta}(c,\gamma) + 2\frac{A}{\varepsilon}$ points of intersection; and $i_{\geq\theta}(c,\gamma) \leq \frac{A}{\eta\theta^2}$. Thus,

$$\mid Int([c],[\gamma]) \mid \geq i(c,\gamma) - \frac{A}{\eta\theta^2} - 2\frac{A}{\varepsilon}$$

As a consequence of Proposition 2.3.4, $Int([c],\xi) = 0$, and the quantity $\mid Int([c],[\gamma]) \mid = \mid Int([T\xi]+\alpha,[c]) \mid$ is bounded independently of T. Lemma 3.1.3 follows. □

We have shown that $\gamma \cap V_I(c)$ has at most K_1 components of type 1 and K_2 components of type 2. Proposition 3.1.1 follows. □

Proposition 3.1.4. *Suppose that $\tilde{M}_\omega$ is a union of closed geodesics $(c_1,\cdots,c_R)$. For each k we chose a neighbourhood of c_k of the form $V_k = V_{I_k}(c_k)$, such that Proposition 3.1.1 holds. Then there exists L_0 such that, for all T, for all $\gamma \in \Gamma(\xi;T)$,*

$$l(\gamma \cap (\cup_{k=1}^R V_k)^c) \leq L_0$$

Proof. Let us suppose, on the contrary, that there exist sequences $T_n \longrightarrow +\infty$ and $\gamma_n \in \Gamma(\xi;T_n)$ such that

$$l(\gamma_n \cap (\cup_{k=1}^R V_k)^c) \longrightarrow +\infty$$

We may furthermore assume that there exists a probability measure ν on SV such that

$$\frac{m_{\gamma_n|(\cup V_k)^c}}{l(\gamma_n \cap (\cup V_k)^c)} \longrightarrow \nu$$

Proposition 3.1.1 implies that $\gamma_n \cap (\cup V_k)$ has at most RK_0 connected components, so that, for all t,

$$\| m_{\gamma_n|(\cup V_k)^c} - \phi_t * m_{\gamma_n|(\cup V_k)^c} \| \leq 2RK_0 t$$

(here $\| \, . \, \|$ denotes the total variation of measures). Consequently, ν is invariant under the action of the geodesic flow. Moreover, $\nu(\cup V_k) = 0$, which implies that ν does not belong to $\mathcal{M}_\omega$, i.e $\langle [\omega], [\nu] \rangle < 1$.

Thus, there exists $\varepsilon > 0$ such that, for large enough n,

$$\int_{\gamma_n \cap (\cup V_k)^c} \omega < (1-\varepsilon) l(\gamma_n \cap (\cup_{k=1}^R V_k)^c)$$

Notice that there exists a constant D such that, for any arc $g : [0,t] \longrightarrow V$ (not necessarily closed) :

$$\int_0^t \langle \omega, \dot{g}_v \rangle dv \leq l(g) + D$$

This is a simple consequence of $\| \, [\omega] \, \|_s = 1$ and of the compactness of V.

Lemma 3.1.5 ends the proof of Proposition 3.1.4 :

Lemma 3.1.5. *For all n,*

$$\int_{\gamma_n \cap (\cup V_k)^c} \omega \geq l(\gamma_n \cap (\cup_{k=1}^R V_k)^c) - A - RK_0 D$$

Proof. We know that $\int_{\gamma_n} \omega \geq l(\gamma_n) - A$ which can be rewritten

$$\begin{aligned} & l(\gamma_n \cap (\cup V_k)) + l(\gamma_n \cap (\cup V_k)^c) - A \\ \leq \quad & \int_{\gamma_n \cap (\cup V_k)} \omega + \int_{\gamma_n \cap (\cup V_k)^c} \omega \\ \leq \quad & l(\gamma_n \cap (\cup V_k)) + RK_0 D + \int_{\gamma_n \cap (\cup V_k)^c} \omega \end{aligned}$$

The last inequality is an consequence of the fact that $\gamma_n \cap (\cup V_k)$ has at most RK_0 connected components in SV. □

□

We can now prove polynomial growth, in the case when M_ω contains only closed geodesics.

3.2. **Counting the geodesics.**

Let us finish the proof of Theorem 1.1.3.

As before, M_ω is the union of a finite number of closed geodesics $c_1, c_2, \cdots, c_R$. For each of them we construct as above a small transverse segment I_k and a neighbourhood $V_{I_k}(c_k) \subset SV$.

Proof of (1). We call $\Gamma(K_0, L_0)$ the set of smooth curves on V, parametrized by arc-length (but not necessarily geodesic nor closed), with endpoints on $\cup_{k=1}^R I_k$, and whose natural lifts to SV satisfy the same conclusion as the geodesics of Propositions 3.1.1 and 3.1.4 : the number of connected components of $\gamma \cap (\cup V_k)$ is less than K_0 for all k, and the time spent outside $\cup V_k$ is less than L_0. Here we always identify the curve $\gamma \in \Gamma(K_0, L_0)$ in V to the curve $(\gamma, \dot{\gamma})$ in SV.

Each curve γ in $\Gamma(K_0, L_0)$ is a succession of segments that we label $a_1, b_1, a_2, b_2, \cdots, a_{n(\gamma)}, b_{n(\gamma)}$ ($b_{n(\gamma)}$ may be reduced to a point) such that :

– the endpoints of the a_i's and b_i's lie on $\cup_{k=1}^R I_k$.

– the a_i's are the connected components of $\gamma \cap \cup_{i=1}^R V_k$, the b_i's are the connected components of $\gamma \cap (\cup_{i=1}^R V_k)^c$.

We identify two elements γ and γ' of $\Gamma(K_0, L_0)$ if $n(\gamma) = n(\gamma')$ and if, for all i, a_i and a'_i (resp. b_i and b'_i) are homotopic in V, with endpoints varying on $\cup_{k=1}^{R} I_k$. We call $\bar{\Gamma}(K_0, L_0)$ the set of such homotopy classes.

Let $\gamma = (a_1, b_1, a_2, b_2, \cdots, a_{n(\gamma)}, b_{n(\gamma)})$ be a representative of an element $\bar{\gamma} \in \bar{\Gamma}(K_0, L_0)$. If $a_i \subset V_k$, we denote $\mid a_i \mid$ the number of intersections of a_i with I_k. It does not depend on the choice of a representative of $\bar{\gamma}$; the same holds for the number $\mid \bar{\gamma} \mid = \sum_{i=1}^{n(\gamma)} \mid a_i \mid$. Besides, there exists a constant $a > 0$ such that, for all $\bar{\gamma} \in \bar{\Gamma}(K_0, L_0)$, for all $\gamma \in \bar{\gamma}$,

$$a^{-1} l(\gamma) - a \leq \mid \bar{\gamma} \mid \leq a l(\gamma) + a$$

(Just note that

$$(\inf l(c_k) - l(I_k)). \mid \bar{\gamma} \mid \leq l(\gamma) \leq (\sup l(c_k) + 2l(I_k)). \mid \bar{\gamma} \mid + L_0$$

by the definition of $\Gamma(K_0, L_0)$.)

Notice also that each element $\bar{\gamma} \in \bar{\Gamma}(K_0, L_0)$ corresponds to at most one closed geodesic (and possibly none, for instance if the end-points of $\bar{\gamma}$ do not lie on the same transverse segment I_k). Thus, in order to prove the first part of Theorem 1.1.3, it is enough to show that $Card\ \{\bar{\gamma} \in \bar{\Gamma}(K_0, L_0), \mid \bar{\gamma} \mid \leq n\}$ grows at most polynomially with n.

(i) Let $\gamma = (a_1, b_1, a_2, b_2, \cdots, a_{n(\gamma)}, b_{n(\gamma)})$ be a representative of an element $\bar{\gamma} \in \bar{\Gamma}(K_0, L_0)$ with $\mid \bar{\gamma} \mid \leq n$. Each a_i is determined up to homotopy by :
– the neighbourhood V_k in which it is contained. There are R choices.
– The number $\mid a_i \mid$. It represents the number of times the component a_i twists around c. One must have $\sum_i \mid a_i \mid \leq n$, so that there are at most $R^{RK_0} n^{RK_0}$ choices.
– the sign of the intersection of a_i with c : there are 2 choices for each a_i, hence 2^{RK_0} choices.

(ii) By definition, $n(\gamma)$ is smaller than RK_0, and $\sum_i l(b_i) \leq L_0$, so that there is only a finite number of ways, say C_{K_0,L_0}, to choose the b_i's, up to homotopy with endpoints varying in $\cup_{i=1}^{R} I_k$.

Finally, (i) and (ii) show that

$$Card\ \{\bar{\gamma} \in \Gamma(K_0, L_0), \mid \bar{\gamma} \mid \leq n\} \leq Q_{K_0,L_0}(n) := C_{K_0,L_0} 2^{RK_0} R^{RK_0} n^{RK_0}$$

Proof of (2). For fixed $n \in \mathbb{N}$, denote $\Gamma(K_0, L_0; n) = \{\gamma \in \Gamma(K_0, L_0), \mid \bar{\gamma} \mid = n\}$ and $\bar{\Gamma}(K_0, L_0; n)$ the associated set of homotopy classes. Choose a representative for each element of $\bar{\Gamma}(K_0, L_0; n)$: $\gamma_1^n, \cdots, \gamma_{p(n)}^n$ with $p(n) = Card\ \bar{\Gamma}(K_0, L_0; n) \leq Q_{K_0,L_0}(n)$. Define $\tilde{L}_0 = \max(L_0, L'_0)$, where L'_0 is the largest possible length of a geodesic segment in one of the V_k's, and intersecting I_k only once. Finally set $t(n) = an + a^2 + 2\tilde{L}_0$.

Consider $T > t(n) + A$ (A is the constant of Observation 1), $\gamma \in \Gamma(\xi, \alpha; T)$ and x belonging to γ in V. We observed earlier that the length $l(\gamma)$ has to be greater than $T - A > t(n)$. In the geodesic γ, consider the segment $\gamma_{[-\frac{t(n)}{2}, \frac{t(n)}{2}]}(x)$ centered at x and of length $t(n)$. Since γ itself belongs to $\Gamma(K_0, L_0)$, and with our choice of $t(n)$, this segment has to contain a smaller segment $\gamma_{[-t_-, t_+]}(x)$ which belongs to $\Gamma(K_0, L_0; n)$. Thus there exists $p \in \{1, \cdots, p(n)\}$ such that $\bar{\gamma}_{[-t_-, t_+]}(x) = \bar{\gamma}_p^n$. Remember that the two segments $\gamma_{[-t_-, t_+]}(x)$ and γ_p^n are homotopic, with end points varying in $\cup_k I_k$. Negative curvature implies that the geometry is hyperbolic : hence, there exists constants $C, \beta > 0$, depending only on the Riemannian metric, such that the point x has to be at distance less than $Ce^{-\beta n}$ from γ_p^n.

We showed that, for all $n \in \mathbb{N}$,

$$\{x \in V, \exists T > t(n), \gamma \in \Gamma(\xi, \alpha; T) \text{ and } x \in \gamma\} \subset \cup_{p=1}^{p(n)} B(\gamma_p^n, Ce^{-\beta n})$$

where $B(\gamma_p^n, Ce^{-\beta n})$ stands for the $Ce^{-\beta n}$-neighbourhood of γ_p^n.

Recall that the curves γ_p^n have length smaller than $an + a^2$, and that $p(n)$ grows at most polynomially with n. Morever, $\cup_{T \leq t(n)} \Gamma(\xi, \alpha; T)$ consists of a finite number of geodesics, for all n. This is enough to show that the closure of $\cup_T \Gamma(\xi, \alpha; T)$ has Lebesgue measure 0 and Hausdorff dimension 1, if it is not empty.

4. The Aubry set; non-polynomial growths

When the Mather set has a minimal exceptional component, it is not so easy to describe the elements of $\Gamma(\xi, \alpha; T)$ and $\Gamma(\omega, A; T)$ using only down-to-earth considerations. In paragraphs 4.1 and 4.2 we derive from Fathi's weak KAM theorem some useful information, similar to Propositions 3.1.1–3.1.4. In paragraph 4.3 we show, however, that it is not sufficient to prove the polynomial growth of $\Gamma(\xi, \alpha; T)$. We also prove that the presence of heteroclinic orbits in the Aubry set implies non-polynomial growth for $\Gamma(\omega, A; T)$.

4.1. More on lagrangian systems : Fathi's weak KAM theorem.

Definition 4.1.1. One defines the Lax-Oleinik semi-groups $(T_t^-)_{t \geq 0}$ and $(T_t^+)_{t \geq 0}$, acting on $C(V, \mathbb{R})$, by

$$T_t^- u(x) = \inf_{\gamma \in C^1([-t,0],V), \gamma(-t)=x} \{u(\gamma(0)) + \int_{-t}^0 \left(\frac{\| \dot{\gamma}_v \|^2}{2} - \omega(\dot{\gamma}_v) \right) dv\}$$

and

$$T_t^+ u(x) = \sup_{\gamma \in C^1([0,t],V), \gamma(0)=x} \{u(\gamma(t)) - \int_0^t \left(\frac{\| \dot{\gamma}_v \|^2}{2} - \omega(\dot{\gamma}_v) \right) dv\}$$

Theorem 4.1.2. *(Fathi,* [Fa1], [Fa2]*)*

(1) There exist fixed points u_-, u_+ for the action of the semi-groups (T_t^-) and (T_t^+) on $C(V, \mathbb{R})$ quotiented by the subspace of constant functions. They necessarily satisfy $T_t^- u_- = u_- - \frac{\|\omega\|_s^2}{2} t$ and $T_t^+ u_+ = u_+ + \frac{\|\omega\|_s^2}{2} t$. Such fixed points are characterized by the two following properties :

– for all C^1 curve $\gamma : [0, t] \longrightarrow V$,

$$u_\pm(\gamma(t)) - u_\pm(\gamma(0)) \leq \int_0^t \left(\frac{\| \dot{\gamma}_v \|^2}{2} - \omega(\dot{\gamma}_v) \right) dv + \frac{\| \omega \|_s^2}{2} t$$

– for all $x \in V$, there exist two curves $\gamma_- : (-\infty, 0] \longrightarrow V$ and $\gamma_+ : [0, +\infty) \longrightarrow V$ with $\gamma_-(0) = \gamma_+(0) = x$, such that, for all $t \geq 0$,

$$u_-(x) - u_-(\gamma_-(-t)) = \int_{-t}^0 \left(\frac{\| \dot{\gamma_-}(v) \|^2}{2} - \omega(\dot{\gamma}_-(v)) \right) dv + \frac{\| \omega \|_s^2}{2} t$$

and

$$u_+(\gamma_+(t)) - u_+(x) = \int_0^t \left(\frac{\| \dot{\gamma_+}(v) \|^2}{2} - \omega(\dot{\gamma}_+(v)) \right) dv + \frac{\| \omega \|_s^2}{2} t$$

(2) For every fixed point of u_- of (T_t^-), there exists a unique fixed point u_+ of (T_t^+) such that u_- and u_+ coincide on M_ω. They satisfy $u_- \geq u_+$. The fixed points u_- and u_+ are then called conjugate.

(3) The Aubry set, defined by

$$A_\omega = \{x \in V, u_-(x) = u_+(x), \forall (u_-, u_+) \text{ a pair of conjugate fixed points}\}$$

is an oriented lamination containing M_ω. There exists a pair of conjugate fixed points that coincide exactly *on A_ω.*

(4) One has the following uniform convergence : for all $\varepsilon > 0$, there exists $T > 0$ such that, for all $x, y \in V$, for all $t \geq T$, for all curve $\gamma : [0,t] \longrightarrow V$, such that $\gamma(0) = x, \gamma(t) = y$, for all pairs (u_-, u_+) of conjugate fixed points,

$$\int_0^t \left(\frac{\| \dot{\gamma}(v) \|^2}{2} - \omega(\dot{\gamma}(v)) \right) dv + \frac{\| \omega \|_s^2}{2} t \geq u_-(y) - u_+(x) - \varepsilon$$

Point (3) implies that A_ω can be lifted, in a unique way, to $\tilde{A}_\omega \subset SV$. Obviously, $\tilde{M}_\omega \subset \tilde{A}_\omega$.

The Ω-limit set of a curve γ defined on $(a, +\infty)$ is the set $\Omega(\gamma) = \cap_{t>a}\overline{\gamma((t, +\infty))}$; the α limit set of γ, defined on $(-\infty, a)$, is $\alpha(\gamma) = \cap_{t<a}\overline{\gamma((-\infty, a))}$. Here one can show ([Fa1]) that the α- and Ω-limit sets $\alpha(g)$ and $\Omega(g)$ of a complete geodesic $g \subset \tilde{A}_\omega$ both contain components of $\tilde{M}_\omega$. On a surface of negative curvature, this implies that the α- and Ω-limit sets actually *are* components of $\tilde{M}_\omega$; indeed, a simple geodesic g which does not intersect the lamination M_ω is either at positive distance from M_ω, or it is asymptotic to a leaf of M_ω ([Ot], Appendix, "La géométrie du complémentaire d'une lamination géodésique").

Remark 4.1.3. We said earlier that the characterization of ω-minimizing measures by their support can be considered a counterpart of the characterization of the equilibrium measures m_ω by their Gibbsian property (Remark 2.2.4). We also mentioned that m_ω could be expressed as the product of the left and right eigenvectors of a "Markovian" operator, the transfer operator (Remark 1.0.2). We see here a similar characterization of the Aubry set, in terms of fixed points of the two Lax-Oleinik semi-groups. We will meet a reason for this analogy in the Appendix : we will explain there how to obtain the Lax-Oleinik semi-groups from the twisted heat semi-group by the procedure of "vanishing viscosity".

Note however that the Aubry set may be larger than the Mather set. The Aubry set is in fact the set of accumulation points of closed, almost-minimizing geodesics, as appears in the following proposition.

Proposition 4.1.4. *(*[Fa1], [Fa2]*) The Aubry set A_ω is the set of points $x \in V$ having the following property : there exists a sequence $\gamma_n : [0, t_n] \longrightarrow V$ of piecewise C^1 paths, such that :*

(a) for all n, $\gamma_n(0) = \gamma_n(t_n) = x$.

(b) $t_n \underset{n \longrightarrow \infty}{\longrightarrow} +\infty$.

(c) $\int_0^{t_n} \left(\frac{\|\dot{\gamma_n}(v)\|^2}{2} - \omega(\dot{\gamma_n}(v)) \right) dv + \frac{\|\omega\|_s^2}{2} t_n \underset{n \longrightarrow \infty}{\longrightarrow} 0$.

The sequence γ_n can be chosen such that the vectors $\dot{\gamma}_n(0)$ and $\dot{\gamma}_n(t_n)$ converge to the unique vector v based at x such that $(x, v) \in \tilde{A}_\omega$.

We first show that Fathi's results impose some restrictions on the elements of $\Gamma(\omega, A; T)$. We give a sufficient condition for Proposition 3.1.1 to hold even when M_ω has an irrational component. However, the observations of the next paragraph show that these restrictions are not strong enough to prove polynomial growth for $\pi(\xi, \alpha; T)$ and $\pi(\omega, A; T)$, so that satisfactory upper bounds for these quantity remain to be found. We notice two things :

– arbitrary small neighbourhoods of the Mather set have positive entropy, if there is an irrational component.

– if the Aubry set contains strictly the Mather set, $\Gamma(\omega, A; T)$ does not grow polynomially.

4.2. **Topological restrictions on quasi-minimizing closed geodesics.** We show that (ω, A)-quasi-minimizing geodesics cannot spend too much time far from the Aubry set.

The following result and its proof were suggested by Massart ([Mas2]). Theorem 4.1.2(4), which is equivalent to the convergence of the Lax-Oleinik semigroup, implies that quasi-minimizing geodesics can spend only bounded time away from the Aubry set. This is a distribution result for optimal orbits, that can be compared to the measure-theoretic equidistribution for general orbits, especially if we remember Remark 4.1.3.

Proposition 4.2.1. *Let W be a neighbourhood of the Aubry set A_ω in V, then there exist constants K and C such that, for every geodesic arc parametrized by arc-length $\gamma : [0, t] \longrightarrow V$, one has*

$$t - \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds \geq Kl(\gamma \cap W^c) + C$$

Proof. We may assume that W is open. Let u_- and u_+ be conjugate fixed points of (T_t^-), (T_t^+) that coincide exactly on A_ω. We apply Theorem 4.1.2 (4) with $\varepsilon = \frac{1}{2} \inf_{W^c}(u_- - u_+)$, which gives us a $T > 0$.

Define a sequence (t_i) in $\mathbb{R} \cup \{+\infty\}$ by $t_0 = \inf\{t \geq 0, \gamma(t) \in W^c$ and $t_{i+1} = \sup\{t \geq t_i + T$ such that $l(\gamma_{|[t_i,t]} \cap W^c) \leq T\}$. If $t_i < +\infty$ then $\gamma(t_i) \in W^c$. Furthermore, $l(\gamma_{|[t_i,t_{i+1}]} \cap W^c) \leq T$.

We have

$$\begin{aligned}
t - \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds &= \sum_{t_{i+1} \leq t} (t_{i+1} - t_i) - \int_{t_i}^{t_{i+1}} \langle \omega, \dot{\gamma}_s \rangle ds \\
&\quad + t_0 - \int_0^{t_0} \langle \omega, \dot{\gamma}_s \rangle ds \\
&\quad + (t - t_n) - \int_{t_n}^t \langle \omega, \dot{\gamma}_s \rangle ds \\
\geq \sum_i (u_-(\gamma_{t_{i+1}}) - u_+(\gamma_{t_i}) - \varepsilon) &+ u_+(\gamma_{t_0}) - u_+(\gamma_0) + u_+(\gamma_t) - u_+(\gamma_{t_n}) \\
\geq -4 \parallel u_+ \parallel_\infty &+ n\varepsilon \\
\geq -4 \parallel u_+ \parallel_\infty &+ \frac{\varepsilon}{T} \sum_i l(\gamma_{|[t_i,t_{i+1}]} \cap W^c) \\
\geq -4 \parallel u_+ \parallel_\infty &+ \frac{\varepsilon}{T} (l(\gamma \cap W^c) - T)
\end{aligned}$$

□

In fact, the stronger proposition holds :

Proposition 4.2.2. *Let $\tilde{W}$ be a neighbourhood of $\tilde{A}_\omega$ in SV, then there exist constants $\tilde{K}$ and $\tilde{C}$ such that, for every geodesic arc parametrized by arclength*

$\gamma : [0,t] \longrightarrow V$, *one has*

$$t - \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds \ \geq \tilde{K} l(\gamma \cap \tilde{W}^c) + \tilde{C}$$

Proof. We may suppose that $\tilde{W}$ is of the form $\cup_{i=1}^n \{(x,v) \in SV, dist\,(x,x_i) \leq \delta, dist\,(v,v_i) \leq 2\rho\delta\}$ where $(x_i,v_i) \in \tilde{A}_\omega$, δ can be chosen as small as needed and ρ is the constant of Lemma 2.1.2. The set $\tilde{W}^c$ is the disjoint union of $A = \cap_{i=1}^n \{x,\ dist\,(x,x_i) > \delta\}$ and of $B = \cup_{i=1}^n \{x,\ dist\,(x,x_i) \leq \delta, dist(v,v_i) > 2\rho\delta\}$. Proposition 4.2.1 shows the existence of K, C such that $t - \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds \geq Kl(\gamma \cap A) + C$.

We now have to prove a similar estimate with B instead of A. We take a as in Lemma 2.1.2, and we partition γ into a succession of intervals of disjoint interiors and length $2a$, and one interval of length less than $2a$. Let $N(\gamma)$ be the number of such segments which contain a point in B; we are going to show the existence of constants C', K' such that $C' + K'N(\gamma) \leq t - \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds$, which is enough to prove the proposition.

Since x_i belongs to the Aubry set, we can use Proposition 4.1.4 and find $T_i \gg a$ and a geodesic $g_i : [0,T_i] \longrightarrow V$ such that

– $g_i(0) = x_i$, $g_i(T_i) = x_i$

– $| \langle [\omega], [g_i] \rangle - T_i | \leq \frac{\eta\delta^2}{2}$

– $dist\,(\dot{g}_i(0), v_i) \leq \rho\delta$ and $dist\,(\dot{g}_i(T), v_i) \leq \rho\delta$.

The definition of $N(\gamma)$ shows that it is possible to find a sequence of times $0 \leq t_1 < \cdots < t_k \leq t$, with $\frac{N(\gamma)}{4} \leq k \leq N(\gamma)$, such that $t_i + a < t_{i+1} - a$, and $(\gamma_{t_i}, \dot{\gamma}_{t_i}) \in B$.

For each $(\gamma_{t_i}, \dot{\gamma}_{t_i})$ we can associate a $j(i)$ such that $dist\,(\gamma_{t_i}, x_{j(i)}) \leq \delta$ and $dist\,(\dot{\gamma}_{t_i}, v_{j(i)}) \geq 2\rho\delta$. We have :

$$t - \int_\gamma \omega + k\frac{\eta\delta^2}{2} \geq t + \sum_{i=1}^k T_{j(i)} - \left(\int_\gamma \omega + \sum_{i=1}^k \langle [\omega], [g_i] \rangle \right) \geq k\eta\delta^2$$

The first inequality comes from our choice of the g_i's; the second, from Lemma 2.1.2 and the fact that $dist\,(\gamma_{t_i}, x_{j(i)}) \leq \delta$, $dist\,(\dot{\gamma}_{t_i}, \dot{g}_i(0)) \geq \rho\delta$ and $dist\,(\dot{\gamma}_{t_i}, \dot{g}_i(T_i)) \geq \rho\delta$. This ends the proof since $k \geq \frac{N(\gamma)}{4}$. □

In particular, the (ω, A)-quasi minimizing geodesics, which satisfy $\langle [\omega], [\gamma] \rangle \geq l(\gamma) - A$, spend a bounded time outside any neighbourhood of $\tilde{A}_\omega$. This is a weaker property than Proposition 3.1.4, which concerned the Mather set $\tilde{M}_\omega$. It is not sufficient to prove polynomial growth, as we will see in paragraph 4.3. In fact, even if we could prove a property similar to Propositions 3.1.1 and 3.1.4 it still would not do.

4.3. **Entropic richness of neighbourhoods of irrational laminations; non-polynomial growth for (ω, A)-minimizing closed geodesics.** We now proceed to the proof of Theorem 1.2.1 :

Proof. If the face F_ω is irrational then there exist a minimal exceptional component c in the Mather set.

There exist two lifts $\tilde{c}_1$ and $\tilde{c}_2$ of leaves of c in $\tilde{V}$ which have no common end, and which can be chosen arbitrarily close from each other, say, at distance $3\epsilon/2$. Let $\tilde{k}$ be the unique geodesic in $\tilde{V}$ perpendicular to $\tilde{c}_1$ and $\tilde{c}_2$: it intersects $\tilde{c}_1$ at a point $\tilde{P}_1$ and $\tilde{c}_2$ at $\tilde{P}_2$, with $d(\tilde{P}_1, \tilde{P}_2) = 3\epsilon/2$. We consider the projections of these objects on V and denote them c_1, c_2, k, P_1, etc...

Let I_1 and I_2 be two small disjoint subsegments of k containing respectively P_1 and P_2; let also $I = I_1 \cup I_2$. We denote T, T_1, T_2 respectively the first return maps on I, I_1, I_2 along leaves of c. Since c is a minimal exceptional component, we note that $T(I_i)$ is not included in I_i for $i = 1, 2$. We use a notation similar to that of paragraph 2.3 : $[x, y]$ for subsegments of k, and $/x, y/$ for pieces of leaves of c.

We first take I_1 and I_2 small enough so that $\inf\{dist(y_1, y_2), y_j \in I_j\} \geq \epsilon$ and $\sup\{dist(y_1, y_2), y_j \in I_j\} \leq 2\epsilon$.

We also apply the following lemma, with θ and $\underline{l}$ to be chosen later. Lemma 4.3.1 is a consequence of the facts that c contains no closed leaf and that tangent vectors to c vary continuously with the base point.

Lemma 4.3.1. *Given $\theta > 0$ and $\underline{l} > 0$ we can choose the intervals I_1 and I_2 small enough so that :*

– for all $y \in I_j \cap c$, $l(/y, T_j(y)/) \geq \underline{l}$.

– for all $y \in I_j \cap c$, the oriented angle between the leaf of c at y and I_j is in $[\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta]$.

We will need to take θ and $\underline{l}$ such that the following lemma holds :

Lemma 4.3.2. *Let ϵ be a positive constant. For $\theta > 0$ small enough, there exists $\underline{l} > 0$ such that, if M, N, P, Q are four points in $\tilde{V}$ satisfying :*

(1) $\epsilon \leq dist(N, P) \leq 2\epsilon$

(ii) The angle $\widehat{MNP}$ between the geodesic segments MN (resp. PQ) and NP is in $[\frac{\pi}{2} - 2\theta, \frac{\pi}{2} + 2\theta]$

(iii) $dist(M, N) \geq 2\underline{l}$ and $dist(P, Q) \geq 2\underline{l}$

Then $dist(M, Q) \geq dist(M, N) + \underline{l}$ and the angle between the geodesic segments MQ and PQ is smaller than θ.

Proof. A proof is given in [Ot] (p 128) in the case of constant negative curvature. We show how to adapt it to the case of variable curvature. The curvature of $\tilde{V}$ is bounded above by a certain negative constant, say -1. We recall the trigonometric formula for a metric of constant curvature -1 : if ABC is a geodesic triangle then $\cosh BC = \cosh AB \cosh AC - \cos \widehat{CAB} \sinh AB \sinh AC$. We also recall the comparison principles : (1) on $\tilde{V}$ a geodesic triangle has smaller angles than the triangle with same side-lengths in the Poincaré disc (ii) for given side-lengths AB, AC and angle $\widehat{CAB}$, the length of the side BC is greater on $\tilde{V}$ than on the Poincaré disc.

In $\tilde{V}$ one can write the following trigonometric inequalities for triangle MPN (we write in boldfaced letters the quantities which are given by the assumptions of the lemma) :

$$\cosh MP \geq \cosh \mathbf{MN} \cosh \mathbf{NP} - \cos \widehat{\mathbf{MNP}} \sinh \mathbf{MN} \sinh \mathbf{NP}$$

$$\cos \widehat{NPM} \geq \frac{\cosh MP \cosh \mathbf{NP} - \cosh \mathbf{MN}}{\sinh MP \sinh \mathbf{NP}}$$

and

$$\widehat{MPQ} \geq \widehat{\mathbf{NPQ}} - \widehat{NPM}$$

For triangle MQP :

$$\cosh MQ \geq \cosh MP \cosh \mathbf{PQ} - \cos \widehat{MPQ} \sinh MP \sinh \mathbf{PQ}$$

and

$$\cos \widehat{MQP} \geq \frac{\cosh MQ \cosh \mathbf{PQ} - \cosh MP}{\sinh MQ \sinh \mathbf{PQ}}$$

So all the equalities used in [Ot] are replaced by convenient inequalities, and the same estimates can be applied. Using the equivalence $\cosh(t) \sim \sinh(t) \sim \frac{e^t}{2}$ as t goes to $+\infty$, one gets that if $MN \geq 2\underline{l}$ and $PQ \geq 2\underline{l}$ with $\underline{l}$ large enough, and if θ is small enough, then $MQ \geq MN + \underline{l}$ and the angle $\widehat{MQP}$ is small as announced.

□

Remark : Once $I = I_1 \cup I_2$ has been fixed, there also exists $\bar{l} > 0$ such that, for all $y \in I_j \cap c$, $l(/y, T_j(y)/) \leq \bar{l}$.

We now pick an arbitrary T-invariant ergodic probability measure m on I, and $x \in I_1$ a generic point for m.

Since c is a minimal exceptional component, necessarily $m(I_j) > 0$ $(j = 1, 2)$; Birkhoff's ergodic theorem shows the existence of $m_j > 0$ such that

$$m_j N \leq Card\,\{0 \leq k < N, T^k(x) \in I_j\} \leq (m_j + 1)N$$

For all N we consider the T-trajectory of length N, $(x, Tx, \cdots, T^N x)$ and denote

$$c_N = [T^N x, x]./x, T^N x/$$

the corresponding portion of leaf of c, closed with a piece of k (the dot denotes the composition of paths, to be read as usual from right to left).

We fix an arbitrary origin O on k, lying between I_1 and I_2. For each $y \in I \cap c$ we construct the loop $g^y = [Ty, O]./y, Ty/.[O, y]$ based at O. We denote $l(g^y)$ its length. We also construct the loops $g_j^y = [T_j y, O]./y, T_j y/.[O, y]$, $j = 1, 2$.

The union of the two transverse segments I_1 and I_2 defines a train-track neighbourhood of the lamination c, independently of the choice of the metric g or g_0; this neighbourhood can be made arbitrarily small. Notice that, for all N and for all permutation σ of the set $\{0, \cdots, N-1\}$, one has, for either of the two metrics :

$$| \, l(g^{T^{\sigma(0)}x}.g^{T^{\sigma(2)}x}. \cdots .g^{T^{\sigma(N-1)}x} - l(c_N) \, | \leq 4N\epsilon \leq 4\frac{l(c_N)}{\underline{l}}\epsilon \tag{4.3.1}$$

This means that permuting pieces of c_N multiplies its length by a factor less than $(1 + \frac{4\epsilon}{\underline{l}})$, which can be made arbitrarily close to 1. Of course these permutations do not change the homology class. We have to be a little careful to see that these permutations will give a lot of different homotopy classes.

Lemma 4.3.3. *Let $a > 0$ be a number such that $a < m_1$ and $a(1 + \frac{\bar{l}}{\underline{l}}) < m_2$. Then, for large enough N, one has :*

- $\{x, T_1 x, \cdots, T_1^{aN} x\} \subset \{x, Tx, \cdots, T^N x\}$
- *if we set $x_2 = T_2(T_1^{aN} x)$, then $\{x_2, T_2 x_2, \cdots, T_2^{aN} x_2\} \subset \{x, Tx, \cdots, T^N x\}$*
- *the leaves $/x, T_1^{aN} x/$ and $/x_2, T_2^{aN} x_2/$ do not intersect.*

Proof. We know that the trajectory $(x, Tx, \cdots, T^N x)$ hits I_1 at least $m_1 N$ times. This implies that, for large enough N, one has $\{x, T_1 x, \cdots, T_1^{aN} x\} \subset \{x, Tx, \cdots, T^N x\}$. The leaf $/x, T_1^{aN} x/$ hits I_2 at most $\frac{\bar{l}}{\underline{l}} aN$ times.

We also know that the trajectory $(x, Tx, \cdots, T^N x)$ hits I_2 at least $m_2 N$ times. If we take $x_2 = T_2 T_1^{aN} x$, this implies that $\{x_2, T_2 x_2, \cdots, T_2^{aN} x_2\} \subset \{x, Tx, \cdots, T^N x\}$. □

After doing this construction we denote

$$g = [T^N x, T_1^{aN} x]./T_2^{aN} x_2, T^N x/.[x_2, T_2^{aN} x_2]./T_1^{aN} x, x_2/$$

Let $\nu = T_{i_1} \circ \cdots \circ T_{i_{2aN}}$ be a word of length $2aN$ in the alphabet $\{T_1, T_2\}$, containing exactly aN times each letter. We associate to ν an element $\gamma_\nu \in \Gamma$; it is obtained by combining the loops $g_j^{T_j^k x_j}$ ($k < aN$, $x_1 = x$, x_2 as above) according to the word ν, then by composing with g : for instance if $aN = 2$ and $\nu = T_1 T_2 T_2 T_1$ we set $\gamma_\nu = g g_1^{T_1 x_1} g_2^{T_2 x_2} g_2^{x_2} g_1^{x}$.

This curve is obtained by permuting pieces of c_N as described above, so it has approximately the same length as c_N. Moreover, all curves obtained this way have different homotopy classes :

Lemma 4.3.4. *If ν_1, ν_2 are two distinct words, then $\gamma_{\nu_1} \neq \gamma_{\nu_2}$*

Proof. The proof is a classical tool in hyperbolic geometry; we extracted this proof from [Ot], p 128. Since we are interested in homotopy classes we can choose to work with the metric g_0. Remember the constants θ and $\underline{l}$ of Lemmas 4.3.1 and 4.3.2.

Suppose that $\gamma_{\nu_1} = \gamma_{\nu_2}$, then $\gamma_{\nu_1}^{-1}\gamma_{\nu_2}$ is a closed path on V which lifts to a closed path in $\tilde{V}$. Besides, this closed path in $\tilde{V}$ is a succession of geodesic segments : every other segment is a lift of a piece of leaf of c, of length greater than $\underline{l}$, and every other segment if a lift of a piece of k, of length in $[\epsilon, 2\epsilon]$. The angles between these segments are in $[\frac{\pi}{2} - \theta, \frac{\pi}{2} + \theta]$.

Call P_{2i}, P_{2i+1} the edges of this polygonal path, cyclically ordered. Applying Lemma 4.3.2 to the points (P_0, P_1, P_2, P_3), we see that

$$dist(P_0, P_3) \geq dist(P_0, P_1) + \underline{l}$$

and the angle between $P_0 P_3$ and $P_3 P_4$ is in $[\frac{\pi}{2} - 2\theta, \frac{\pi}{2} + 2\theta]$. Applying inductively the same argument to the points $P_0 P_{2i-1} P_{2i} P_{2i+1}$ shows that $dist(P_0, P_{2i+1})$ is strictly increasing. This is impossible if the sequence (P_n) is periodic. □

This way we get $C_{2aN}^{aN} = \frac{(2aN)!}{(aN)!^2} \sim \frac{2^{2aN}}{\sqrt{\pi a N}}$ distinct homotopy classes with same homology class and represented by closed geodesics of same length, up to a multiplicative constant close to 1. It remains to show that the constructed curves satisfy the condition of Theorem 1.2.1, that is to say, they belong to some set $\Gamma(\omega, \varepsilon, A; T)$. Lemma 4.3.5 below, as well as inequality (4.1), show that this is true for $T = l(c_N)(1 + \frac{4\epsilon}{\underline{l}})$ and $1 - \varepsilon = \frac{1}{1 + \frac{4\epsilon}{\underline{l}}}$.

Proposition 4.3.5. *There exists a constant A such that, for all N,*

$$\langle [\omega], [c_N] \rangle - l(c_N) \geq -A$$

Remember that c_N is a piece of a leaf of c, closed with a small piece of the transversal k. Proposition 4.3.5 is a consequence of the following result of Mañe :

Theorem 4.3.6. ([Mn1]) *There exists a Lipschitz function u on V such that, if $v \in \tilde{M}_\omega$ and γ is the geodesic with initial velocity v, then for all $t \in \mathbb{R}$,*

$$u(\gamma(0)) - u(\gamma(t)) = \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds - t$$

Proof. Fathi showed that one can actually take for u any fixed point of the Lax-Oleinik semi-group (T_t^-) : Theorem 4.2 says that such a function u satisfies

$$u(p(\phi_{t'}v)) - u(p(\phi_t v)) \leq \int_t^{t'} \left(\frac{\| \phi_u v \|^2}{2} - \omega(\phi_u v) \right) du + \frac{\| \omega \|_s^2}{2}(t'-t)$$

for all $v \in SV$, for $t \leq t'$. Here in fact $\| \phi_u v \|= 1$ and $\| [\omega] \|_s = 1$.

Now notice that the integrals of both sides of the previous inequality with respect to any ω-minimizing measure μ vanish : for the left-hand side, this comes from the (ϕ_t)-invariance of μ, and for the right-hand side one needs to remember that $\int_{SV} \omega d\mu = 1$. Hence, the above inequality is in fact an equality on the support of μ, for all ω-minimizing measure μ; so that it is an equality on the Mather set. This proves Theorem 4.3.6. □

In particular, there exists A such that $\mid \int_0^t \langle \omega, \dot{\gamma}_s \rangle ds - t \mid \leq A$.
This ends the proof of Theorem 1.2.1. □

Eventually we prove Theorem 1.1.5 :

Proof. Let us prove (1). Assume that F_ω is a rational face and that $A_\omega = M_\omega$. We showed in paragraph 4.2 that (ω, A)-minimizing geodesics spend bounded time outside any neigbourhood of $\tilde{A}_\omega = \tilde{M}_\omega$; this means that Proposition 3.1.4 still holds for geodesics $\gamma \in \Gamma(\omega, A; T)$, and the argument of paragraph 3.2 remains the same. (Besides, if A is small enough, Proposition 4.2.2 also shows that the elements of $\Gamma(\omega, A; T)$ have to stay in a union of collar neighbourhoods of the components of M_ω; so, each element of $\Gamma(\omega, A; T)$ is a multiple of one of these components).

We now turn to the second statement. Let $x \in A_\omega \setminus M_\omega$. By Proposition 4.1.4, there exists a sequence $t_n \longrightarrow \infty$ and curves (which can be taken to be geodesics parametrized by arclength) $\gamma_n : [0, t_n] \longrightarrow V$ such that $\gamma_n(0) = \gamma_n(t_n) = x$, and

$$\langle [\omega], [\gamma_n] \rangle - l(\gamma_n) \geq -\frac{A}{n}$$

Besides, the two tangent vectors to γ_n at x will converge to the tangent vector to A_ω at x. Remember that leaves of $A_\omega \setminus M_\omega$ spiral towards components of M_ω, so there exists a component c of M_ω such that $dist(\gamma_n, c) \longrightarrow 0$.

Let I be a small segment transverse to c, and of length smaller than A. For all n we divide I into $2n$ segments $I_1^{(n)}, \cdots, I_{2n}^{(n)}$ of equal size, and we require the following on the curve γ_n : if $c \cap I_j^{(n)} \neq \emptyset$ then $\gamma_n \cap I_j^{(n)} \neq \emptyset$.

Recall that t_n is the length of γ_n and that $t_n \underset{n \longrightarrow \infty}{\longrightarrow} +\infty$.

Let l be the maximum of the first return time to I along c. There exists a portion of a leaf of c of length less than $t_n.l$ and with endpoints distant at most from $\frac{A}{t_n}$. Closing this leaf along I we get a closed curve c_n; using Theorem 4.3.6, we have

$$\mid \langle [\omega], [c_n] \rangle - l(c_n) \mid \leq \frac{CA}{t_n}$$

where $C = \mid u \mid_{lip} + \sup_x \mid \omega_x \mid +1$.

From our construction of γ_n, there exist two points $P_n \in c_n \cap I$ and $Q_n \in \gamma_n \cap I$ such that $dist(P_n, Q_n) \leq \frac{A}{n}$. For n sufficiently large, the closed curves c_n and $\gamma_n' = [P_n, Q_n].\gamma_n.[Q_n, P_n]$, based at P_n, generate a free group in the fundamental group : otherwise they would commute and since these two curves are almost closed geodesics, they would come very close in Hausdorff distance as $n \longrightarrow \infty$. Hence it would be possible to find points of M_ω arbitrarily close to x, which is not the case.

We now consider all the following closed curves, distinct in the fundamental group :

$$c_n^{k_0}.\gamma_n'.c_n^{k_1}\cdots\gamma_n'.c_n^{k_n}$$

with $k_0+\cdots+k_n=t_n$. Their lengths are less than $l.t_n^2+nt_n+2A$; they all belong to $\Gamma(\omega,A(2C+1);l.t_n^2+nt_n+2A)$. To conclude the proof of Theorem 1.1.5 it suffices to check that the number of such distinct curves we have constructed, namely $\frac{(t_n+n)!}{n!t_n!}$, grows faster than any polynomial in l_n; this results from an application of the Stirling formula.

□

Appendix : on twisted Laplace operators and Lax-Oleinik semi-groups

In this appendix, independent from the rest of the paper, we establish a relation between Lax-Oleinik semi-groups and twisted heat semi-groups, by the method of "vanishing viscosity". We obtain a relation between the stable norm and the largest eigenvalue of heat semi-groups twisted by a real valued 1-form. We also prove a large deviation upper bound for the stationary measure of the twisted Brownian motion; the rate function is the second Peierls barrier of Hamiltonian mechanics.

Let V be a smooth, connected, compact manifold, without boundary, endowed with a Riemannian metric. This metric gives rise to a Laplace operator and to a stochastic process, the Brownian motion on V.

We denote Δ the usual Laplace operator on V, and $\tilde{\Delta}$ the corresponding operator on the universal covering $\tilde{V}$.

For every $x\in V$, we denote $\mathbb{P}_x$ the Wiener measure on $C(\mathbb{R},V)$, started at x. Under $\mathbb{P}_x$, the coordinate functions on $C(\mathbb{R},V)$, $X_t:\gamma\mapsto\gamma(t)$, are a realization of the Brownian motion started at x. We will denote $\mathbb{E}_x$ the expectation with respect to the measure $\mathbb{P}_x$. These objects can also be defined on $\tilde{V}$, where they will be denoted similarly with a *tilde*.

Brownian motion and the operator Δ are linked the following way : if f is a smooth function on V,

$$\exp\frac{t\Delta}{2}.f(x)=\mathbb{E}_x(f(X_t))$$

for all $x\in V$, $t\in\mathbb{R}$. The semi-group of operators $P^t=\exp\frac{t\Delta}{2}$ is the heat semi-group on V. We refer the reader to [C] for a very complete account of the spectral properties of the heat semi-group on a Riemannian manifold.

We now define twisted Laplace operators and semi-groups. If $\omega=\Re\omega+\sqrt{-1}\Im\omega$ is a closed 1-form on V with complex values, and $\tilde{\omega}$ its lift to the universal cover $\tilde{V}$, the twisted Laplace operator $\tilde{\Delta}_\omega$ is defined as follows :

$$\tilde{\Delta}_\omega\tilde{f}(\tilde{x})=e^{-\int_{\tilde{o}}^{\tilde{x}}\tilde{\omega}}\,\tilde{\Delta}\big(e^{\int_{\tilde{o}}^{\tilde{x}}\tilde{\omega}}\tilde{f}(\tilde{x})\big)$$

where $\tilde{o}$ is a fixed origin on $\tilde{V}$.

This operator acts on smooth functions on $\tilde{V}$ and preserves the set of $\pi_1(V)$-invariant functions. Thus we can define a twisted Laplace operator Δ_ω acting on smooth functions on V, by

$$\Delta_\omega f(z)=\tilde{\Delta}_\omega\tilde{f}(\tilde{z})$$

where $\tilde{f}$ is the lift of f and $\tilde{z}$ lies over z. If ω is harmonic, Δ_ω admits the expression :

$$(4.3.2) \qquad \Delta_\omega f = \Delta f + 2\langle \omega, df\rangle + \parallel \omega \parallel^2 f$$

where $\parallel . \parallel^2 = \langle ., . \rangle$ is the Riemannian metric.

Twisted Laplace operators have been classically studied for $\Re\omega = 0$ (i.e ω taking values in $\sqrt{-1}\mathbb{R}$), on a manifold with constant negative sectional curvatures ([KS1], [PhS]). In these papers they are used to compute the asymptotics of the number of closed geodesics in a fixed homology class, via Selberg's trace formula; see also [CV] for an investigation of the trace formula in variable curvature.

Here we are mainly interested in the case $\Im\omega = 0$ (i.e ω taking values in $\mathbb{R}$). The twisted Laplace operators are of very different natures when $\Re\omega = 0$ and $\Im\omega = 0$. We recall their spectral properties in both cases, as well as those of the twisted heat semigroups

$$P^t_\omega = \exp \frac{t\Delta_\omega}{2}$$

which give the solutions of

$$(4.3.3) \qquad \frac{\partial u}{\partial t} = \frac{\Delta_\omega}{2} u$$

These are kernel operators : the kernel of P^t_ω is given by the expression

$$(4.3.4) \qquad K^t_\omega(x,y) = \sum_{\gamma\in\pi_1(V)} e^{\int_{\tilde{x}}^{\gamma\tilde{y}} \tilde{\omega}} \tilde{K}^t(\tilde{x}, \gamma\tilde{y})$$

$$(4.3.5) \qquad = e^{\int_{\tilde{x}}^{\tilde{y}} \tilde{\omega}} \sum_{\gamma\in\pi_1(V)} e^{\langle\omega,[\gamma]\rangle} \tilde{K}^t(\tilde{x}, \gamma\tilde{y})$$

where $\tilde{K}$ is the heat kernel on $\tilde{V}$.

For $\Re\omega = 0$, Δ_ω has negative, self-adjoint compact inverse on $L^2(V)$; so its spectrum is a sequence of negative real numbers which tends to $-\infty$ (we have not renormalized the Laplace operator to be positive, as is often the case).

For $\Im\omega = 0$, our operators are no longer self-adjoint : the adjoint of P^t_ω is $P^{t\,*}_\omega = P^t_{-\omega}$. On the other hand, heat semi-groups twisted by real 1-forms have the property that $f \geq 0 \Rightarrow P^t_\omega f \geq 0$. They can be seen as transition operators for a Markov process, except that they are not stochastic ($P^t_\omega 1 \neq 1$ if $[\omega] \neq 0$ in $H^1(V, \mathbb{R})$). It is natural to consider the dual action of the semi-group on Borel measures on V, given by

$$\int_V f d(P^{t*}_\omega \mu) = \int_V P^t_\omega f d\mu$$

Notice that it preserves the set of positive measures. Moreover, it extends the action of the adjoint $P^{t*}_\omega = P^t_{-\omega}$ already defined on $L^2(V)$, in the sense that $P^{t*}_\omega(f dx) = (P^{t*}_\omega f)dx$ if dx is the Lebesgue measure on V.

By a classical fixed point argument, there exists a real $\Lambda(\omega)$ and a probability measure μ_ω on V such that $P^{t*}_\omega \mu_\omega = e^{t\Lambda(\omega)}\mu_\omega$ for all $t \geq 0$. If we let

$$h_\omega(x) = \int_V K^1_{-\omega}(y,x) d\mu_{-\omega}(y)$$

we get a smooth positive function such that $P^t_\omega h_\omega = e^{t\Lambda(-\omega)} h_\omega$ for all $t \geq 0$. It follows that $\Lambda(\omega) = \Lambda(-\omega)$, and that the measure $d\nu_\omega = h_\omega h_{-\omega} dx$ is (up to a normalizing factor) the stationary probability measure for the "twisted Brownian motion", i.e the diffusion defined by the (stochastic) transition semi-group :

$$Q^t f(x) = e^{-t\Lambda(\omega)} h_\omega(x)^{-1} P^t_\omega(h_\omega f)(x)$$

For a continuous function f on V, one has the following convergence :

$$e^{-t\Lambda(\omega)} P^t_\omega f(x) \underset{t\longrightarrow+\infty}{\longrightarrow} h_\omega(x) \int_V f h_{-\omega}(y) dy$$

uniformly in x. This implies that $e^{\Lambda(\omega)}$ is the eigenvalue of largest modulus for the twisted semi-group, and it is simple, isolated.

For general ω it is difficult to say something about the spectrum of (P^t_ω). Clearly the spectral radius of P^t_ω on continuous functions is less than that of $P^t_{\Re\omega}$. Also, $(P^t_\omega)_{\omega\in H^1(V,\mathbb{C})}$ is an analytic family of semi-groups, which have a simple isolated eigenvalue of maximum modulus for ω real; by perturbation theory ([K]), one can say that in a complex neighbourhood of $\{\Im\omega = 0\}$ the semi-group $(P^t_\omega)_{t\geq 0}$ will have a simple isolated eigenvalue of maximum modulus, which depends analytically on ω.

We want to relate these semi-groups to the two Lax-Oleinik (or Hopf-Lax) semi-groups, acting on the space of continuous functions $C(V,\mathbb{R})$ the following way : if $u \in C(V,\mathbb{R})$,

$$T^-_t u(x) = \inf_\gamma \{u(\gamma(0)) + \int_{-t}^0 \left(\frac{\| \dot\gamma_v \|^2}{2} - \omega(\dot\gamma_v) \right) dv\}$$

where the inf is taken over all curves $\gamma : [-t,0] \longrightarrow V$ with square integrable derivatives, and such that $\gamma(-t) = x$. Similarly,

$$T^+_t u(x) = \sup_{\gamma:[0,t]\longrightarrow V, \gamma(0)=x} \{u(\gamma(t)) - \int_0^t \left(\frac{\| \dot\gamma_v \|^2}{2} - \omega(\dot\gamma_v) \right) dv\}$$

(These semi-groups depend on a real-valued 1-form ω, which will be fixed from now on.)

In [Fa2] Fathi demonstrated the importance of the "fixed points" of Lax-Oleinik semi-groups in the description of globally minimizing orbits for the lagrangian $\frac{\|v\|^2}{2} - \omega(v)$ – we mean fixed points for the action of T^+ and T^- on $C(V,\mathbb{R})$ quotiented by the subspace of constant functions. Fathi also showed that these fixed points provide weak solutions to the KAM problem for this Hamiltonian system.

We give a proof of the following relation :

Theorem 4.3.7. *For all $t \geq 0$, for every continuous function u, for all $x \in V$,*

$$\lim_{\lambda\longrightarrow+\infty} \frac{1}{\lambda} \log \left(P^{t/\lambda}_{\lambda\omega} . e^{\lambda u} \right)(x) = T^+_t u(x)$$

and

$$\lim_{\lambda\longrightarrow+\infty} \frac{1}{\lambda} \log \left(P^{t/\lambda}_{-\lambda\omega} . e^{-\lambda u} \right)(x) = -T^-_t u(x)$$

Theorem 4.3.7 is an application of the method of "vanishing viscosity", a technique originally studied by E. Hopf ([Ho]) for the Burgers equation and other PDE. Here, the logarithm of the solution of the heat equation (4.3.3), $P^{t/\lambda}_{\lambda\omega} . e^{\lambda u}(x)$, satisfies after correct scaling a Hamilton-Jacobi equation with a viscosity term, which tends to 0 as λ tends to $+\infty$. At the limit, one gets a lispchitz viscosity solution to the Hamilton-Jacobi equation :

$$\frac{\partial u}{\partial t} + \frac{\| d_x u + \omega_x \|^2}{2} = 0$$

Given the initial condition u, it is known that the Lax-Oleinik semi-group gives such a solution : $(x,t) \mapsto T_t^+ u(x)$. In various cases ([PLL], [Ba]), the uniqueness of lipschitz solutions allows to conclude the proof of Theorem 4.3.7. However we do not know if such a uniqueness theorem has been written down in the general case of a Hamiltonian system on a manifold; one can expect that the ideas presented in [PLL], [Ba], can be adapted. We give a probabilistic proof of 4.3.7 in the spirit of the works of Schilder ([S]) and Varadhan ([Var]), which allows to overlook this gap.

Proof. We use the following estimates on the heat kernel on $\tilde{V}$:

Theorem 4.3.8. *(see* [Var] *for results on* $\mathbb{R}^n$*,* [C]*,* [LY] *for much stronger estimates)*

(i) Let $n = \dim V$*, and let* κ *be a non-positive lower bound on the Ricci curvature of* $\tilde{V}$*. Then, there exists a constant* $c = c(n)$ *and, for all* $\delta > 0$*, a constant* $C = C(n, \delta)$ *such that, for all* $\tilde{x}, \tilde{y} \in \tilde{M}$*,*

$$\tilde{K}^t(\tilde{x}, \tilde{y}) \leq \frac{C}{\Big(Vol\ B(\tilde{x}, \sqrt{t}) Vol\ B(\tilde{y}, \sqrt{t})\Big)^{1/2}} \exp\Big\{ - \frac{d^2(\tilde{x}, \tilde{y})}{(2+\delta)t} - c\delta\kappa t \Big\}$$

(ii)

$$\lim_{t \longrightarrow 0} t \log \tilde{K}^t(\tilde{x}, \tilde{y}) = -\frac{1}{2} d^2(\tilde{x}, \tilde{y})$$

uniformly for all $\tilde{x}, \tilde{y}$ *such that the distance* $d(\tilde{x}, \tilde{y})$ *is bounded.*

These estimates imply in particular Schilder's large deviation theorem – but this latter was proved earlier, for Brownian motion on $\mathbb{R}^n$:

Theorem 4.3.9. *(*[S]*)*

For $\varepsilon > 0$*, define* $\alpha_\varepsilon : C(\mathbb{R}, \tilde{V}) \longrightarrow C(\mathbb{R}, \tilde{V})$ *by* $\alpha_\varepsilon \gamma(t) = \gamma(\varepsilon t)$*.*

Let F *be a positive, continuous function on* $C([0,T], \tilde{V})$*, endowed with the topology of uniform convergence. Then, for all* $\tilde{x} \in \tilde{V}$*,*

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon \log \tilde{\mathbb{E}}_{\tilde{x}}((F \circ \alpha_\varepsilon)^{1/\varepsilon}) = \sup_\gamma F(\gamma) - \frac{1}{2} \int_0^T \| \dot{\gamma}_s \|^2 \, ds$$

where the supremum is taken over the set of curves $\gamma : [0,T] \longrightarrow \tilde{V}$ *with square integrable derivative, and such that* $\gamma(0) = \tilde{x}$*.*

We can now prove Theorem 4.3.7; we use Schilder's theorem as well as the relation between Brownian motion and the heat semi-group on $\tilde{V}$. Let $t \geq 0$, u a continuous function on V, and $\tilde{u}$ its lift to $\tilde{V}$.

$$\begin{aligned}
\lim_{\lambda\longrightarrow+\infty}\frac{1}{\lambda}\log\big(P^{t/\lambda}_{\lambda\omega}.e^{\lambda u}\big)(x) &= \lim\frac{1}{\lambda}\log\tilde{\mathbb{E}}_{\tilde{x}}\big(e^{\lambda\tilde{u}(\tilde{X}_{t/\lambda})+\lambda\int_{\tilde{X}_0}^{\tilde{X}_{t/\lambda}}\tilde{\omega}}\big)\\
&= \sup_{\gamma:[0,t]\longrightarrow\tilde{V},\gamma(0)=\tilde{x}}\{\tilde{u}(\gamma(t))-\int_0^t\left(\frac{\|\dot{\gamma}_v\|^2}{2}-\tilde{\omega}(\dot{\gamma}_v)\right)dv\}\\
&= \sup_{\gamma:[0,t]\longrightarrow V,\gamma(0)=x}\{u(\gamma(t))-\int_0^t\left(\frac{\|\dot{\gamma}_v\|^2}{2}-\omega(\dot{\gamma}_v)\right)dv\}
\end{aligned}$$

where we recognize the expression of the Lax-Oleinik semi-group (T_t^+). Similarly,

$$\begin{aligned}
\lim_{\lambda\longrightarrow+\infty}\frac{1}{\lambda}\log\big(P^{t/\lambda}_{-\lambda\omega}.e^{-\lambda u}\big)(x) &= \sup_{\gamma(0)=x}\{-u(\gamma(t))-\int_0^t\left(\frac{\|\dot{\gamma}_v\|^2}{2}+\omega(\dot{\gamma}_v)\right)dv\}\\
&= -\inf_{\gamma(-t)=x}\{u(\gamma(0))+\int_{-t}^0\left(\frac{\|\dot{\gamma}_v\|^2}{2}-\omega(\dot{\gamma}_v)\right)dv\}
\end{aligned}$$

where the second equality is obtained by reversing time. □

We now establish the asymptotic behaviour of the eigenvalue $\Lambda(\omega)$. We denote $\|\,.\,\|_s$ the stable norm on $H^1(V,\mathbb{R})$. For a real-valued 1-form ω, $\frac{\|[\omega]\|_s^2}{2}$ is the *critical value* – introduced by Mañe ([Mn2]) – of the Lagrangian $v\mapsto\frac{\|v\|^2}{2}-\omega(v)$.

Theorem 4.3.10.

$$\frac{\Lambda(\lambda\omega)}{\lambda^2}\underset{\lambda\longrightarrow\infty}{\longrightarrow}\frac{\|[\omega]\|_s^2}{2}$$

Lemma 4.3.11. *The family $(\frac{1}{\lambda}\log h_{\lambda\omega})_{\lambda>0}$ is uniformly equicontinuous on V.*

Proof. Up to a multiplicative constant, $h_{\lambda\omega}$ is given by the expression

$$h_{\lambda\omega}(x)=\int_V K^t_{-\lambda\omega}(y,x)d\mu_{-\lambda\omega}(y)$$

for any t (remember that $\mu_{-\lambda\omega}$ is the invariant measure for the semi-group $(P^t_{-\lambda\omega})$, and that its density is precisely $h_{\lambda\omega}$). Take $t=1/\lambda$. The expression (4.3.4), and Theorem 4.3.8, show that

$$\lim_{\lambda\longrightarrow+\infty}\frac{1}{\lambda}\log K^{1/\lambda}_{-\lambda\omega}(y,x)=-\inf_{\gamma\in\pi_1(V)}\frac{d^2(\tilde{y},\gamma\tilde{x})}{2}+\int_{\tilde{y}}^{\gamma\tilde{x}}\tilde{\omega}$$

uniformly in $x,y\in V$. The function

$$\begin{aligned}
I(y,x) &:= \inf_{\gamma\in\pi_1(V)}\frac{d^2(\tilde{y},\gamma\tilde{x})}{2}+\int_{\tilde{y}}^{\gamma\tilde{x}}\tilde{\omega}\\
&= \inf_{\gamma\in C^1([0,1],V),\gamma(0)=y,\gamma(1)=x}\int_0^1\left(\frac{\|\dot{\gamma}_s\|^2}{2}+\omega(\dot{\gamma}_s)\right)ds
\end{aligned}$$

is lipschitz. Write, for all $x, y \in V$

$$\begin{aligned}
\frac{1}{\lambda}\log h_{\lambda\omega}(x) - \frac{1}{\lambda}\log h_{\lambda\omega}(y) &= \frac{1}{\lambda}\log\left(\frac{\int_V K^{1/\lambda}_{-\lambda\omega}(z,x)d\mu_{-\lambda\omega}(z)}{\int_V K^{1/\lambda}_{-\lambda\omega}(z,y)d\mu_{-\lambda\omega}(z)}\right)\\
&\leq \frac{1}{\lambda}\log\left(\sup_z \frac{K^{1/\lambda}_{-\lambda\omega}(z,x)}{K^{1/\lambda}_{-\lambda\omega}(z,y)}\right)\\
&\underset{\lambda\longrightarrow+\infty}{\longrightarrow} \sup_z I(z,x) - I(z,y)
\end{aligned}$$

uniformly in x, y. This shows that the family $(\frac{1}{\lambda}\log h_{\lambda\omega})_{\lambda>0}$ is uniformly equicontinuous on V.

Thus, if we normalize $h_{\lambda\omega}$ so that $h_{\lambda\omega}(x_0) = 1$ for some fixed origin x_0, we get a relatively compact family in $C(V, \mathbb{R})$.

Let $\bar\Lambda$ be a limit (in $\mathbb{R}\cup\{-\infty,+\infty\}$) of $\frac{\Lambda(\lambda_n\omega)}{\lambda_n^2}$, for some sequence $\lambda_n \longrightarrow +\infty$; we may assume that $\frac{1}{\lambda_n}\log h_{\lambda\omega_n}$ converges to a continuous function u in the uniform topology.

One has, for all x and t,

$$\begin{aligned}
&u(x) + \bar\Lambda t = \lim \frac{1}{\lambda_n}\log\left(e^{\frac{t}{\lambda_n}\Lambda(\lambda_n\omega)}h_{\lambda_n\omega}(x)\right)\\
&= \lim\frac{1}{\lambda_n}\log\left(P^{t/\lambda_n}_{\lambda_n\omega}.h_{\lambda_n\omega}\right)(x) = \sup_{\gamma(0)=x}\{u(\gamma(t)) - \int_0^t\left(\frac{\|\dot\gamma_v\|^2}{2} - \omega(\dot\gamma_v)\right)dv\}\\
&= T_t^+u(x)
\end{aligned}$$

where the last equality is once again a consequence of Schilder's theorem and of Varadhan's lemma ([DZ], p 137).

Thus, u is a "fixed point" of T_t^+. Fathi ([Fa1],[Fa2]) proved that this is only possible for $\bar\Lambda = \frac{\|[\omega]\|_s^2}{2}$, which proves Theorem 4.3.10 . □

Remark 4.3.12. If V has negative sectional curvature, let P be the pressure function defined on $H^1(V,\mathbb{R})$, and extended analytically to a neighbourhood of $\{\Im\omega = 0\}$ (see [BL]). In *constant* negative curvature Theorem 4.3.10 could also be obtained from the equality

$$\Lambda(\lambda\omega) = \frac{1}{2}(P(\lambda\omega)^2 - P(\lambda\omega))$$

proved for instance in [KS2] (p12) for $\lambda \in \sqrt{-1}\mathbb{R}$. Since both sides of the equations are analytic functions of λ in a complex neighbourhood of the real line, the equality also holds for $\lambda \in \mathbb{R}$.

Remark 4.3.13. I thank the referee for pointing out to me a paper by G. Paternain ([Pat]) where the following relation is proved, for real ω :

$$\lim_{\varepsilon\longrightarrow 0}\frac{\Lambda(\sqrt{-1}\varepsilon\omega)}{\varepsilon^2} = -l$$

for some non-negative l which is less (and, in general, strictly less) than $\frac{\|[\omega]\|_s^2}{2}$. This quantity l is called the harmonic value of the Lagrangian $v \mapsto \frac{\|v\|^2}{2} - \omega(v)$.

These two results have different interpretations. In one case one considers a quantized perturbation of the classical Laplace operator, and studies its behaviour when the perturbation tends to 0. Paternain's result relates the spectrum of these

Schödinger-type operators (more precisely, the second derivative at 0 of the first eigenvalue) to some quantities coming from classical Hamiltonian dynamics.

In the present paper we do not work in the setting of perturbation theory, but, on the contrary, of large deviation theory. We consider $\lambda\omega$, where $\lambda \in \mathbb{R}$ tends to $+\infty$; this parameter plays the role of the inverse of the temperature in statistical physics, or of the viscosity coefficient in fluid mechanics.

We conclude this paper by a large deviation upper bound for the family of stationary measures $(\nu_{\lambda\omega})_{\lambda\geq 0}$. The second Peierls barrier is defined by

$$h_*(x,y) = \inf u_-(y) - u_+(x)$$

where the inf is taken over all pairs of conjugate fixed points of T_t^-, T_t^+ (see [Fa2]). It is a real valued, lipschitz function.

Theorem 4.3.14. *Let ν_ω denote the stationary probability measure for Brownian motion twisted by the real valued 1-form ω. Then, for every closed subset $F \subset V$,*

$$\limsup_{\lambda\longrightarrow+\infty} \frac{1}{\lambda} \log \nu_{\lambda\omega}(F) \leq -\inf_{x\in F} h_*(x,x)$$

Proof. We have seen that $\nu_{\lambda\omega} = h_{\lambda\omega}h_{-\lambda\omega}dx$ up to a normalizing factor.

Let $\lambda_n \longrightarrow +\infty$ be a sequence such that $\frac{1}{\lambda_n}\log \nu_{\lambda_n\omega}(F)$ converges in $\mathbb{R}\cup\{-\infty\}$. We may assume that $(\frac{1}{\lambda_n}\log h_{\lambda_n\omega})$ and $(\frac{1}{\lambda_n}\log h_{-\lambda_n\omega})$ converge uniformly on V, to continuous functions u_+, $-v_-$ respectively. It follows easily that

$$\limsup_{n\longrightarrow+\infty} \frac{1}{\lambda_n} \log \nu_{\lambda_n\omega}(F) \leq \sup_{x\in F} u_+(x) - v_-(x) - J_0$$

where $J_0 := \sup_{x\in V} u_+(x) - v_-(x)$ comes from the normalization of $\nu_{\lambda_n\omega}$. From the considerations above, u_+ is a fixed point of (T_t^+) and v_- is a fixed point of (T_t^-). Let u_- be the fixed point of T^- conjugate to u_+. Fathi proved that $u_+ \leq v_- + J_0$ implies $u_- \leq v_- + J_0$. Thus we have $u_+(x) - v_-(x) - J_0 \leq u_+(x) - u_-(x) \leq -h_*(x,x)$ for all x; and

$$\lim_{n\longrightarrow+\infty} \frac{1}{\lambda_n} \log \nu_{\lambda_n\omega}(F) \leq -\inf_{x\in F} h_*(x,x)$$

as soon as $\frac{1}{\lambda_n}\log \nu_{\lambda_n\omega}(F)$ converges, which finishes the proof. □

The set $\{x \in V, h_*(x,x) = 0\}$ is called the Mañe set. It is the set of globally minimizing orbits for the Lagrangian $\frac{\|v\|^2}{2} - \omega(v)$. It contains the Aubry and Mather sets associated to ω, but since it is chain transitive, it will be larger than the Aubry (or Mather) set if the latter is not itself chain transitive.

We do not know if a full large deviation theorem holds – i.e, a lower estimate for the measures of open subsets, with identical rate functions on both sides. This is actually equivalent to the existence of the uniform limit of $\frac{1}{\lambda}\log h_{\lambda\omega}$. We conjecture the following large deviations lower bound – at least for low dimensional systems and when the Aubry set is chain transitive : for every open set $G \subset F$,

$$\liminf_{\lambda\longrightarrow+\infty} \frac{1}{\lambda} \log \nu_{\lambda\omega}(G) \geq -\inf_{x\in G} h(x,x)$$

where h is the first Peierls barrier : $h(x,y) = \sup u_-(y) - u_+(x)$ (so that $h(x,x)$ vanishes precisely on the Aubry set).

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