ENTROPY AND THE LOCALIZATION OF EIGENFUNCTIONS.

NALINI ANANTHARAMAN

Abstract. We study the large eigenvalue limit for the eigenfunctions of the Laplacian, on a compact manifold of variable negative curvature – or more generally, assuming only that the geodesic flow has the Anosov property. We prove that the Wigner measures associated to eigenfunctions cannot concentrate entirely on sets of small topological entropy under the action of the geodesic flow, such as, for instance, closed geodesics.

1. Introduction, statement of results

We consider a compact Riemannian manifold M of dimension $d \geq 2$, and assume that the geodesic flow $(g^t)_{t\in\mathbb{R}}$, acting on the unit tangent bundle of M, has a "chaotic" behaviour; this refers to certain asymptotic properties of the flow when time t tends to infinity: ergodicity, mixing, hyperbolicity... Here we mean that the geodesic flow has the Anosov property. The name "quantum chaos" expresses the belief that the chaotic properties of the flow should still be visible in the corresponding quantized dynamical system: that is, according to the Schrödinger equation, the unitary flow $\left(\exp(i\hbar t\frac{\Delta}{2})\right)_{t\in\mathbb{R}}$ acting on the Hilbert space $L^2(M)$ – where Δ stands for the Laplacian on M and $\hbar$ is something proportional to the Planck constant. At the quantum level, one expects that the chaotic features should express themselves in certain behaviours of the eigenfunctions of the Laplacian, or in the distribution of its eigenvalues (see [Sa95]). These ideas rely on the fact that the quantum flow $\left(\exp(i\hbar t\frac{\Delta}{2})\right)_{t\in\mathbb{R}}$ converges, in a sense to be precised below, to the classical flow (g^t) in the so-called "semi-classical limit" $\hbar \longrightarrow 0$: one likes to imagine that "for $\hbar$ small" the qualitative behaviour of quantum system will be related to that of the classical flow.

The convergence of the quantum flow to the classical flow is stated precisely in the Egorov theorem. Let us consider one of the usual quantization procedures, say $Op_\hbar$, which associates an operator $Op_\hbar(a)$ acting on $L^2(M)$ to every smooth compactly supported function $a \in C_c^\infty(TM)$ on the tangent bundle TM. The Egorov theorem says that, for fixed t,

$$||\exp(-it\frac{\hbar\Delta}{2}).Op_\hbar(a).\exp(it\frac{\hbar\Delta}{2}) - Op_\hbar(a\circ g^t)||_{L^2(M)} = \underset{\hbar\to 0}{O}(\hbar)\,.$$

In this paper, we focus our attention on the behaviour of the eigenfunctions on the Laplacian,

$$-h^2\Delta\psi_h = \psi_h$$

in the large energy limit $h \longrightarrow 0$ (we simply use the notation h instead of $\hbar$, and $-\frac{1}{h^2}$ ranges over the spectrum of the Laplacian). Let us consider an orthonormal basis

Received by the editors July 2004, revised November 2005.

of eigenfunctions in $L^2(M) = L^2(M, dVol)$ where Vol is the Riemannian volume. Each wave function ψ_h defines a probability measure on M:

$$|\psi_h(x)|^2 dVol(x),$$

that can be lifted to the tangent bundle by considering the distribution

$$\nu_h : a \in C_c^\infty(TM) \mapsto \langle Op_h(a)\psi_h, \psi_h \rangle_{L^2(M)},$$

usually called Wigner measure or Husimi measure (depending on the choice of the quantization) associated to the eigenfunction ψ_h; or also, sometimes, "microlocal lift" of the probability measure $|\psi_h(x)|^2 dx$. If the quantization procedure was chosen positive, which can be done using Friedrichs symmetrization (see [Ze86], Section 3, or [Co85], 1.1), then the distributions ν_hs are actually probability measures. It is possible to extract converging subsequences of the family $(\nu_h)_{h\to 0}$, and the limit, say ν_0, of such a subsequence is necessarily a probability measure carried by the unit tangent bundle $S^1M \subset TM$. In addition, the Egorov theorem implies that ν_0 is invariant under the (classical) geodesic flow. We will call such a measure ν_0 a *semi-classical invariant measure* of the flow. The question of identifying all such measures ν_0 arises naturally: the Snirelman theorem ([Sn74], [Ze87], [Co85], [HM87]) answers that the Liouville measure is one of them, in fact it is a limit along a subsequence "of density one" of the family (ν_h), as soon as the geodesic flow acts ergodically on S^1M with respect to the Liouville measure. It is not known in such a general context whether there can be exceptional subsequences which converge to other invariant measures, like, for instance, measures carried by closed geodesics. It was conjectured in [RS94] that the whole sequence actually converges to the Liouville measure, if M has negative sectional curvature: this is called the "Quantum Unique Ergodicity" conjecture.

The problem was solved recently by Lindenstrauss ([Li03]) in the case of an arithmetic surface of constant negative curvature, when the functions ψ_h are common eigenstates for the Laplacian and the Hecke operators; but little is known for other Riemann surfaces or in higher dimension. In the setting of discrete time dynamical systems, and in the very particular case of linear Anosov diffeomorphisms of the torus, Faure, Nonnenmacher and De Bièvre provided counter-examples to the conjecture: they constructed semi-classical invariant measures formed by a convex combination of the Lebesgue measure on the torus and of the measure carried by a closed orbit ([FNDB03]). However, it was shown in [BDB03], for the same discrete time model, that semi-classical invariant measures cannot be *entirely* carried on a closed orbit.

1.1. Non-concentration on sets of small topological entropy.

We work in the general context of Anosov geodesic flows for manifolds of arbitrary dimension, and we are interested in the entropy of semi-classical invariant measures. The Kolmogorov entropy, also called metric entropy, of a (g^t)-invariant probability measure ν_0 is a nonnegative number $h_g(\nu_0)$ that measures, in some sense, the asymptotic complexity of a generic orbit of the flow when time tends to infinity. For example, a measure carried on a closed geodesic has zero entropy; said the other way round, a measure having positive entropy cannot be entirely carried on a closed geodesic. On the other hand, an upper bound on entropy is given by the Ruelle inequality: since the geodesic flow has the Anosov property, the unit tangent bundle S^1M is foliated into unstable manifolds of the flow, and for any invariant probability measure ν_0

one has

$$h_g(\nu_0) \leq \int_{S^1M} |\log J^u(v)| d\nu_0(v), \tag{1.1.1}$$

where $J^u(v)$ is the unstable jacobian of the flow at v, defined as the jacobian of g^{-1} restricted to the unstable manifold of $g^1 v$. In (1.1.1) equality holds if and only if ν_0 is the Liouville measure on S^1M ([LY85]). Thus, proving Quantum Unique Ergodicity is equivalent to proving that $h_g(\nu_0) = \int_{S^1M} |\log J_u| d\nu_0$ for any semi-classical invariant measure ν_0. But already a non-trivial lower bound on the entropy of ν_0 would be nice.

Denote

$$\chi = -\sup_{v \in S^1M} \log J^u(v).$$

For instance, for a d-dimensional manifold of constant sectional curvature -1, $\chi = d - 1$. We will prove the following theorems:

Theorem 1.1.1. *Let F be a closet subset of S^1M, invariant under the geodesic flow, with a topological entropy $h_{top}(F) < \frac{\chi}{2}$. Then, under Assumption (I) below,*

$$\nu_0(F) < 1.$$

In other words, the support of ν_0 has topological entropy greater than $\frac{\chi}{2}$.

Remark 1.1.2. Assumption (I) is a technical assumption of real analyticity for the propagator of the Schrödinger equation, which is straighforward to check for a manifold of constant curvature -1.

Remark 1.1.3. The so-called Variational Principle ([KH]) asserts that, if F is a (g^t)-invariant closed subset of S^1M,

$$\sup_{\nu(F)=1} h_g(\nu) = h_{top}(F),$$

where the supremum runs over the set of (g^t)-invariant probability measures supported on F ([KH]). Thus, the conclusion of Theorem 1.1.1 is weaker than the statement that $h_g(\nu_0) \geq \frac{\chi}{2}$.

Conjecture 1.1.4. *For any semi-classical measure ν_0,*

$$h_g(\nu_0) \geq \frac{1}{2} \int_{S^1M} |\log J^u(v)| d\nu_0(v).$$

Theorem 1.1.1 is to be compared to the results in [BDB03], according to which the semi-classical invariant measures cannot be entirely carried on a closed geodesic. See also [CP94], where it is proved, in constant negative curvature, the concentration on a closed geodesic cannot be too fast.

The proof of Theorem 1.1.1 is based on the following ideas: if ν_0 is a semi-classical measure, we construct in paragraph 1.3 a sequence of invariant "pseudo-measures" converging to ν_0 and for which we prove nice exponential estimates for the measures of the so-called cylinder sets, at the heart of the concept of entropy (Theorem 1.3.3). Conjecture 1.1.4 would follow immediately from the semi-continuity of entropy, were our pseudo-measures genuine probability measures. This is unfortunately not the case since they are not positive. What we obtain at the end is not an estimate of the metric entropy of ν_0, but a lower bound for the topological entropy of the support of ν_0. The method should work for more general hyperbolic hamiltonian systems. In [AN05] it is implemented for the toy model of the Baker's map, for which Quantum Unique Ergodicity is known to fail. The analogue of Theorem 1.1.1, as well as Conjecture 1.1.4 are proved (with considerable simplifications due to the nature of the model); is also shown that the bound of Theorem 1.1.1 is achieved, for this toy model. Thus, Theorem 1.1.1 should not be interpreted as a step in the direction of Quantum Unique Ergodicity, but rather as a general fact which holds even when Quantum Unique Ergodicity is known to fail.

In the next paragraph we recall the definition of metric entropy. Then, in paragraph 1.3, we construct our pseudo-measures, and state Theorems 1.3.3 on the decay of the measures of cylinder sets, the key to Theorem 1.1.1. All these results rely on Theorem 4.0.1, which – speaking very roughly – uses the uniform hyperbolicity of the classical flow to estimate the kernel of $\exp(ih\frac{\Delta}{2})^n$ when $h \longrightarrow 0$, and for large n.

1.2. Entropy of the geodesic flow.

Topological entropy. We denote $h_{top}(S^1M)$ the topological entropy of the action of (g^t) on S^1M. More generally, if $F \subset S^1M$ is closed and invariant under (g^t), we denote $h_{top}(F)$ the topological entropy of the flow restricted to F: we refer to [KH] for the definition.

We linger more on the definition of metric entropy:

Metric entropy.

Recall the definition of metric entropy, defined by Kolmogorov and Sinai. Let $S^1M = P_1 \sqcup ... \sqcup P_l$ be a finite measurable partition of the unit tangent bundle S^1M. The entropy of ν_0 with respect to the action of geodesic flow and to the partition P is defined by

$$\begin{aligned} &h_g(\nu_0, P)\\ &= \lim_{n\longrightarrow+\infty} -\frac{1}{n}\sum_{(\alpha_j)\in\{1,\ldots,l\}^{n+1}} \nu_0(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n}P_{\alpha_n})\log\nu_0(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n}P_{\alpha_n})\\ &= \inf_{n\in\mathbb{N}} -\frac{1}{n}\sum_{(\alpha_j)\in\{1,\ldots,l\}^{n+1}} \nu_0(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n}P_{\alpha_n})\log\nu_0(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n}P_{\alpha_n}).\end{aligned}$$

The existence of the limit, and the fact that it coincides with the inf follow from a subadditivity argument. Then, the entropy of ν_0 itself with respect to the action of the geodesic flow is defined as

$$h_g(\nu_0) = \sup_P h_g(\nu_0, P),$$

the supremum running over all finite measurable partitions P. Rather often, this supremum is actually reached for a well-chosen partition P.

The entropy is non-negative, and bounded a priori from above; for instance, on a compact d-dimensional riemannian manifold of constant sectional curvature -1, the entropy of any measure is smaller than $d-1$; more generally, for an Anosov geodesic flow, one has an a priori bound in terms of the unstable Jacobian, called the Ruelle inequality (see [KH]):

$$h_g(\nu_0) \leq \int_{S^1M} |\log J^u| d\nu_0,$$

with equality if and only if ν_0 is the Liouville measure on S^1M ([LY85]) – let us also mention the so-called Variational Principle, asserting that, if F is a (g^t)-invariant closed subset of S^1M,

$$\sup_{\nu(F)=1} h_g(\nu) = h_{top}(F),$$

where the supremum runs over the set of (g^t)-invariant probability measures supported on F ([KH]).

For later purposes, we reformulate slightly the definition of entropy. The following definition, although equivalent to the usual one, is a bit different, in that we only use partitions of the base M[1].

Let $P = (P_1, ...P_l)$ be a finite measurable partition of M (instead of S^1M); we denote $\varepsilon/2$, $(\varepsilon > 0)$ an upper bound on the diameter of the P_is. We can also consider P as a partition of the tangent bundle, simply by lifting it to TM.

Let $\Sigma = \{1, ...l\}^{\mathbb{Z}}$. To each tangent vector $v \in S^1M$ one can associate a unique element $I(v) = (\alpha_j)_{j\in\mathbb{Z}} \in \Sigma$, by requiring $g^j v \in P_{\alpha_j}$ for all integers j. Thus, one defines a "coding map" $I : S^1M \longrightarrow \Sigma$. If we define the shift σ acting on Σ by

$$\sigma((\alpha_j)_{j\in\mathbb{Z}}) = (\alpha_{j+1})_{j\in\mathbb{Z}},$$

we have $I \circ g^1 = \sigma \circ I$.

We introduce the probability measure μ_0 on Σ, image of ν_0 under the coding map I. More explicitly, the finite-dimensional marginals of μ_0 are given by

$$\mu_0\big([\alpha_0, ..., \alpha_{n-1}]\big) = \nu_0(P_{\alpha_0} \cap g^{-1}P_{\alpha_1} ... \cap g^{-n+1}P_{\alpha_{n-1}}).$$

We have denoted $[\alpha_0, ..., \alpha_{n-1}]$ the subset of Σ, formed of sequences in Σ beginning with the letters $(\alpha_0, ..., \alpha_{n-1})$; such a set is called a *cylinder set* of length n. We will denote Σ_n the set of cylinder sets of length n; they form a partition of Σ.

Since ν_0 is carried by the unit tangent bundle, and (g^t)-invariant, its image μ_0 is σ-invariant. The entropy of μ_0 with respect to the action of the shift σ is

$$h_\sigma(\mu_0) = \lim_{n\longrightarrow+\infty} -\frac{1}{n}\sum_{\mathcal{C}\in\Sigma_n} \mu_0(\mathcal{C})\log\mu_0(\mathcal{C}) \tag{1.2.1}$$

$$= \inf_n -\frac{1}{n}\sum_{\mathcal{C}\in\Sigma_n} \mu_0(\mathcal{C})\log\mu_0(\mathcal{C}). \tag{1.2.2}$$

The fact that the limit exists and coincides with the inf comes from the remark that the sequence $(-\sum_{\mathcal{C}\in\Sigma_n}\mu_0(\mathcal{C})\log\mu_0(\mathcal{C}))_{n\in\mathbb{N}}$ is subadditive, which follows from the concavity of the log and the σ-invariance of μ_0 (see [KH]). Then, $h_g(\nu_0)$ is the sup, over all partitions P of M, of the entropies $h_\sigma(\mu_0)$ obtained by the previous construction: we can indeed restrict our attention to partitions P depending only on the base M, and work with time one of the geodesic flow g^1, if the injectivity radius is greater than one (a harmless assumption we will make in the rest of the paper).

The advantage of definition (1.2.2) is that the entropy, defined on the set of σ-invariant probability measures on Σ, is the infimum of a family of continuous functions, and thus is an upper semi-continuous function (for the weak topology). In other words, if we could find a sequence (μ_k) of σ-invariant probability measures

[1]The reason for doing so is to be able to work with multiplication operators in paragraph 1.3, instead of having to deal with more general pseudo-differential operators.

converging to μ_0 on Σ, and satisfying – for some $\beta \geq 0$ and some positive real numbers (C_k),

$$\mu_k(\mathcal{C}^{(n)}) \leq C_k e^{-\beta n}$$

for every $n \in \mathbb{N}$, every cylinder set $\mathcal{C}^{(n)}$ of length n, and every k, this would imply that $h_\sigma(\mu_0) \geq \beta$ and thus $h_g(\nu_0) \geq \beta$.

This motivates the following attempt to find a lower bound on the entropy of the semi-classical measure μ_0, by "quantizing" the construction above, and estimating the rate of decay of the quantum measures of cylinder sets.

1.3. **The quantized construction: estimates on the decay of the measures of cylinder sets.**

1.3.1. *The measure μ_h.* Since we will resort to microlocal analysis we have to replace characteristic functions $\mathbf{I}_{P_i}$ by smooth functions. We will assume that the P_i have smooth boundary, and will consider a smooth partition of unity obtained by smoothing the characteristic functions $\mathbf{I}_{P_i}$; that is, a finite family of C^∞ functions $A_i \geq 0$ $(i = 1, ..., l)$, such that

$$\sum_{i=1}^{l} A_i = 1.$$

We can consider the A_is as functions on TM, depending only on the base point. For each i, denote $\hat{P}_i$ a set that contains the support of A_i in its interior. Throughout the paper we denote $\varepsilon > 0$ an upper bound on the diameters of the $\hat{P}_i$s.

Actually, the way we smooth the $\mathbf{I}_{P_i}$s to obtain A_i is rather crucial, and will be discussed in paragraph 2.1. Let us just say, for the moment, that the A_i will depend on h in a way that

$$A_i^h \underset{h \longrightarrow 0}{\longrightarrow} 1 \tag{1.3.1}$$

uniformly in every compact subset in the interior of P_i, and

$$A_i^h \underset{h \longrightarrow 0}{\longrightarrow} 0 \tag{1.3.2}$$

uniformly in every compact subset outside P_i. We also assume that the smoothing is done at a scale h^κ $(\kappa \in [0, 1/2))$, so that the derivatives of A_i^h are controlled as

$$||D^n A_i^h|| \leq C(n) h^{-n\kappa}.$$

This implies that the results of pseudo-differential calculus are applicable to the functions A_i^h (see Appendix A1).

For technical reasons, we will need to control more precisely the derivatives of the A_is. We assume that M is a real-analytic manifold, and we take the A_is in a Gevrey class $\mathcal{G}_s$ for some $s > 1$. Let us recall the definition of this class of functions.

The Gevrey class $\mathcal{G}_s$:

Let f be a function of class C^∞, going from an open subset Ω of a normed vector space to some other normed vector space – say, of finite dimensions. We say that f is in the Gevrey class of order s, denoted $\mathcal{G}_s$,if for every compact set $K \subset \Omega$, there exist $C, R > 0$ such that for every k in $\mathbb{N}$, for every $x \in K$,

$$||D^k f(x)|| \leq C R^k (k!)^s. \tag{1.3.3}$$

Note that $\mathcal{G}_1$ coincides with the class of real analytic functions. If a function f satisfies (1.3.3) *for all $x \in \Omega$, we write $f \in \mathcal{G}_s(R, C)$, and we call the real number $\max\{R, C\}$ a "Gevrey constant" (or "analyticity constant", in the case $s = 1$) for f. Obviously, these numbers depend on the choice of the norms on the two vector spaces in question.*

Each Gevrey class $\mathcal{G}_s$ is invariant by composition with an analytic function ([Ho]*, Chapter 8), so that one can also speak of the Gevrey class on a real analytic manifold. By Theorem 1.3.5 in* [Ho]*, on a real analytic compact manifold, there exist partitions of unity in the class $\mathcal{G}_s$ for any $s > 1$.*

We construct a functional μ_h defined on a certain class of functions on Σ. We see the functions A_i as multiplication operators on $L^2(M)$; and we denote $A_i(t)$ their evolutions under the quantum flow:

$$A_i(t) = \exp\big(-it\frac{\hbar\Delta}{2}\big) \circ A_i \circ \exp\big(it\frac{\hbar\Delta}{2}\big).$$

We define the "measures" of cylinder sets under μ_h, by the expressions:

$$\begin{aligned}(1.3.4)\qquad \mu_h\big([\alpha_0, ..., \alpha_n]\big) &= \langle\, A_{\alpha_n}(n)....A_{\alpha_1}(1)A_{\alpha_0}(0)\,\psi_h, \psi_h\,\rangle_M \\ (1.3.5)\qquad &= \langle\, A_{\alpha_n}(0)....A_{\alpha_1}(-1)A_{\alpha_0}(-n)\,\psi_h, \psi_h\,\rangle_M.\end{aligned}$$

For $\mathcal{C} = [\alpha_0, ..., \alpha_{n-1}] \in \Sigma_n$, we will denote $\hat{\mathcal{C}}_h$ the operator $\hat{\mathcal{C}}_h = A_{\alpha_{n-1}}(0)....A_{\alpha_1}(-n+2)A_{\alpha_0}(-n+1)$.

The functional μ_h is defined on the vector space spanned by characteristic functions. Note that μ_h is not a positive measure, because the operator $\hat{\mathcal{C}}_h$ used in (1.3.4) are not positive. The first part of the following proposition is a compatibility condition; the second part says that μ_h is σ-invariant. The proof is obvious and uses the fact that ψ_h is en eigenfunction. The third condition holds if ψ_h is normalized in $L^2(M)$.

Proposition 1.3.1. *(i) For every n, for every cylinder $[\alpha_0, ..., \alpha_{n-1}] \in \Sigma_n$,*

$$\sum_{\alpha_n} \mu_h\big([\alpha_0, ..., \alpha_n]\big) = \mu_h\big([\alpha_0, ..., \alpha_{n-1}]\big).$$

(ii) For every n, for every cylinder $[\alpha_0, ..., \alpha_{n-1}] \in \Sigma_n$,

$$\sum_{\alpha_{-1}} \mu_h\big([\alpha_{-1}, .., \alpha_{n-1}]\big) = \mu_h\big([\alpha_0, .., \alpha_{n-1}]\big).$$

(iii) For every $n \geq 0$,

$$\sum_{[\alpha_0, ..., \alpha_{n-1}]} \mu_h\big([\alpha_0, ..., \alpha_{n-1}]\big) = 1.$$

We assume in the rest of the paper that we have extracted from the sequence $(\nu_h)_{-1/h^2 \in Sp(\Delta)}$ a sequence $(\nu_{h_k})_{k\in\mathbb{N}}$ that converges to ν_0 in the weak topology: $\langle Op_{h_k}(a)\psi_{h_k}, \psi_{h_k}\rangle_{L^2(M)} \underset{k\longrightarrow+\infty}{\longrightarrow} \int_{S^1M} a\, d\nu_0$, for every $a \in C_c^\infty(TM)$. To simplify notations, we forget about the extraction, and simply consider that $\nu_h \underset{h\longrightarrow 0}{\longrightarrow} \nu_0$.

If the partition of unity (A_i) does not depend on h, the usual Egorov theorem shows that μ_h converges, as $h \longrightarrow 0$, to a same σ-invariant probability measure defined by $\mu_0^{(A)}$ on Σ, defined by

$$\mu_0^{(A)}\big([\alpha_0,...,\alpha_n]\big) = \nu_0\left(A_{\alpha_0}.A_{\alpha_1}\circ g^1...A_{\alpha_n}\circ g^n)\right).$$

By "convergence", we mean that the measure of each cylinder set converges. Now, suppose the partition of unity depends on h so as to satisfy (1.3.1), (1.3.2); we may, and will also assume that ν_0 does not charge the boundary of P.

Proposition 1.3.2. *The family (μ_h) converges to μ_0 as $h \longrightarrow 0$.*

Proof. Let $\mathcal{C} = [\alpha_0,...,\alpha_n]$ be a cylinder set. By the Egorov theorem 5.0.3,

$$||\hat{\mathcal{C}}_h - Op_h\Big(A_{\alpha_0}\,A_{\alpha_1}\circ g^1...A_{\alpha_{n-1}}\circ g^{n-1}\Big)||_{L^2(M)} = O(h^{1-2\kappa}). \tag{1.3.6}$$

The function $A_{\alpha_0}\,A_{\alpha_1}\circ g...A_{\alpha_{n-1}}\circ g^{n-1}$ is nonnegative, and, as $h\longrightarrow 0$, it converges uniformly to 1 on every compact subset in the interior of $P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n+1}P_{\alpha_{n-1}}$, since A_i converges uniformly to 1 on every compact subset in the interior of P_i (1.3.1).

If we chose a positive quantization procedure Op_h, it follows from (1.3.6) that

$$\liminf_{h\longrightarrow 0}\mu_h(\mathcal{C}) = \liminf_{h\longrightarrow 0}\langle Op_h\big(A_{\alpha_0}\,A_{\alpha_1}\circ g...A_{\alpha_{n-1}}\circ g^{n-1}\big)\psi_h,\psi_h\rangle$$
$$\geq \liminf \nu_h\big(int(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n+1}P_{\alpha_{n-1}})\big) \geq \nu_0\big(int(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n+1}P_{\alpha_{n-1}})\big)$$

We have assumed that ν_0 does not charge the boundary of the P_is, and thus the last term is also $\nu_0\big(P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n+1}P_{\alpha_{n-1}}\big)$.

Similarly, using (1.3.2) one can prove that

$$\limsup_{h\longrightarrow 0}\mu_h(\mathcal{C}) \leq \nu_0\big(\overline{P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n+1}P_{\alpha_{n-1}}}\big).$$

This ends the proof since we assumed ν_0 does not charge the boundary of the partition P. □

The key technical result of this paper, proved in Section 4, is an upper bound on μ_h, valid for cylinder sets of all lengths.

1.3.2. *Decay of the measures of cylinder sets.* Remember that ε is an upper bound on the diameter of the support of the A_is. We denote $G_s(A)$ a common Gevrey constant for all the A_is. We also recall the definition of the unstable Jacobian: since the geodesic flow is Anosov, each energy layer $S^\lambda M = \{v\in TM, ||v|| = \lambda\}$ $(\lambda > 0)$ is foliated into strong unstable manifolds of the geodesic flow. The unstable Jacobian $J^u(v)$ at $v\in TM$ is defined as the jacobian of g^{-1}, restricted to the unstable leaf at the point g^1v.

Theorem 1.3.3. *Under Assumption (I), one has the following estimates on the measures of cylinder sets.*

There exists a function $R(n,h)$ such that:

For every $\eta > 0$, there exists $\varepsilon_0 > 0$, $h_0 > 0$, such that, if $0 < \varepsilon \leq \varepsilon_0$, if $0 < h < h_0$, one has an upper bound of the form:

$$|\mu_h\big([\alpha_0,...,\alpha_n]\big)| \leq \Big(1+\eta\Big)^n\left(\frac{\prod_{i=0}^{n-1}J^u(v_i)}{(2\pi h)^d}\right)^{1/2}[1+R(n,h)], \tag{1.3.7}$$

valid for every cylinder $[\alpha_0,...,\alpha_n]$.

Here v_i is a vector in $supp\, A_i$, such that $g^1 v_i \in suppA_{i+1}$ and $d(v_i, g^1 v_i) = d(supp\, A_i, supp\, A_{i+1})$.

The function $R(n,h)$ tends to 0 as $h \longrightarrow 0$, uniformly for $n \leq \mathcal{K}|\log h|$ (for any arbitrary $\mathcal{K}$).

Assumption (I) is given in paragraph 1.4.

The proof of Theorem 1.3.3 does not use the fact that ψ_h is an eigenfunction; it relies on an estimate of the kernel of the operator $A_{\alpha_0} A_{\alpha_1}(1)...A_{\alpha_n}(n)$, given by Theorem 4.0.1 in Section 4.

The fact that ψ_h is an eigenfunction is used through the invariance of μ_h under the shift, which is crucial to go from Theorem 1.3.3 to Theorem 1.1.1. The proof of Theorem 1.1.1 may be roughly summarized by two observations:

(a) For all $n \in \mathbb{N}$, Theorem 1.3.3 tells us that, for every cylinder $\mathcal{C} \in \Sigma_n$, $|\mu_h(\mathcal{C})| \leq \frac{e^{-n(\chi/2-\eta)}}{(2\pi h)^{d/2}}[1 + R(n,h)]$, where $R(n,h)$ is a remainder term that remains small when n says of order $|\log h|$.

Thus, for any $\theta \in (0,1)$, a set of μ_h-measure greater than $(1-\theta)$ cannot be covered by less than $(1-\theta)(2\pi h)^{d/2} e^{n(\chi/2-\eta)}[1 + R(n,h)]^{-1}$ cylinders of length n.

(b) If $\tilde{F} \subset \Sigma$ is a σ-invariant set of topological entropy strictly less than $\chi/2$, then there exists $C, \delta > 0$ such that, for all n, $\tilde{F}$ can be covered by $Ce^{n(\chi/2-\delta)}$ cylinder sets of length n.

These two simple remarks encourage the intuition that the limit μ_0 cannot be carried by a set of topological entropy less than $\chi/2$.

The problem we will have to face when passing to the limits $h \longrightarrow 0$ is that the inequality $|\mu_h(\mathcal{C})| \leq \frac{e^{-n(\chi/2-\eta)}}{(2\pi h)^{d/2}}[1 + R(n,h)]$ only contains information when $\frac{e^{-n(\frac{\chi}{2}-\eta)}}{(2\pi h)^{d/2}} << 1$, that is, $n \geq \vartheta|\log h|$ for a certain ϑ (roughly speaking, $\vartheta = \frac{d}{\chi}$). On the other hand, observation (b) is only exploitable if μ_h is close to being a probability measure; semi-classical analysis tells us that this is the case on the set of cylinders of length $\leq \bar{\kappa}|\log h|$, where $\bar{\kappa}$ is also somehow controlled by the Lyapunov exponents of the flow (this time scale is called the Ehrenfest time). A priori, $\bar{\kappa} < \vartheta$, and our task will be to link the two regimes $n \leq \bar{\kappa}|\log h|$ and $n \geq \vartheta|\log h|$. This will be done by a certain sub-multiplicative argument presented in paragraph 2.2. The fact that ψ_h is an eigenfunction will be used through the σ-invariance of μ.

The paper is organized as follows:

– in Section 1 we prove Theorems 1.1.1, admitting the estimates provided by Theorem 1.3.3.

– in Section 2 we recall the main theorem of the paper [AMB92] by Aubry-McKay-Baesens, with a an aim to applying it to the proof of Theorems 4.0.1 and 1.3.3. The result translates the uniform hyperbolicity of the geodesic flow into certain properties of its generating function.

– in Section 3, we use the stationary phase method combined to the result of [AMB92] to prove Theorems 4.0.1 and 1.3.3.

The paper has two appendices. In A1 we construct the partition of unity A_i^h. In A2 we collect some facts about small scale pseudo-differential operators.

The last paragraph of this section is devoted to the statement and discussion of the assumptions.

1.4. **Assumptions.**

Assumption (I) (regularity assumptions): Let χ_h be a pseudo-differential operator whose symbol has compact support, localized in a small neighbourhood of the energy layer S^1M. If the injectivity radius is greater than 1, it follows from the theory of Fourier Integral operators that the propagator $\exp(\frac{ih\Delta}{2})\chi_h$ can be written as $E_h.\chi_h$, where the kernel of E_h is

$$e^{(h)}(x,y,1) = \frac{1}{(2\pi h)^{d/2}} e^{\frac{id_M^2(x,y)}{2h}} a^{(h)}(x,y),$$

and $a^{(h)}(x,y) \sim \sum_j h^j a_j(x,y)$ is in a good class of symbols.

We assume that the manifold M is real-analytic, endowed with a real-analytic metric. Besides, we assume that $(a^{(h)})_{0<h\leq 1}$ is a family of analytic functions on the set $\{(x,y), \frac{1}{2} \leq d_M(x,y) \leq \frac{3}{2}\}$, *with uniform analyticity constants.* In other words, on this set, there exists for all h an analyticity constant $G_1(a^{(h)})$ for $a^{(h)}$, such that $\sup_{0<h\leq 1} G_1(a^{(h)}) < +\infty$.

Remark 1.4.1. We check that Assumption (I) is satisfied for a surface of constant negative curvature at the end of Section 4. In a more general context, I do not know if anything has been proved concerning the analyticity of the propagator of the Schrödinger equation on analytic manifolds. For the heat kernel, the corresponding result is proved in [LGS96].

Let us fix a few notations:

– for distances: we will denote d_M, d_{TM}, etc... the distances on $M, TM, ...$ induced by the Riemannian metric. When there is no ambiguity, we will omit the subscripts $M, TM, ...$, and simply denote with the letter d any of these distances.

– for projections: the letter π will denote the natural projection $TM \longrightarrow M$

– and for the quantum evolution of operators: if $\hat{A}$ is an operator, we will denote $\hat{A}(t)$ its evolution under the quantum flow, that is, $\hat{A}(t) = \exp(-it\frac{h\Delta}{2}).\hat{A}.\exp(it\frac{h\Delta}{2})$.

2. Proof of Theorem 1.1.1.

Here we admit Theorem 1.3.3 and prove Theorem 1.1.1. The proof of the theorem may be roughly summarized in two observations:

(a) For all $n \in \mathbb{N}$, $\sum_{\mathcal{C}\in\Sigma_n} \mu_h(\mathcal{C}) = 1$, and we know that, for every cylinder $\mathcal{C} \in \Sigma_n$,

$$|\mu_h(\mathcal{C})| \leq \frac{e^{-n(\chi/2-\eta-Ch)}}{(2\pi h)^{d/2}}[1 + R(n,h)];$$

we have $\lim_{h\longrightarrow 0} R(n,h) = 0$ uniformly for $n \leq \mathcal{K}|\log h|$ (for arbitrary $\mathcal{K}$).

Thus, for any $\theta \in (0,1)$, a set of μ_h-measure greater than $(1-\theta)$ cannot be covered by less than $(1-\theta)(2\pi h)^{d/2} e^{n(\chi/2-\eta-Ch)}[1 + R(n,h)]^{-1}$ cylinders of length n (see Paragraph 2.2).

(b) If $\tilde{F} \subset \Sigma$ is a σ-invariant set of topological entropy strictly less than $\chi/2$; say $h_{top}(\tilde{F}) \leq \chi/2 - 11\delta$ for some positive δ. Then there exists C such that, for every

$n \in N$, $\tilde{F}$ can be covered by $Ce^{n(h_{top}(\tilde{F})+\delta)} \leq Ce^{n(\chi/2-\delta)}$ cylinder sets of length n (see Paragraph 2.3.)

The two observations (a) and (b) lead us to form the intuition that it is difficult for the limit measure μ_0 to concentrate on a set of topological entropy less than $\chi/2$.

Sketch of the proof. We will start with a variant of observation (b), proved in paragraph 2.3:

(b') Let $\tilde{F} \subset \Sigma$ be a σ-invariant set of topological entropy $h_{top}(\tilde{F}) \leq \chi/2 - 11\delta$. Then there exists a neighbourhood $\Sigma(W_{n_1})$ of $\tilde{F}$, formed of cylinders of length n_1, such that, for N large enough, for every $\mu \in [0,1]$,

$$\sharp\Sigma_N(W_{n_1}, \mu) \leq e^{8\delta N} e^{N h_{top}(F)} e^{(1-\mu)N(1+n_1)\log l},$$

where l is the number of elements of the partition P.

Here we denote $\Sigma_N(W_{n_1}, \mu)$ the set of N-cylinders $[\alpha_0, ..., \alpha_{N-1}]$ such that

$$\frac{\sharp\big\{j \in [0, N-n_1], [\alpha_j, ..., \alpha_{j+n_1-1}] \in \Sigma(W_{n_1})\big\}}{N - n_1 + 1} \geq \mu.$$

They correspond to orbits that spend a lot of time in the neighbourhood of $\tilde{F}$.

Comparing (a) and (b'), we see that, if $\eta < \delta$ and μ is sufficiently close to 1, one can find ϑ large enough so that, for $N \geq \vartheta|\log h|$,

$$(1-\theta)(2\pi h)^{d/2} e^{N(\chi/2-\eta-Ch)}[1 + R(N,h)]^{-1} > e^{8\delta N} e^{N h_{top}(F)} e^{(1-\mu)N h_{top}(S^1 M)}.$$

Hence:

$$|\mu_h\big(\Sigma_N(W_{n_1}, \mu)\big)| \leq 1 - \theta. \tag{2.0.1}$$

Then, using the σ-invariance of μ_h, we want to write, for $N = \vartheta|\log h|$,

$$\begin{aligned}
|\mu_h\left(\Sigma(W_{n_1})\right)| &= |\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mu_h\left(\sigma^{-k}\Sigma(W_{n_1})\right)| && (2.0.2)\\
&= |\mu_h\Big(\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbf{I}_{\sigma^{-k}\Sigma(W_{n_1})}\Big)| && (2.0.3)\\
&\leq \mu_h\big(\Sigma_N(W_{n_1}, \mu)\big) + \mu\,\mu_h\big(\Sigma_N(W_{n_1}, \mu)^c\big) && (2.0.4)\\
&\leq (1-\mu)\mu_h\big(\Sigma_N(W_{n_1}, \mu)\big) + \mu && (2.0.5)\\
&\leq (1-\mu)(1-\theta) + \mu, && (2.0.6)
\end{aligned}$$

and, passing to the limit $h \longrightarrow 0$, we get $\mu_0(\Sigma(W_{n_1})) \leq (1-\mu)(1-\theta) + \mu$ and hence

$$\mu_0(\tilde{F}) \leq (1-\mu)(1-\theta) + \mu < 1.$$

For (2.0.4), we have used the fact that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbf{I}_{\sigma^{-k}\Sigma(W_{n_1})} \leq 1$$

in general, and that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1} \mathbf{I}_{\sigma^{-k}\Sigma(W_{n_1})} \leq \mu$$

on $\Sigma_N(W_{n_1},\mu)^c$, the complement of $\Sigma_N(W_{n_1},\mu)$. The problem is that this line is not correct since μ_h is not a probability measure !

We know however that μ_h converges weakly to a probability measure, and we may try to make this statement more quantitative. Semi-classical analysis will tell us that μ_h is close to being a probability measure when restricted to the set of cylinders of length $N \leq \bar{\kappa}|\log h|$, for $\bar{\kappa}$ not too large. To sum up, the inequality (2.0.1) only holds for $N \geq \vartheta|\log h|$ whereas the heuristics (2.0.2)–(2.0.6) only makes sense for $N \leq \bar{\kappa}|\log h|$; and a priori, $\bar{\kappa} < \vartheta$. Our job will be to pass from one time-scale to the other; this will be done thanks to the sub-multiplicative lemma of paragraph 2.2.

In paragraph 2.1 we give certain important precisions about the partitions of unity we want to use. In 2.2, we come back to observation (a) and prove the crucial sub-multiplicativity lemma. Paragraph 2.3 is dedicated to proving (b'). In paragraph 2.4 we show that, until a certain time $\bar{\kappa}|\log h|$, the measure μ_h can be treated as a probability measure. Finally, we conclude as in (2.0.2)–(2.0.6).

2.1. **Nice partitions of unity.** For reasons that will become clear later, we need to be more specific about our partitions of unity (A_i). In order to apply semi-classical methods we need the A_i to be smooth, and on the other hand we would like the family A_i to behave almost like a family of orthogonal projectors: $A_i^2 \simeq A_i$, $A_iA_j \simeq 0$ for $i \neq j$.

Take a finite partition $M = P_1 \sqcup ... \sqcup P_l$ by sets of diameter less than $\varepsilon/2$, and such that each P_i contains a ball of radius $\varepsilon/4$. By modifying slightly the P_is we may assume that the semi-classical measure ν_0 does not charge the boundary of the partition. Denoting $\overline{P_i}$ the closure of P_i, we also have $M = \overline{P_1} \cup ... \cup \overline{P_l}$, this union is no longer disjoint but two different sets may intersect only at boundary points.

Our partition of unity will be defined by taking

$$\tilde{A}_i^h(x) = \frac{1}{h^\kappa}\mathbf{I}_{P_i} * \zeta\big(x/h^\kappa\big); \tag{2.1.1}$$

that is,

$$\tilde{A}_i^h(x) = \frac{1}{h^\kappa}\int \zeta\left(\frac{y}{h^\kappa}\right)\mathbf{I}_{P_i}(x-y)dy,$$

where ζ is a nonnegative, compactly supported function in the Gevrey class $\mathcal{G}_s$, of integral 1; the convolution is to be unterstood in a local chart, and $\kappa \geq 0$ will be chosen later. Then, we take as a partition of unity the family

$$A_i = \frac{\tilde{A}_i^h}{\sum_{j=1}^l \tilde{A}_j^h}.$$

The partition of unity $(A_i)_{1\leq i\leq l}$ depends on h, and if $\kappa > 0$ it converges weakly to $(\mathbf{I}_{P_i})_{1\leq i\leq l}$ when $h \longrightarrow 0$. It has the following properties:

- $\overline{P_i} \subset supp\, A_i \subset B(\overline{P_i}, \varepsilon/2)$ for all i, for h small enough. In accordance with the notations of the previous sections, we denote $\hat{P}_i = B(\overline{P_i}, \varepsilon/2)$.
- ${A_i}^2 = A_i$ except on a set of measure of order h^κ.
- for $i \neq j$, $A_iA_j = 0$ except on a set of measure of order h^κ.

- for all i, $A_i \in \mathcal{G}_s(Ch^{-\kappa}, C)$, for some C depending only on the cut-off function ζ used in the definition of $\tilde{A}_i$.

We must choose κ so that semi-classical methods still work: that is, $h^2 G_s(A) \underset{h \longrightarrow 0}{\longrightarrow} 0$, in other words $\kappa < 1/2$ (see Appendix A1).

In addition, we need to assume that there exists some $p > 0$ such that

- For all i, $||(A_i^2 - A_i)\psi_h||_{L^2(M)} = O(h^{p/2})$.
- For $i \neq j$, $||A_i A_j \psi_h||_{L^2(M)} = O(h^{p/2})$.

In other words, the operators A_i act on ψ_h almost as a family of orthogonal projectors. Because $||\psi_h||_{L^2(M)} = 1$, it is always possible to construct the A_is in order to satisfy all the requirements above; this requires to modify slightly the partition P_i before applying the convolution (2.1.1). The construction is described in detail in Appendix A2.

2.2. **Counting $(h, (1-\theta), n)$-covers: a sub-multiplicative property.** We now try to exploit observation (a). As already mentioned, we will have to face the problem that the inequality $|\mu_h(\mathcal{C})| \leq \frac{e^{-n(\chi/2-\eta-Ch)}}{(2\pi h)^{d/2}}[1 + R(n,h)]$ only contains information when $\frac{e^{-n(\frac{\chi}{2}-\eta-Ch)}}{(2\pi h)^{d/2}} << 1$, that is, $n \geq \vartheta |\log h|$ for a certain ϑ. On the other hand, observation (a) is only exploitable if μ_h is close to being a probability measure; semi-classical analysis tells us that this is the case on the set of cylinders of length $\leq \bar{\kappa}|\log h|$. A priori, $\bar{\kappa} < \vartheta$, and to reconcile the two regimes $n \leq \bar{\kappa}|\log h|$ and $n \geq \vartheta|\log h|$ we will need a certain sub-multiplicativity property (Lemma 2.2.3).

Definition 2.2.1. (i) Let W be a subset of Σ_n, the set of n-cylinders in Σ; we denote $W^c \subset \Sigma_n$ its complement. For a given $h > 0$ and $\theta \in [0,1]$, we say that W is a $(h, (1-\theta), n)$-cover of Σ if

$$||\sum_{\mathcal{C} \in W^c} \hat{\mathcal{C}}_h \psi_h||_{L^2(M)} \leq \theta. \tag{2.2.1}$$

(ii) We define

$$N_h(n,\theta) = \min\{\sharp W, W \text{ is a } (h,(1-\theta),n)\text{-cover of } \Sigma\},$$

the minimal cardinality of an $(h, (1-\theta), n)$-cover of Σ.

Remember the notation: for $\mathcal{C} = [\alpha_0, ..., \alpha_{n-1}] \in \Sigma_n$, $\hat{\mathcal{C}}_h$ stands for the operator $\hat{\mathcal{C}}_h = A_{\alpha_{n-1}}(0)....A_{\alpha_1}(-n+2)A_{\alpha_0}(-n+1)$.

In some sense, (2.2.1) means that the measure of the complement of W is small. The reason why we measure this by $||\sum_{\mathcal{C}\in W^c} \hat{\mathcal{C}}_h\psi_h||_{L^2(M)}$, and not by $|\sum_{\mathcal{C}\in W^c} \mu_h(\mathcal{C})| = |\sum_{\mathcal{C}\in W^c}\langle \hat{\mathcal{C}}_h \psi_h, \psi_h\rangle_{L^2(M)}|$, is that we need the sub-multiplicative property of $N_h(n,\theta)$ given by the next lemma, and that it only works with definition 2.2.1.

We will use the following lemma, proved in Appendix A1:

Lemma 2.2.2. *Let χ_h be a pseudo-differential operator, whose symbol is an energy cut-off, supported in a neighbourhood of the energy layer $||v|| = 1$. There exist $\bar{\kappa}$ and $\alpha > 0$ such that, for all $n \leq \bar{\kappa}|\log h|$, for every subset $W \subset \Sigma_n$,*

$$||\chi_h^* \sum_{\mathcal{C}\in W} \hat{\mathcal{C}}_h \chi_h||_{L^2(M)} \leq 1 + O(h^\alpha).$$

Lemma 2.2.3. (Sub-multiplicativity) *If $\bar\kappa$ and α are as in Lemma 2.2.2, then for every $n \leq \bar\kappa|\log h|, k\in\mathbb{N}$ and $\theta\in]0,1[$,*

$$N_h\Big(kn, k\theta(1+O(nh^\alpha)\Big)\leq N_h\big(n,\theta\big)^k.$$

Proof. Given a $(h,(1-\theta),n)$-cover of Σ, denoted W, we show that $W^k\subset\Sigma_{kn}$, defined as the set of kn-cylinders $[\alpha_0,...,\alpha_{kn-1}]$ such that $[\alpha_{jn},...,\alpha_{(j+1)n-1}]\in W$ for all $j\in[0,k-1]$, is a $(h,1-k\theta,kn)$-cover:

Each $\mathcal{C}\in(W^k)^c$ may be decomposed into the concatenation of k cylinders of length n, $\mathcal{C}=\mathcal{C}^0\mathcal{C}^1...\mathcal{C}^{k-1}$, one of which is not in W. Thus, we have

$$(2.2.2)\quad ||\sum_{\mathcal{C}\in(W^k)^c}\hat{\mathcal{C}}_h\psi_h||_{L^2(M)}$$
$$=||\sum_{j=0}^{k-1}\sum_{\mathcal{C}^i\in W\text{ for }i<j,\mathcal{C}^j\in W^c,\mathcal{C}^i\in\Sigma_n\text{ for }i>j}\hat{\mathcal{C}}_h^0...\hat{\mathcal{C}}_h^j(-jn)...\hat{\mathcal{C}}_h^{k-1}(-(k-1)n)\psi_h||$$
$$=||\sum_{j=0}^{k-1}\sum_{\mathcal{C}^i\in W\text{ for }i<j,\mathcal{C}^j\in W^c}\hat{\mathcal{C}}_h^0...\hat{\mathcal{C}}_h^j(-jn)\psi_h||.$$

Using Lemma 2.2.2 to bound the norm of $\sum_{\mathcal{C}^i\in W\text{ for }i<j}\hat{\mathcal{C}}_h^0...\hat{\mathcal{C}}_h^{j-1}(-(j-1)n)$ by $(1+O(h^\alpha))^j$, we see that (2.2.2) is less than

$$(1+O(h^\alpha))^n\sum_{j=0}^{k-1}||\sum_{\mathcal{C}^j\in W^c}\hat{\mathcal{C}}_h^j(-jn)\psi_h||$$
$$=(1+O(h^\alpha))^n\sum_{j=0}^{k-1}||\sum_{\mathcal{C}^j\in W^c}\hat{\mathcal{C}}_h^j\psi_h||\leq k\theta(1+O(nh^\alpha)).$$

We used here the fact that ψ_h is an eigenfunction. □

The next proposition is just an expression of Observation (a).

Proposition 2.2.4. *For every N large enough and for $h\leq 1$, we have*

$$N_h(N,\theta)\geq(1-\theta)(2\pi h)^{d/2}e^{N(\frac{\chi}{2}-\eta-Ch)}\big[1+R(n,h)\big]^{-1}.$$

Proof. Let W be a $(h,(1-\theta),N)$-cover of Σ. We have

$$|\sum_{\mathcal{C}\in W^c}\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle|\leq||\sum_{\mathcal{C}\in W^c}\hat{\mathcal{C}}_h\psi_h||\,||\psi_h||\leq\theta,$$

so that

$$|\sum_{\mathcal{C}\in W}\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle|\geq 1-\theta.$$

Thus,

$$1-\theta\leq\sum_{\mathcal{C}\in W}|\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle|\leq\sharp W\frac{e^{-N(\frac{\chi}{2}-\eta-Ch)}}{(2\pi h)^{d/2}}\big[1+R(n,h)\big],$$

where the last line comes from Theorem 1.3.3. □

This immediately implies:

Lemma 2.2.5. *Given any $\delta > 0$, we may choose ϑ large enough, and η small enough, so that, for $N = \vartheta|\log h|$, we have*

$$N_h(N,\theta) \geq \big(1-\theta\big)e^{N(\frac{\chi}{2}-\delta)}.$$

We will choose δ later in the proof of Theorem 1.1.1 (actually, at the beginning of the next paragraph). It will depend on the set F appearing in the theorems.

As we said, semi-classical analysis is usually only valid until a certain time $\bar{\kappa}|\log h|$, and in general $\bar{\kappa} < \vartheta$. Lemma 2.2.3 is precisely the tool that will allow us to reduce the time scale: starting from Lemma 2.2.5, it tells us that, for $N = \bar{\kappa}|\log h|$, $0 \leq \bar{\kappa} \leq \vartheta$,

$$N_h(N, \frac{\bar{\kappa}}{\vartheta}\theta) \geq (1-\theta)^{\bar{\kappa}/\vartheta} e^{N(\frac{\chi}{2}-\delta)}. \tag{2.2.3}$$

Note that the σ-invariance of μ_h was absolutely crucial to prove Lemma 2.2.3 and hence (2.2.3).

2.3. **Covering sets of small topological entropy.** The aim of this paragraph is to put a precise statement behind observation (b). Lemma 2.3.2 belowsays that, if F is a set of small topological entropy, then the set of orbits spending a lot of time near F also has a small rate of exponential growth.

We denote $h_{top}(S^1M)$ the topological entropy of the action of (g^t) on S^1M. More generally, if $F \subset S^1M$ is closed and invariant under (g^t), we denote $h_{top}(F)$ the topological entropy of the flow restricted to F (see the definition in [KH], Chapter 3).

To prove Theorem 1.1.1, let us consider an invariant subset $F \subset S^1M$ of topological entropy $h_{top}(F) < \frac{\chi}{2}$; let $\delta > 0$ be such that $h_{top}(F) + 11\,\delta \leq \frac{\chi}{2}$. It is for this real number $\delta > 0$ that we will later apply Lemma 2.2.5 and (2.2.3).

Let us now denote $N_n(\overline{P}, F)$: the minimal number of cylinders

$$[\alpha_0, \alpha_1, ..., \alpha_{n-1}] \in \Sigma_n$$

such that the corresponding sets $\overline{P_{\alpha_0}} \cap ... \cap g^{-n+1}\overline{P_{\alpha_{n-1}}}$ cover F. Without loss of generality, we may assume

$$\limsup \frac{\log N_n(\overline{P}, F)}{n} \leq h_{top}(F) + 2\delta. \tag{2.3.1}$$

Remark 2.3.1. Here is why: We know from the definition of topological entropy that, for $\varepsilon > 0$ small enough and every N large enough, there exists a set $\{\xi_1, ..., \xi_{\exp(N(h_{top}(F)+\delta))}\}$ of geodesic arcs $[0, N] \longrightarrow M$, lying in F, and which is (ε, N)-spanning for F in the following sense:

For every geodesic arc $\xi : [0, N] \longrightarrow M$, in F, there exists $j \in \big[1, \exp\big(N(h_{top}(F)+\delta)\big)\big]$ such that $d_M\big(\xi(t), \xi_j(t)\big) \leq \varepsilon$ for all $t \in [0, N]$.

Fix $T \geq 1$ such that $8^{2d} < e^{T\delta}$, and replace the alphabet $\{1, ..., l\}$ by a new alphabet A, whose letters are triples

$$(\alpha_0, \alpha_{T-1}, \gamma) = \cup_{\alpha_1, ...\alpha_{T-2}}[\alpha_0, ..., \alpha_{T-1}],$$

where

– $\alpha_j \in \{1, ..., l\}$.

– $\gamma : [0, T] \longrightarrow M$ is a given geodesic (parametrized with arc-length) with $\gamma(0) \in \overline{P}_{\alpha_0}$, $\gamma(T-1) \in \overline{P}_{\alpha_{T-1}}$.

– $\alpha_1, ..., \alpha_{T-2}$ run over the set of letters such that: there exists $\gamma' : [0, T-1] \longrightarrow M$, a geodesic path, with $\gamma(j) \in \overline{P}_{\alpha_j}$ $(j = 0, ..., T-1)$ and γ' is homotopic to γ with endpoints staying in $P_{\alpha_0}, P_{\alpha_T}$.

We claim that the minimum number of sequences of length n in this new alphabet, such that the corresponding cylinders cover F, is less than

$$\exp\left(nT(h_{top}(F)+\delta)\right) \times 8^{2dn} \leq \exp(nT(h_{top}(F)+2\delta)). \tag{2.3.2}$$

In fact, take $N = nT$ in the choice of the (ε, N)-spanning set above. Let $\xi : [0, nT] \longrightarrow M$ be a geodesic in F. There exists $j \in \left[1, \exp\left(nT(h_{top}(F)+\delta)\right)\right]$ such that ξ stays ε-close to ξ_j. In particular, $\xi(kT)$ and $\xi(kT-1)$ are respectively ε-close to $\xi_j(kT)$ and $\xi_j(kT-1)$ (for $k = 0, 1, ..., n$).

Because each P_i has volume greater than $(\varepsilon/4)^d$, at most $(8\varepsilon)^d$ can fit into a ball a radius 2ε. We see thus that, given ξ_j $(j \in \left[1, \exp\left(nT(h_{top}(F)+\delta)\right)\right])$, we need at most $(8\varepsilon)^{2dn}$ words in the new alphabet A to describe the cylinders of length n, covering the geodesics ξ in F staying ε-close to ξ_j in $[0, nT]$. This shows (2.3.2). If $T = 1$ we get exactly what we claimed, (2.3.1). If $T > 1$, we could work on a different symbolic space Σ defined with the new alphabet A instead of $\{1, ..., l\}$, and have (2.3.1). Thus we may assume without loss of generality that (2.3.1) holds.

In particular, there exists n_0 such that

$$N_n(\overline{P}, F) \leq e^{n(h_{top}(F)+3\delta)},$$

for all $n \geq n_0$. We denote W_n a cover of minimal cardinality of F by sets of the form $\overline{P_{\alpha_0}} \cap ... \cap g^{-n+1}\overline{P_{\alpha_{n-1}}}$, and $\Sigma(W_n) \subset \Sigma_n$ the set of the corresponding cylinders $[\alpha_0, ..., \alpha_{n-1}]$.

Given $N \in \mathbb{N}$, $n \leq N$ and $\mu \in [0, 1]$, we denote $\Sigma_N(W_n, \mu)$: the set of N-cylinders $[\alpha_0, ..., \alpha_{N-1}]$ such that

$$\frac{\sharp\left\{j \in [0, N-n], [\alpha_j, ..., \alpha_{j+n-1}] \in \Sigma(W_n)\right\}}{N-n+1} \geq \mu.$$

They correspond to orbits that spend too much time in the neighbourhood of F.

The next lemma bounds the cardinality of $\Sigma_N(W_n, \mu)$.

Lemma 2.3.2. (Counting cylinder sets) *There exist $n_1 \geq n_0$, and N_0 such that, for every $N \geq N_0$ and for every $\mu \in [0, 1]$,*

$$\sharp\Sigma_N(W_n, \mu) \leq e^{8\delta N} e^{N h_{top}(F)} e^{(1-\mu)N(1+n_1)\log l}.$$

Proof. Take $n_1 \geq n_0$ large enough so that

$$\lim_{N \longrightarrow +\infty} \frac{1}{N} \log \binom{\lfloor \frac{N}{n_1} \rfloor}{N} \leq \frac{\delta}{10}$$

(we denote $\binom{k}{N}$ the binomial coefficients); n_1 is now fixed.

Given a sequence $[\alpha_0 ..., \alpha_{N-1}] \in \Sigma_N$, define a sequence of “stopping times”:

$$\begin{aligned}
\tau_0 &= \inf\left\{0 \leq j \leq N - n_1, [\alpha_j, ..., \alpha_{j+n_1-1}] \in \Sigma(W_{n_1})\right\}, \\
\tau_0' &= \inf\left\{\tau_0 \leq j \leq N - n_1, [\alpha_j, ..., \alpha_{j+n_1-1}] \notin \Sigma(W_{n_1})\right\}, \\
\tau_1 &= \inf\left\{\tau_0' - 1 + n_1 \leq j \leq N - n_1, [\alpha_j, ..., \alpha_{j+n_1-1}] \in \Sigma(W_{n_1})\right\},
\end{aligned}$$

and so on:

$$\tau_{k+1} = \inf\left\{\tau'_k - 1 + n_1 \le j \le N - n_1, [\alpha_j, ..., \alpha_{j+n_1-1}] \in \Sigma(W_{n_1})\right\},$$
$$\tau'_{k+1} = \inf\left\{\tau_k \le j \le N - n_1, [\alpha_j, ..., \alpha_{j+n_1-1}] \notin \Sigma(W_{n_1})\right\}.$$

The sequence (τ_k) becomes stationary, equal to $N - n_1$, for $k \ge \left\lfloor \frac{N}{n_1} \right\rfloor$. Define the intervals $I_0 = [\tau_0, \tau'_0 - 1 + n_1 - 1]$,...,$I_k = [\tau_k, \tau'_k - 1 + n_1 - 1]$. If $\mathcal{C} = [\alpha_0, ..., \alpha_{N-1}]$ is in $\Sigma_N(W_{n_1}, \mu)$, then the complement of $\cup I_k$ has cardinality less than $(1-\mu)(N - n_1 + 1) + n_1 \le (1-\mu)N + n_1$.

A cylinder $\mathcal{C} = [\alpha_0, ..., \alpha_{N-1}] \in \Sigma_N(W_{n_1}, \mu)$ is completely determined by the following data:
(i) the intervals $(I_k)_{0 \le k \le \lfloor N/n_1 \rfloor}$
(ii) the restriction of $\mathcal{C}$ to the union of the I_ks.
(iii) the values of $\mathcal{C}$ outside the I_ks.

Let us count in each case the number of possibilities:
(i) There are at most $\binom{\lfloor N/n_1 \rfloor}{N}^2$ possibilities, corresponding to the choices of the endpoints of the intervals I_k; by our choice of n_1, for N large enough this is less than $e^{\delta N}$.
(ii) Each I_k can be split into a disjoint union of intervals of length n_1 and at most one interval of length less than n_1. The intervals of length (exactly) n_1 thus obtained are at most N/n_1, and they correspond to cylinders covering F: there are at most $(\sharp\Sigma(W_{n_1}))^{N/n_1}$ possibilities. If $n_1 \ge n_0$ this is less than $\left(e^{n_1(h_{top}(F)+3\delta)}\right)^{N/n_1} = e^{N(h_{top}(F)+3\delta)}$ by (2.3.1). For the remaining intervals, of length strictly less than n_1, there can be at most $(1-\mu)N$ of them; this gives $l^{(1-\mu)Nn_1}$ possibilities.
(iii) For the values of α outside the I_ks, the number of possible choices is bounded by $l^{(1-\mu)N+n_1}$. Choose N_0 such that $l^{n_1} \le e^{N_0\delta/2}$.

This ends the proof of the lemma. Note that our estimates are very rough, since we argued as if all choices in (i), (ii) and (iii) were independent.

□

In particular, if we choose $\mu \in (0,1)$ close enough to 1 so that

$$h_{top}(F) + (1-\mu)N(1+n_1)\log l + 8\delta \le \frac{\chi}{2} - 2\delta,$$

we have

$$\sharp\Sigma_N(W_n, \mu) \le e^{(\frac{\chi}{2} - 2\delta)N}, \tag{2.3.3}$$

for all N large enough.

Now, comparing (2.3.3) with (2.2.3), for h small enough and $N = \bar{\kappa}|\log h|$, we have necessarily:

$$\Big|\Big| \sum_{\mathcal{C} \in \Sigma_N(W_{n_1}, \mu)^c} \hat{\mathcal{C}}_h \psi_h \Big|\Big|_{L^2} \ge \frac{\bar{\kappa}}{\vartheta}\theta. \tag{2.3.4}$$

This says, in a certain sense, that the measure of the complement of $\Sigma_N(W_{n_1}, \mu)$ cannot be too small. There remains to relate (2.3.4) with

$$|\mu_h(\Sigma_N(W_{n_1}, \mu)^c)| = \Big| \sum_{\mathcal{C} \in \Sigma_N(W_{n_1}, \mu)^c} \langle \hat{\mathcal{C}}_h \psi_h, \psi_h \rangle \Big|.$$

This is done in the next two paragraphs, and goes roughly as follows:

Imagine that the operators $\hat{\mathcal{C}}_h$ were orthogonal projectors, with orthogonal images for distinct cylinders $\mathcal{C}$. Ideally, this would be the case if:

– the operators A_i were a family of orthogonal projectors (that is, if the A_is were characteristic functions of disjoint sets);

– the operators $A_i(t)$ commuted with one another for all t.

If that was the case, we could write

$$\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)^c}\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle = \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)^c}||\hat{\mathcal{C}}_h\psi_h||^2_{L^2} = ||\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)^c}\hat{\mathcal{C}}_h\psi_h||^2_{L^2} \tag{2.3.5}$$

so that (2.3.4) would imply the lower bound

$$|\mu_h(\Sigma_N(W_{n_1},\mu)^c)| \geq \left(\frac{\bar{\kappa}}{\vartheta}\theta\right)^2.$$

Unfortunately, the A_is are not characteristic functions of disjoint sets, they form a smooth partition of unity; and the operators $A_i(t)$ do not commute.

However,

– we have constructed the A_i so that they act on ψ_h almost as an orthogonal family of projectors.

– there exists $\bar{\kappa}>0$ such that the operators $A_i(t)$ almost commute for $|t|\leq\bar{\kappa}|\log h|$:

Proposition 2.3.3. *There exists $\bar{\kappa}>0$ such that, for every $N\leq 2\bar{\kappa}|\log h|$, for every permutation τ of $\{0,...,N\}$, for every sequence $t_0,...,t_N$ such that $|t_i|\leq\bar{\kappa}|\log h|$, for every sequence $\alpha_0,...,\alpha_N$,*

$$||A_{\alpha_N}(t_N).....A_{\alpha_1}(t_1)A_{\alpha_0}(t_0)\psi_h - A_{\alpha_{\tau N}}(t_{\tau N}).....A_{\alpha_{\tau 1}}(t_{\tau 1})A_{\alpha_{\tau 0}}(t_{\tau 0})\psi_h||_{L^2(M)} = O(h^{\bar{\kappa}})$$

The proof is given in Appendix A1.

So, there is hope to prove (2.3.5), at least up to a negligible remainder term. That is the object of the next paragraph.

2.4. Relating $||\sum\hat{\mathcal{C}}_h\psi_h||$ and $\sum\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle$.

Remember that we constructed the partition of unity (A_i^h) in such a way that:

There exists $p>0$ such that

$$||(A_i^2-A_i)\psi_h||_{L^2(M)} = O(h^{p/2})$$

and

$$||A_iA_j\psi_h||_{L^2(M)} = O(h^{p/2}),$$

for all i and all $j\neq i$.

Let us choose the parameter $\bar{\kappa}$ so that the conclusion of Proposition 2.3.3 holds. This ensures that there is no harm in treating the $\hat{\mathcal{C}}_h$ as orthogonal projectors in (2.3.4).

Using Proposition 2.3.3, which allows commutation of the operators $A_i(t)$ for $|t|\leq\bar{\kappa}|\log h|$ (up to $O(h^{\bar{\kappa}})$), we find that, for $N\leq\bar{\kappa}|\log h|$, for $\mathcal{C},\mathcal{C}'\in\Sigma_N,\mathcal{C}\neq\mathcal{C}'$,

$$|\langle\hat{\mathcal{C}}_h\psi_h,\hat{\mathcal{C}}'_h\psi_h\rangle| = O(h^{\bar{\kappa}})+O(h^{p/2}),$$

and

$$|\langle\hat{\mathcal{C}}_h\psi_h,\psi_h\rangle - \langle\hat{\mathcal{C}}_h\psi_h,\hat{\mathcal{C}}_h\psi_h\rangle| = N\big(O(h^{\bar{\kappa}})+O(h^{p/2})\big).$$

Then, for $N \le \bar\kappa|\log h|$,

$$\sum_{C,C'\in\Sigma_N, C\neq C'} |\langle \hat{\mathcal{C}}_h\psi_h, \hat{\mathcal{C}}'_h\psi_h\rangle| = \big(O(h^{\bar\kappa}) + O(h^{p/2})\big)(\sharp\Sigma_n)^2$$

and

$$\sum_{\mathcal{C}\in\Sigma_N} |\langle \hat{\mathcal{C}}_h\psi_h, \psi_h\rangle - \langle \hat{\mathcal{C}}_h\psi_h, \hat{\mathcal{C}}_h\psi_h\rangle| = N\big(O(h^{\bar\kappa}) + O(h^{p/2})\big)\sharp\Sigma_N.$$

Since the cardinality of Σ_N grows exponentially, we can adjust $\bar\kappa$ so that, for $N \le \bar\kappa|\log h|$,

$$\sum_{C,C'\in\Sigma_N, C\neq C'} |\langle \hat{\mathcal{C}}_h\psi_h, \hat{\mathcal{C}}'_h\psi_h\rangle| = O(h^{\bar\kappa})$$

and

$$\sum_{\mathcal{C}\in\Sigma_N} |\langle \hat{\mathcal{C}}_h\psi_h, \psi_h\rangle - \langle \hat{\mathcal{C}}_h\psi_h, \hat{\mathcal{C}}_h\psi_h\rangle| = O(h^{\bar\kappa}).$$

The two properties above imply that, for $N \le \bar\kappa|\log h|$, for every subset $S \subset \Sigma_N$,

$$\sum_{\mathcal{C}\in S} |\mu_h(\mathcal{C})| = |\sum_{\mathcal{C}\in S} \mu_h(\mathcal{C})| + O(h^{\bar\kappa}) \tag{2.4.1}$$

$$= \sum_{\mathcal{C}\in S} ||\hat{\mathcal{C}}_h\psi_h||^2 + O(h^{\bar\kappa}) \tag{2.4.2}$$

$$= ||\sum_{\mathcal{C}\in S} \hat{\mathcal{C}}_h\psi_h||^2 + O(h^{\bar\kappa}). \tag{2.4.3}$$

The point is that, when working on cylinders of size $\bar\kappa|\log h|$, the measure μ_h is non-negative, up to a negligible remainder term. The first line implies in particular that

$$\sum_{\mathcal{C}\in\Sigma_N} |\mu_h(\mathcal{C})| = 1 + O(h^{\bar\kappa}) \tag{2.4.4}$$

Now, coming back to (2.3.4), we get for $N = \bar\kappa|\log h|$, and n_1 as in Lemma 2.3.2,

$$\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)^c} |\mu_h(\mathcal{C})| \ge \left(\frac{\bar\kappa}{\vartheta}\theta\right)^2 + O(h^{\bar\kappa})$$

and, because of (2.4.4),

$$\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)} |\mu_h(\mathcal{C})| \le 1 - \left(\frac{\bar\kappa}{\vartheta}\theta\right)^2 + O(h^{\bar\kappa}) \tag{2.4.5}$$

2.5. **End of the proof.** We can now conclude the proof, following the strategy given at the beginning of this section.

We use again the σ-invariance of μ_h (Proposition 1.3.1 (ii), and we get, for $N = \bar{\kappa}|\log h|$,

$$|\mu_h(\Sigma(W_{n_1}))| = |\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mu_h(\sigma^{-k}\Sigma(W_{n_1}))| + O(h^\infty) \tag{2.5.1}$$

$$= |\mu_h\Big(\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mathbb{I}_{\sigma^{-k}\Sigma(W_{n_1})}\Big)| + O(h^\infty) \tag{2.5.2}$$

$$\leq \sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)}|\mu_h(\mathcal{C})| + \mu\sum_{\mathcal{C}\notin\Sigma_N(W_{n_1},\mu)}|\mu_h(\mathcal{C})| + O(h^\infty) \tag{2.5.3}$$

$$\leq (1-\mu)\sum_{\mathcal{C}\in\Sigma_N(W_{n_1},\mu)}|\mu_h(\mathcal{C})| + \mu + O(h^{\bar{\kappa}}) \tag{2.5.4}$$

$$\leq (1-\mu)\left(1-\left(\frac{\bar{\kappa}}{\vartheta}\theta\right)^2\right) + \mu + O(h^{\bar{\kappa}}). \tag{2.5.5}$$

For (2.5.3), we have used the fact that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mathbb{I}_{\sigma^{-k}\Sigma(W_{n_1})} \leq 1,$$

in general, and that

$$\frac{1}{N-n_1}\sum_{k=0}^{N-n_1-1}\mathbb{I}_{\sigma^{-k}\Sigma(W_{n_1})} \leq \mu$$

on $\Sigma_N(W_{n_1},\mu)^c$. In the next line, we have used (2.4.4); and we conclude thanks to (2.4.5).

Passing to the limit in (2.5.5) (and using 1.3.2), we obtain

$$\nu_0\big(\cup_{\Sigma(W_{n_1})} P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n_1+1}P_{\alpha_{n_1-1}}\big) \leq (1-\mu)\left(1-\left(\frac{\bar{\kappa}}{\vartheta}\theta\right)^2\right) + \mu < 1.$$

By definition of W_{n_1}, one has $F\subset \cup_{\Sigma(W_{n_1})}P_{\alpha_0}\cap g^{-1}P_{\alpha_1}...\cap g^{-n_1+1}P_{\alpha_{n_1-1}}$: we obtain finally

$$\nu_0(F) \leq (1-\mu)\Big(1-\Big(\frac{\bar{\kappa}}{\vartheta}\theta\Big)^2\Big) + \mu < 1.$$

Noting that this last estimate holds for every $\theta < 1$, we get

$$\nu_0(F) \leq (1-\mu)\Big(1-\Big(\frac{\bar{\kappa}}{\vartheta}\Big)^2\Big) + \mu,$$

which proves Theorem 1.1.1

Remark 2.5.1. The method gives a uniform estimate for all F such that $h_{top}(F) \leq \frac{\chi}{2} - 11\delta$.

The rest of the paper is devoted to the proof of the technical estimate, Theorem 1.3.3. First we need to recall a few facts about hyperbolic dynamics, and to introduce some notations.

3. About Anosov geodesic flows

3.1. **What does an Anosov geodesic flow look like ?** The action of the geodesic flow $(g^t)_{t\in\mathbb{R}}$ on an energy layer $S^\lambda M$ is said to be Anosov if there is a splitting of the tangent bundle $T(S^\lambda M)$ into three sub-bundles, $T(S^\lambda M) = E^s \oplus E^0 \oplus E^u$, invariant under the flow, and such that:

– E^0 is 1-dimensional and is generated by the tangent to the orbits of the geodesic flow;

– there exist $C > 0, \theta \in [0,1)$ such that, for all $v \in S^\lambda M$, for all $\mathbf{v} \in E^s_v \subset T_v(S^\lambda M)$, for all $t \geq 0$,

$$||dg^t(v).\mathbf{v}|| \leq C\theta^t ||\mathbf{v}||;$$

and, for all $v \in S^\lambda M$, for all $\mathbf{v} \in E^u_v \subset T_v(S^\lambda M)$, for all $t \geq 0$,

$$||dg^{-t}(v).\mathbf{v}|| \leq C\theta^t ||\mathbf{v}||.$$

For a geodesic flow, the distributions E^s and E^u, called stable and unstable distributions, are both $(d-1)$-dimensional. They are automatically integrable, and we denote respectively W^s and W^u the integral foliations. The leaves are called stable and unstable manifolds of the flow. Each energy layer $S^\lambda M$ in TM is also foliated into $(d-1)$-dimensional spheres

$$S^\lambda_x M = \{v \in S^\lambda M, \pi(v) = x\}.$$

Any two of these three foliations are transverse in $S^\lambda M$. In fact, the sphere foliation is transverse, not only to the strong stable and unstable foliations W^s, W^u, but also to the weak stable and unstable foliation W^{0s}, W^{0u}, whose leaves are the orbits under the geodesic flow of the leaves of W^s, W^u: a proof of this fact will be provided at the end of this section.

We will also need the following properties of Anosov geodesic flows, proved in [Eb73], [Kl74], [An85], [Ru91]:

– the injectivity radius of $\tilde{M}$, the universal cover of M, is infinite; $\tilde{M}$ is diffeomorphic to $\mathbb{R}^d$, via the exponential map.

– there are no conjugate points; in $\tilde{M}$, every geodesic achieves the infimum of the distance between any two points.

Finally, we will use the Shadowing Lemma in the following form:

For $\theta \geq 0$ and $T \geq 1$, we will call "(θ, T)-pseudogeodesic" (parametrized by $[0, nT]$) any piecewise C^1 curve $\xi : [0, nT] \longrightarrow M$ such that, on every interval $[kT, (k+1)T]$, ξ is a geodesic, parametrized with constant speed in $[1-\theta, 1+\theta]$, and such that, for all $k = 1, 2, ..., n-1$, $d_{TM}\big(\dot{\xi}_-(kT), \dot{\xi}_+(kT)\big) \leq \theta$ (these notations mean the left- and right-hand side derivatives with respect to time). Note that a (θ, mT)-pseudogeodesic is, in particular, a (θ, T)-pseudogeodesic, if m is an integer.

The Shadowing Lemma says that there exists θ_0 and $J > 0$ such that, for every $\theta \in (0, \theta_0]$, if $\xi : [0, nT] \longrightarrow M$ is a (θ, T)-pseudogeodesic for some $T \geq 1$, then there exists a geodesic $\xi_S : [0, nT] \longrightarrow M$ such that

$$d_{TM}\big(\dot{\xi}_S(t), \dot{\xi}(t)\big) \leq J\theta$$

for all $t \in [0, nT]$.

3.2. **A theorem of Aubry-McKay-Baesens .** The main theorem in [AMB92] translates the uniform hyperbolicity of a trajectory of a twist map, in terms of the hessian of the generating function. Here is a way to reformulate the result:

Let Ω be an open subset of $\mathbb{R}^d$, containing 0, and, for all $k \in \mathbb{Z}$, a function of class C^2:

$$L_k : \Omega \times \Omega \longrightarrow \mathbb{R}.$$

For $i, j \in \{1, 2\}$, we denote $D_i L_k$ the derivative of L_k with respect to the variable i, and, similarly, $D_{ij} L_k$ is the derivative of order 2 with respect to the variables i, j. We assume that the second derivatives of the L_ks are bounded uniformly in k; we also assume that, for all $y \in \Omega$, for all k, $D_1 L_k(y, .)$ is a diffeomorphism onto its image, that $D_2 L_k(0, 0) + D_2 L_{k+1}(0, 0) = 0$, and that $||D_{12} L_k(0, 0)^{-1}||$ is bounded uniformly in k.

By the implicit functions theorem, there exists for all k a diffeomorphism T_k, defined in a neighbourhood of $(0, 0) \in \Omega \times \Omega$ such that:

$$T_k(0, 0) = (0, 0)$$

and

$$\frac{\partial}{\partial y} \left(L_k(x, y) + L_{k+1}(y, z) \right) = 0 \Longleftrightarrow (y, z) = T_k(x, y).$$

We say that the sequence of functions (L_k) are generating functions for the diffeomorphisms T_k.

Let $H(0)$ be the hessian at $(0, ...0, ...)$ of the (formal) sum

$$(x_k)_{k \in \mathbb{Z}} \in \Omega^{\mathbb{Z}} \mapsto \sum_k L_k(x_k, x_{k+1}).$$

In other words, $H(0)$ is the infinite symmetric matrix, tridiagonal by blocks of size $d \times d$, these blocks being given by

$$H(0)_{k,k} = D^2_{22} L_{k-1}(0, 0) + D^2_{11} L_k(0, 0),$$

$$H(0)_{k,k+1} = D^2_{21} L_k(0, 0),$$

$$H(0)_{k,j} = 0 \text{ if } j > k + 1.$$

The space $\mathbb{R}^d$ is endowed with its canonical euclidean structure denoted $\langle ., . \rangle$ and the corresponding euclidean norm $||.||$. The space $l^2(\mathbb{Z}, \mathbb{R}^d)$ is the space of sequences $(v_n) \in (\mathbb{R}^d)^{\mathbb{Z}}$ such that $\sum ||v_n||^2 < +\infty$, endowed with the corresponding euclidean norm, denoted $||.||_2$. Similarly, the space $l^\infty(\mathbb{Z}, \mathbb{R}^d)$ is the space of sequences $(v_n) \in (\mathbb{R}^d)^{\mathbb{Z}}$ such that $\sup ||v_n|| < +\infty$, endowed with the corresponding norm $||.||_\infty$. The notations $||.||_2$ and $||.||_\infty$ will also be used to denote the norms of bounded linear operators, respectively from l^2 to l^2 and from l^∞ to l^∞.

Theorem 3.2.1. *(*[AMB92]*) The operator $H(0)$ is invertible in $l^2(\mathbb{Z}, \mathbb{R}^d)$ (or $l^\infty(\mathbb{Z}, \mathbb{R}^d)$) if and only if the family of diffeomorphisms T_k satisfies the following "uniform hyperbolicity" condition:*

There exist $C > 0$, $0 < \lambda < 1$, and, for all k, a decomposition $\mathbb{R}^d \times \mathbb{R}^d = E^s_k \oplus E^u_k$ such that

$$\text{for all } v \in E^s_k, l \geq 0, |DT_{k+l}(0, 0).DT_{k+l-1}(0, 0)...DT_k(0, 0).v| \leq C\lambda^l |v|$$

and

$$\text{for all } v \in E^u_k, l \geq 0, |DT_{k-l}(0, 0).DT_{k-l+1}(0, 0)...DT_k(0, 0).v| \leq C\lambda^l |v|$$

with, in addition, a uniform lower bound on the angles between the spaces E^s_k and E^u_k.

The constants C, λ, as well as the uniform lower bound on the angles between E_k^s et E_k^u, can be expressed explicitly in function of the norm of the inverse $\| H(0)^{-1} \|_2$, and vice-versa.

A crucial remark in the proof of this result is that there is an isomorphism between $\mathbb{R}^d \times \mathbb{R}^d$ (the tangent space of $\Omega \times \Omega$ at $(0,0)$) and the kernel of $H(0)$ in $(\mathbb{R}^d)^{\mathbb{Z}}$. This isomorphism identifies $(x_0, x_1) \in \mathbb{R}^d \times \mathbb{R}^d$ with the sequence $(x_k)_{k\in\mathbb{Z}}$ defined by

$$(x_k, x_{k+1}) = DT_{k-1}(0)...DT_1(0).DT_0(0).(x_0, x_1).$$

System of charts adapted to a geodesic:

Back to the setting of this paper, let us consider a geodesic ξ in M, parametrized with constant speed in $[1-\varepsilon, 1+\varepsilon]$, say for instance 1. Let $\eta > 0$ be small enough, and $\Omega = B(0,\eta) \subset \mathbb{R}^d$.

For all $k \in \mathbb{Z}$, let us choose a real analytic chart Φ_k sending Ω to a neighbourhood of $\xi(k)$, mapping 0 to $\xi(k)$, and such that $\Phi_k(\Omega)$ contains the ball of center $\xi(k)$ and of radius η in M. This charts should in addition be chosen so as to map the segment $]-\eta,\eta[\times\{0\}^{d-1}$ to the geodesic segment $\xi_{|]k-\eta,k+\eta[}$ (the parametrization being preserved), and, finally, so that, for all $t \in]-\eta,\eta[$, $D\Phi_k(t,0,...,0)$ maps the hyperplane $\{0\} \times \mathbb{R}^{d-1}$ to the orthogonal of $\dot{\xi}(k+t)$ in $T_{\xi(k+t)}M$.

For later purposes, let us note that this can be done so that

$$||D^n\Phi_k|| \leq CR^n n!$$

and

$$||D^n\Phi_k^{-1}|| \leq CR^n n!,$$

with constants C, R independent of the geodesic ξ (parametrized with constant speed in $[1-\varepsilon, 1+\varepsilon]$) and of the integer k. To be more precise, let us work with a given, finite system of charts on M, and say that all the estimates on the derivatives of the Φ_ks, Φ_k^{-1}s, as well as other functions, will be performed in this system of charts.

We define

$$L_k(x,y) = \frac{1}{2} d_M^2(\Phi_k(x), \Phi_{k+1}(y))$$

and

$$\tilde{L}_k(x,y) = d_M(\Phi_k(x), \Phi_{k+1}(y)).$$

Let $V = \{0\} \times \mathbb{R}^{d-1} \subset \mathbb{R}^d$. Theorem 3.2.1 says here that the hessian matrix $\tilde{H}(0)$ of the formal sum $\sum_k \tilde{L}_k(x_k, x_{k+1})$ restricted to $V^{\mathbb{Z}}$, is invertible in $l^2(\mathbb{Z}, V)$. In fact, the functions $\tilde{L}_k$, restricted to $(V\cap\Omega)\times(V\cap\Omega)$, generate diffeomorphisms $\tilde{T}_k$ that can be naturally interpreted as the first return maps on transversals to the direction of the geodesic flow at $\dot{\xi}(k) \in S^1M$; the Anosov property of the flow precisely means that these first return maps on transversals satisfy the uniform hyperbolicity condition of Theorem 3.2.1.

It follows also that the hessian matrix $H(0)$ of the formal sum $\sum_k L_k(x_k, x_{k+1})$ restricted to $V^{\mathbb{Z}}$, is invertible in $l^2(\mathbb{Z}, V)$. Actually, $H(0) = \tilde{H}(0)$ thanks to our choice of local coordinates that send the hypersurfaces $t + V$ to hypersurfaces orthogonal to ξ in M.

3.3. **A variant of Theorem 3.2.1.** For later purposes, we will need a variant of the main result of [AMB92]: for all n, let $H_n(0)$ be the hessian matrix corresponding to the second variation at 0 of the energy $\sum L_k(x_k, x_{k+1})$ on $V^{[1,n-1]} = V^{n-1}$. In other words, $H_n(0)$ represents, in the local coordinates, the second variation of the energy $\frac{1}{2}\sum d_M^2(\zeta_k, \zeta_{k+1})$ at $\zeta_1 = \xi(1), ..., \zeta_{n-1} = \xi(n-1)$, when the endpoints $\zeta_0 = \xi(0), \zeta_n = \xi(n)$ are fixed, and $\zeta_1, ..., \zeta_{n-1}$ vary in the orthogonal direction to the geodesic ξ. The matrix $H_n(0)$ is of size $(n-1)(d-1) \times (n-1)(d-1)$, it is the restriction of $H(0)$ to the space V^{n-1}.

Proposition 3.3.1. *The norms* $||H_n(0)^{-1}||_2$ *and* $\| H_n(0)^{-1} \|_\infty$ *are bounded uniformly in* n.

Remark 3.3.2. Now, we work on $l^2([1, n-1], V)$ and $l^\infty([1, n-1], V)$, defined like $l^2(\mathbb{Z}, V)$, $l^\infty(\mathbb{Z}, V)$ except that the sequences are indexed by $[1, n-1]$ instead of $\mathbb{Z}$. For all n, we will use the notation $\langle ., . \rangle$ for the scalar product in l^2 spaces, and $||.||_2$ and $||.||_\infty$ for the norms on l^2 or l^∞ spaces. As previously, the notations $||.||_2$ and $||.||_\infty$ will also stand for the corresponding operator norms.

Proof. The fact that $H_n(0)$ is invertible for all n follows from the fact that an Anosov geodesic flow has no conjugate points.

We start with l^2 norms. Since every geodesic has index 0, the symmetric matrix $H(0)$ is nonnegative, and we already know that it is invertible in $l^2(\mathbb{Z}, V)$. Thus, the spectrum of $H(0)$ is contained in an interval $[\varepsilon, 1/\varepsilon]$, where $0 < \varepsilon < 1$. This implies that

$$\langle H(0)v, v \rangle \geq \varepsilon \langle v, v \rangle$$

for all $v \in l^2(\mathbb{Z}, V)$.

Now, if $v \in V^{[1,n-1]}$ is seen as a vector in $l^2(\mathbb{Z}, V)$ with only $n-1$ non-zero coordinates, this can be written as

$$\langle H_n(0)v, v \rangle \geq \varepsilon \langle v, v \rangle,$$

which proves Proposition 3.3.1 for the l^2 norm.

In the l^∞ norm, the boundedness of $H_n(0)^{-1}$ does not follow in such a straightforward manner from Theorem 3.2.1. But one only has to do a slight modification in the original proof given in [AMB92].

First note that the action of $H_n(0)$ on $V^{[1,n-1]}$ is the same as the action of $H(0)$ on $\{(v_j)_{j\in\mathbb{Z}} \in V^{\mathbb{Z}}, v_0 = 0, v_n = 0\}$, restricted to the coordinates $1, ..., n-1$. Then, the proof of Proposition 2 in [AMB92] can be transcripted word by word, replacing the spaces l^2 or l^∞ by $\{(v_j)_{j\in\mathbb{Z}} \in V^{\mathbb{Z}}, v_0 = 0, v_n = 0\}$, and their decomposition (2.44) of $KerH(0)$ into unstable and stable manifold, by another decomposition:

(3.3.1)

$$KerH(0) = \Big\{(v_j)_{j\in\mathbb{Z}} \in KerH(0), v_0 = 0\Big\} \bigoplus \Big\{(v_j)_{j\in\mathbb{Z}} \in KerH(0), v_n = 0\Big\}.$$

In other words, every $v = (v_k) \in KerH(0)$ can be decomposed into $v = s^{(n)} + u^{(n)}$ with $s^{(n)} = (s_k^{(n)})_{k\in\mathbb{Z}} \in KerH(0)$, $u^{(n)} = (u_k^{(n)})_{k\in\mathbb{Z}} \in KerH(0)$, $u_0^{(n)} = 0$ and $s_n^{(n)} = 0$. The fact that the decomposition is unique follows from the property that the flow has no conjugate point.

Now, if $v \in V^{[1,n-1]}$ has only one non-vanishing coordinate, namely the i-th where $i \in [1, n-1]$, then $H_n(0)^{-1}v$ may be constructed by the following procedure:

– let $v^* \in V^{\mathbb{Z}}$ be the unique vector such that $v_i^* = 0$, $v_{i+1}^* = \big(H(0)_{i,i+1}\big)^{-1} v_i$ and $v^* \in KerH(0)$.

– decompose as above $v^* = s^{(n)} + u^{(n)}$ with $s^{(n)} = (s_k^{(n)})_{k\in\mathbb{Z}} \in KerH(0)$, $u^{(n)} = (u_k^{(n)})_{k\in\mathbb{Z}} \in KerH(0)$, $u_0^{(n)} = 0$ and $s_n^{(n)} = 0$. Let $\zeta \in V^{\mathbb{Z}}$ be defined by $\zeta_k = -u_k^{(n)}$ $(k \leq i)$ and $\zeta_k = s_k^{(n)}$ $(k \geq i+1)$. Then $\zeta_0 = \zeta_n = 0$ and $H(0).\zeta$ coincides with v on the coordinates $1, ..., n-1$. In other words, the restriction of ζ to the coordinates $1, ..., n-1$ is exactly $H_n(0)^{-1}v$.

For a general $v = (v_1, ..., v_{n-1}) \in V^{[1,n-1]}$, $H_n(0)^{-1}v$ is the superposition of the $H_n(0)^{-1}(0, 0, ..., v_j, 0, ..., 0)$ $(j \in [1, n-1])$ constructed above.

Finally, we will check in the next paragraph that the weak stable and unstable foliations are transverse to the sphere foliation, which implies estimates analogous to inequalities (2.46) and (2.49) of [AMB92]: there exists $C > 0$ and $\lambda \in (0, 1)$ such that, for every $n \in \mathbb{N}$, every $i \in [1, n-1]$, if $v \in V^{[1,n-1]}$ has only the i-th non-zero coordinate and $v^*, s^{(n)}$ and $u^{(n)}$ are defined as above, then

$$\left(||s_i^{(n)}||^2 + ||s_{i+1}^{(n)}||^2\right)^{1/2} \leq C||v||,$$

$$\left(||u_i^{(n)}||^2 + ||u_{i+1}^{(n)}||^2\right)^{1/2} \leq C||v||,$$

(this expresses the fact that, in the splitting (3.3.1) the angle is bounded away from 0),

$$||s_{n+k}^{(n)}||^2 + ||s_{n-1+k}^{(n)}||^2 \geq C^{-2}\lambda^{2k}\left(||s_n^{(n)}||^2 + ||s_{n-1}^{(n)}||^2\right)$$

for $k \leq 0$, and

$$||u_k^{(n)}||^2 + ||u_{k+1}^{(n)}||^2 \geq C^{-2}\lambda^{-2k}\left(||u_0^{(n)}||^2 + ||u_1^{(n)}||^2\right)$$

for $k \geq 0$ (this expresses the fact that a tangent vector to the sphere foliation, when evolved under the linearized flow, grows exponentially).

As proved in [AMB92] (Proposition 3), these estimates imply that

$$||H_n(0)^{-1}||_\infty \leq \frac{2C^2}{\lambda^{-1} - 1}.$$

□

3.4. **The sphere foliation is transverse to the weak stable foliation.** We prove that the kernel of $H(0)$, acting on $V^{\mathbb{Z}}$, cannot contain a non-zero vector $v = (v_k)_{k\in\mathbb{Z}}$ such that $v_0 = 0$ and $||v_n||_2$ decays exponentially fast as $n \longrightarrow +\infty$.

Let v be such a vector and let $v_+ := (..., 0, 0, v_0, v_1, ...)$: it belongs to $l^2(\mathbb{Z}, V)$. Because the flow has no conjugate points, the matrix $H(0)$ is non-negative, as an operator on the Hilbert space $l^2(\mathbb{Z}, V)$. One has $\langle H(0)v_+, v_+\rangle = 0$, since the only non-zero coordinate of $\tilde{H}(0)v_+$ is the 0-th coordinate, whereas $(v_+)_0 = 0$ by assumption.

The function $\langle H(0)., .\rangle$, defined on $l^2(\mathbb{Z}, \mathbb{R}^d)$ has a minimum at v_+, which implies that its derivative vanishes: $H(0)v_+ = 0$. Hence, $v_+ = 0$, and $v = 0$, since an element of $KerH(0)$ is entirely determined by two successive coordinates.

4. Proof of Theorem 1.3.3.

This section is devoted to proving the estimates on the decay of the measures of cylinder sets, given by Theorem 1.3.3.

Let, as above, χ_h be a pseudo-differential operators with compactly supported symbol χ, localized in a neighbourhood of the energy layer S^1M (say, on $S^{[1-\varepsilon,1+\varepsilon]} :=$

$\{v \in TM, ||v|| \in [1-\varepsilon, 1+\varepsilon]\}$). We also use ε for an upper bound on the diameter of the sets $\hat{P}_\alpha$, containing the supports of the functions A_α in their interiors. Assume that the injectivity radius of M is much larger than 1. Consider two functions A_i, A_j in our partition of unity. If all points $\xi_j \in supp\, A_j$ and $\xi_{j+1} \in supp\, A_i$ satisfy $|d(\xi_j, \xi_{j+1}) - 1| > \varepsilon$, then $||A_j \exp(\frac{ih\Delta}{2})\chi_h A_i||_{L^2(M)} = O(h^\infty)$. Otherwise, it follows from the general theory of Fourier integral operators ([Ho71], [DH72]) that $A_j \exp(\frac{ih\Delta}{2})\chi_h A_i$ can be written as $A_j E_h \cdot \chi_h A_i$ (up to $O(h^\infty)$), where the kernel of E_h takes the form

$$e^{(h)}(x, y, 1) = \frac{1}{(2\pi h)^{d/2}} e^{i \frac{d_M^2(x,y)}{2h}} a^{(h)}(x, y), \tag{4.0.1}$$

where a is a symbol of order 0. This means that $a^{(h)}$ is a C^∞ function with an asymptotic expansion of the form

$$a^{(h)}(x, y) = \sum_{j=0}^{+\infty} h^j a_j(x, y).$$

This expansion is valid in all the C^k norms. The Van Vleck formula says that $a_0(x, y) = e^{-\frac{i\pi}{4} d} Jac[\exp^1](v(x, y))^{-1/2}$: this means the jacobian of the exponential map $\exp_x^1$ at the unique vector $v(x, y) \in S_x^{[1-\varepsilon, 1+\varepsilon]} M$ such that $\exp_x^1(v(x, y)) = y$. In Assumption (I) we ask that the family $(a^{(h)})$ be uniformly analytic.

Consider a cylinder $[\alpha_0, ..., \alpha_n] \in \Sigma_{n+1}$. In Theorem 1.3.3 we wish to estimate $A_{\alpha_0} \exp(\frac{ih\Delta}{2}) A_{\alpha_1} ... \exp(\frac{ih\Delta}{2}) A_{\alpha_n} \psi_h$. Since ψ_h is microlocalized on the energy layer $S^1 M$, this is the same, up to $O(h^\infty)$, as $A_{\alpha_0} \exp(\frac{ih\Delta}{2}) A_{\alpha_1} ... \exp(\frac{ih\Delta}{2}) A_{\alpha_n} \chi_h \psi_h$, also the same as $A_{\alpha_0} \exp(\frac{ih\Delta}{2}) \chi_h A_{\alpha_1} ... \exp(\frac{ih\Delta}{2}) \chi_h A_{\alpha_n} \chi_h \psi_h$. Since the localization in energy is preserved by mulitplication by A_j and $\exp(ih\Delta/2)$, we may take χ_h freely in or out of the operator. It suffices to restrict our attention to the case when, for all j, there is a vector in the support of χ joining a point in $supp\, A_{\alpha_j}$ to a point in $supp\, A_{\alpha_{j+1}}$; otherwise $||A_{\alpha_0} \exp(\frac{ih\Delta}{2}) A_{\alpha_1} ... \exp(\frac{ih\Delta}{2}) A_{\alpha_n} \chi_h \psi_h|| = O(h^\infty)$, and the estimate of Theorem 1.3.3 is trivial. Finally, what we need to estimate is $A_{\alpha_0} E_h A_{\alpha_1} ... E_h A_{\alpha_n} \psi_h$, always up to $O(h^\infty)$. Note that terms of order $O(h^\infty)$ do not bother us, since they remain negligible with respect to the right-hand side of (1.3.7) as long as n is of order $|\log h|$.

What we actually estimate is the kernel of the operator $A_{\alpha_0} E_h A_{\alpha_1} ... E_h A_{\alpha_n}$. Consider some given ξ_0 and ξ_n in the supports, respectively, of A_{α_0} and A_{α_n}.

The full expression of the kernel at (ξ_0, ξ_n) is

$$\frac{A_{\alpha_0}(\xi_0) A_{\alpha_n}(\xi_n)}{(2\pi h)^{(n-1)d/2}} \int A_{\alpha_1}(\xi_1) ... A_{\alpha_{n-1}}(\xi_{n-1}) e^{\frac{i}{2h} \sum_{k=0}^{n-1} d^2(\xi_k, \xi_{k+1})} a^{(h)}(\xi_0, \xi_1) ... a^{(h)}(\xi_{n-1}, \xi_n) dVol(\xi_1) ... dVol(\xi_{n-1}); \tag{4.0.2}$$

According to the principle of the stationary phase method, to estimate such an integral we have to look for critical points of the function

$$F_{n-1}((\xi_1, ..., \xi_{n-1})) = \frac{1}{2} \sum_{k=0}^{n-1} d^2(\xi_k, \xi_{k+1}),$$

the endpoints ξ_0 and ξ_n being fixed.

Suppose that, for some j, and for all $\xi_{j-1} \in suppA_{j-1}$, $\xi_j \in suppA_j$, $\xi_{j+1} \in suppA_{j+1}$, the angle between the two geodesic segments (ξ_{j-1}, ξ_j) and (ξ_j, ξ_{j+1}) is greater than ε. Then (4.0.2) is $O(h^\infty)$ and the estimate of Theorem 1.3.3 is again trivial.

Finally, we have to prove Theorem 1.3.3 in the case of a cylinder $[\alpha_0, ..., \alpha_n]$ such that:

(A) for all j, for all $\xi_j \in supp\, A_j$, $\xi_{j+1} \in supp\, A_{j+1}$, $|d(\xi_j, \xi_{j+1}) - 1| \leq 3\varepsilon$.

(B) for all j, and for all $\xi_{j-1} \in suppA_{j-1}$, $\xi_j \in suppA_j$, $\xi_{j+1} \in suppA_{j+1}$, the angle between the two geodesic segments (ξ_{j-1}, ξ_j) and (ξ_j, ξ_{j+1}) is less than 3ε.

Theorem 4.0.1. *For every $\beta > 0$, there exists $C = C(\beta, d) \geq 0$ such that:*

For every $\eta > 0$, there exists $\varepsilon_0 > 0$ such that, if $0 < \varepsilon \leq \varepsilon_0$, one has an upper bound of the form:

$$(4.0.3) \quad \frac{1}{(2\pi h)^{d(n-1)/2}} \mid \int_{M^{n-1}} A_{\alpha_0}(\xi_0) A_{\alpha_1}(\xi_1) ... A_{\alpha_{n-1}}(\xi_{n-1}) A_{\alpha_n}^{\gamma_n}(\xi_n) \prod_{j=0}^{n-1} e^{(h)}(\xi_j, \xi_{j+1}, 1) dVol(\xi_1) dVol(\xi_2) ... dVol(\xi_{n-1}) \mid$$

$$\leq A_{\alpha_0}(\xi_0) A_{\alpha_n}^{\gamma_n}(\xi_n) \Big(1 + C(\eta + h)\Big)^{n-1} Jac[\exp^n](v_\alpha(\xi_0, \xi_n))^{-1/2}.$$

$$\Big[\exp(Chn^{2s+3} G_s(A)^2) + C^n \eta^{(n-1)d} n^{(n-1)d/2} h^{\beta n}\ n^{((2\beta+d)n+5)s} G_s(A)^{(2\beta+d)n+3}\Big]$$

for every n, for $h \in (0, 1]$, for every cylinder $[\alpha_0, ..., \alpha_n]$ satisfying (A), (B), for every $\xi_0, \xi_n \in M$.

The notation $Jac[\exp^n](v_\alpha(\xi_0, \xi_n))$ stands for the Jacobian of the exponential map $\exp^n : T_{\xi_0}M \longrightarrow M$, taken at the unique vector $v_\alpha(\xi_0, \xi_n)$ such that

– $\exp^n(v_\alpha(\xi_0, \xi_n)) = \xi_n$;

– the geodesic from ξ_0 to ξ_n, generated by v_α, is homotopic (with fixed endpoints) to a piecewise geodesic path going successively through the supports of A_{α_1},...,$A_{\alpha_{n-1}}$.

Remark 4.0.2. More generally, let M, N be two Riemannian manifolds of the same dimension, and Φ a differentiable map from M to N. Then for all $x \in M$, the jacobian of Φ at x, denoted $Jac[\Phi](x)$, is defined as the determinant of the derivative $D\Phi(x)$ (that is to say, the determinant of the matrix of $D\Phi(x)$ with respect to two arbitrary orthonormal bases of T_xM, $T_{\Phi(x)}N$). If $L \subset M$ is a submanifold, and Φ restricted to L is an immersion at $x \in L$, we will denote $Jac_L[\Phi](x)$ the jacobian at x of the restricted map, $\Phi : L \longrightarrow \Phi(L)$.

To prove Theorem 4.0.1, we follow the principle of the stationary phase method, and look for critical points of the function

$$F_{n-1}((\xi_1, ..., \xi_{n-1})) = \frac{1}{2} \sum_{k=0}^{n-1} d^2(\xi_k, \xi_{k+1}),$$

the endpoints ξ_0 and ξ_n being fixed. There is a unique geodesic ξ^c joining ξ_0 and ξ_n, homotopic to any of the polygonal paths $(\xi_0, \xi_1, ..., \xi_n)$ with $\xi_j \in supp\, A_j$. Then the sequence $(\xi_1^c, ..., \xi_{n-1}^c) \in M^{n-1}$, corresponding to the positions at times $1, 2, ..., n-1$ of ξ^c, gives a non-degenerate critical point of F_{n-1}.

This critical point may not belong to the support of $A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1})$. Remember, however, the Shadowing Lemma: given $\eta > 0$, there exists $\theta > 0$ such that, if ζ is a $(\theta, 1)$-pseudogeodesic, then the unique geodesic in M joining $\zeta(0)$ to

$\zeta(n)$ in time n, and homotopic to ζ, stays at distance at most η from ζ in TM; its (constant) speed lying in the interval $[1-\eta, 1+\eta]$. Now, given $\theta > 0$, we can take ε small enough so that, if $[\alpha_0, ..., \alpha_n]$ satisfies (A) and (B), then every $(\xi_0, ..., \xi_n) \in \hat{P}_{\alpha_0} \times ... \times \hat{P}_{\alpha_n}$ defines a $(\theta, 1)$-pseudogeodesic: namely, the piecewise geodesic curve joining, for all j, the points ξ_j and ξ_{j+1} in the time interval $[j, j+1]$ by a geodesic of constant speed. It follows that the unique geodesic ξ^c in M going from ξ_0 to ξ_n in time n – homotopic to any of the polygonal paths $(\xi_0, \xi_1, ..., \xi_n)$ with $\xi_j \in supp\, A_j$ – has speed in $[1-\eta, 1+\eta]$ and satisfies $d_M(\xi^c(k), \xi_k) < \eta$ for $k = 0, 1, ..., n$.

Thus, if ε is well chosen, the support of the function $A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1})$ is included in a uniform neighbourhood of $(\xi_1^c, ..., \xi_{n-1}^c)$ of radius η, that is to say:

$$A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1}) \neq 0 \Longrightarrow d(\xi_k, \xi_k^c) < \eta, \text{ for } k = 1, ..., n-1.$$

Then, the integral (4.0.2) runs over $B(\xi_1^c, \eta) \times ... \times B(\xi_n^c, \eta)$, where $B(x, \eta)$ stands for the ball of center x and radius η in M.

The parameter η will be chosen in paragraph 4.2, depending on the parameters controlling the uniform hyperbolicity of the flow; ε should then be chosen accordingly.

4.1. **Generalities on the stationary phase method: dependence on the dimension.** In this paragraph, we describe the stationary phase method and its dependence on the dimension, with the aim of applying our discussion to the study of integral (4.0.2).

Suppose we want to use the stationary phase method to study the asymptotic behaviour of the family of integrals

$$\frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} G_n(\xi) e^{\frac{iF_n(\xi)}{h}} d\xi \tag{4.1.1}$$

in the limit $h \longrightarrow 0$. The integrals run over $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, and we also want to control the dependence of the estimates with respect to the dimension n; we would like to get estimates that are significant for n of the order $|\log h|$ at least (the Ehrenfest time-scale). The families of functions (G_n) and (F_n) are of class C^∞. For all n, G_n is supported in $B(0, \eta)^n$, where $B(0, \eta)$ stands for the ball of center 0 and radius η in $\mathbb{R}$. We assume that F_n has a unique critical point at 0, which is non-degenerate.

We follow Hörmander's approach of the stationary phase method.

The first step is to perform a change of coordinates in which F_n becomes quadratic, using the Morse lemma. If one wants to perform this change of coordinates on $B(0, \eta)^n$, for all n, keeping control of the derivatives of the change of coordinates, one needs some additional knowledge on the family (F_n). For instance, in the case $F_n(\xi) = \sum_{j=0}^n d^2(\xi_j, \xi_{j+1})$, the hyperbolicity of the geodesic flow and the discussion of Section 2 will have to come into play: see paragraph 4.2.

Suppose now that we have managed to transform (4.1.1), by a suitable change of coordinates, into

$$\frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} \tilde{G}_n(\xi) e^{\frac{i \sum_{j=1}^n \varepsilon_j \xi_j^2}{2h}} d\xi \tag{4.1.2}$$

where $\varepsilon_j = \pm 1$ and $\tilde{G}_n$ is of class C^∞, supported in $B(0, \tilde{\eta})^n$ for some $\tilde{\eta} > 0$.

Using the fact that the Fourier tranform of the distribution $\frac{1}{h^{n/2}}e^{\frac{i\sum_{j=1}^{n}\varepsilon_j\xi_j^2}{2h}}$ is $e^{\frac{i\pi}{4}\sum_{j=1}^{n}\varepsilon_j}e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2}$, we can rewrite (4.1.2) in the form

$$\frac{e^{\frac{i\pi}{4}\sum_{j=1}^{n}\varepsilon_j}}{(2\pi)^{n/2}}\int dx_1...dx_n e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2}\mathcal{F}(\tilde{G}_n)(x_1,...,x_n) \tag{4.1.3}$$

Remark 4.1.1. We used the notation $\mathcal{F}$ for the Fourier transform, defined on $\mathbb{R}^n$ by:

$$\mathcal{F}u(x) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle}u(\xi)d\xi.$$

With this convention, the Fourier transform $\mathcal{F}$ is an isometry of $L^2(\mathbb{R}^n, d\xi)$. The Fourier inversion formula reads:

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\mathcal{F}u(x)e^{i\langle x,\xi\rangle}dx = \mathcal{F}u(\xi). \tag{4.1.4}$$

The next step in the study of (4.1.3) is to expand $e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2}$ in powers of h:

$$e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2} = 1 - \left(\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2\right) + \frac{1}{2}\left(\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2\right)^2 + \cdots$$

Usually, if one wants an expansion of (4.1.3) in powers of h up to order K, one just needs the expansion of $e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2}$ up to order K. Suppose, for instance, that one wants to find the term of order 0 in the expansion of (4.1.2) or (4.1.3) in powers of h, and to control the error term of order 1; one may try to write

$$e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2} = 1 + hf_h(x_1,...,x_n) \tag{4.1.5}$$

where (f_h) is a family of functions such that, for all h,

$$|f_h(x_1,...,x_n)| \leq \sum_{j=1}^{n} x_j^2 = ||x||_2^2.$$

Doing so, one gets

$$\begin{aligned} &\frac{1}{(2\pi)^{n/2}}\int dx_1...dx_n e^{-\frac{ih}{2}\sum_{j=1}^{n}\varepsilon_j x_j^2}\mathcal{F}(\tilde{G}_n)(x_1,...,x_n) \quad (4.1.6)\\ &= \frac{1}{(2\pi)^{n/2}}\left(\int dx_1...dx_n\mathcal{F}(\tilde{G}_n)(x_1,...,x_n) + h\int dx_1...dx_n f_h(x_1,...,x_n)\mathcal{F}(\tilde{G}_n)(x_1,...,x_n)\right) \end{aligned}$$

By the Fourier inversion formula, the first term is $\tilde{G}_n(0)$, which is, up to the phase factor $e^{\frac{i\pi}{4}\sum_{j=1}^{n}\varepsilon_j}$, the leading term in the expansion of (4.1.2). To bound the remainder term, we use

$$|\int dx_1...dx_n\, f_h(x_1,...,x_n)\mathcal{F}(\tilde{G}_n)(x_1,...,x_n)| \leq \int dx_1...dx_n ||x||_2^2\, |\mathcal{F}(\tilde{G}_n)(x_1,...,x_n)|$$

and the following estimate:

Lemma 4.1.2. *For all $n \in \mathbb{N}$, for every smooth compactly supported function G on $\mathbb{R}^n$, for all $k \in \mathbb{N}$,*

$$\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} dx_1...dx_n(|x_1|^2+...+|x_n|^2)^k|\mathcal{F}(\tilde{G})(x_1,...,x_n)|$$
$$\leq 2U_n\Big(||(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2})^k\tilde{G}||_{L^2(\mathbb{R}^n,dx)}$$
$$+||(\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2})^{k+\lfloor\frac{n+1}{4}\rfloor+1}\tilde{G}||_{L^2(\mathbb{R}^n,dx)}\Big)$$

where the constant U_n is

$$U_n = \left(\frac{2\pi^{n/2}}{(\frac{n}{2}-1)!}\right)^{1/2},$$

that is, the square-root of the area of the unit $(n-1)$-dimensional sphere.

Proof. By the Cauchy-Schwarz inequality, we have for all $\delta > 0$ (and denoting $||x||_2^2 = |x_1|^2+...+|x_n|^2$ for $x=(x_1,...,x_n)\in\mathbb{R}^n$)

$$\int dx_1...dx_n||x||_2^{2k}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|$$
$$\leq \left(\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k}\min(1,||x||_2^{-\delta})^{-2}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2\right)^{1/2}(\int_{\mathbb{R}^n} dx_1...dx_n \min(1,||x||_2^{-\delta})^2)^{1/2}.$$

For $2\delta \geq n+1$, we have

$$\int_{\mathbb{R}^n} dx_1...dx_n\min(1,||x||_2^{-\delta})^2 \leq \int_{\mathbb{R}}\min(1,r^{-n-1})r^{n-1}dr\times\frac{2\pi^{n/2}}{(\frac{n}{2}-1)!}\leq 2U_n^2$$

The other term $\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k}\min(1,||x||_2^{-\delta})^{-2}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2$ can be bounded by

$$\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2+\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k+2\delta}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2.$$

If $\delta/2$ is an integer ($\delta = 2\lfloor\frac{n+1}{4}\rfloor+2$) then $\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k+2\delta}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2$ is the L^2 norm of $(\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2})^{k+\delta/2}\tilde{G}$, and $\int_{\mathbb{R}^n} dx_1...dx_n||x||_2^{4k}|\mathcal{F}(\tilde{G})(x_1,...,x_n)|^2$ is the L^2 norm of $(\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2})^k\tilde{G}$. □

Let us denote $||G||_0 = \sup_\zeta|G(\zeta)|$, and

$$||D^lG||_0 = \sup_\zeta||D^lG(\zeta)||_\infty$$
$$=\sup_\zeta\sup\Big\{\frac{|D^lG(\zeta).(v^{(1)},...,v^{(l)})|}{||v^{(1)}||_\infty...||v^{(l)}||_\infty},\ (v^{(1)},...,v^{(l)})\in(\mathbb{R}^n\setminus\{0\})^l\Big\},$$

where $||v||_\infty = \max_{j=1}^n|v_j|$ for $v\in\mathbb{R}^n$. If G is a smooth function supported in $B(0,\tilde{\eta})^n$, we have

$$||(\sum_{j=1}^n\frac{\partial^2}{\partial x_j^2})^k\tilde{G}||_{L^2(\mathbb{R}^n,dx)}\leq Vol(B(0,\tilde{\eta}))^n n^k\max_{j=0}^{2k}||D^jG||_0.$$

Thus, applying Lemma 4.1.2 for $k = 1$, we find that for a smooth function $\tilde{G}_n$ supported in $B(0, \tilde{\eta})^n$,

$$\int dx_1...dx_n ||x||_2^2 |\mathcal{F}(\tilde{G}_n)(x_1, ..., x_n)| \leq 4U_n n^{\lfloor \frac{n+1}{4} \rfloor + 2} VolB(0, \tilde{\eta})^n \max_{l=0}^{2\lfloor \frac{n+1}{4} \rfloor + 4} ||D^l \tilde{G}_n||_0$$

Thus, to control the remainder term in (4.1.6), one needs to control $n/2$ derivatives of $\tilde{G}_n$.

Assume, for instance, that the functions $\tilde{G}_n$ are uniformly Gevrey, that is, there exist $C > 0$ such that

$$||D^k \tilde{G}_n||_0 \leq C^k (k!)^s,$$

for all $k, n \in \mathbb{N}$. This is certainly the best one can hope, and it gives an estimate of the form $hC^n \left(\frac{n}{2}!\right)^s$ for the remainder term, which grows much too fast in n to be interesting.

To get estimates that take into account the dependence on n, more care is needed. In turns out more judicious to develop $e^{-\frac{ih}{2} \sum_{j=1}^n \varepsilon_j x_j^2}$ to the order n (or a multiple of n):

$$e^{-\frac{ih}{2} \sum_{j=1}^n \varepsilon_j x_j^2} = \sum_{k=0}^{\beta n - 1} \left(\frac{-ih}{2}\right)^k \frac{1}{k!} \left(\sum_{j=1}^n \varepsilon_j x_j^2\right)^k + O\left(\left(\frac{h}{2}\right)^{\beta n} \frac{1}{(\beta n)!} \left(\sum_{j=1}^n x_j^2\right)^{\beta n}\right)$$

where β is a fixed positive parameter (we should actually have the *integer part* $\lfloor \beta n \rfloor$ in the formulas, instead of βn). One obtains the following estimates:

Proposition 4.1.3.

$$\begin{aligned} &\frac{1}{(2\pi h)^{n/2}} \int_{\mathbb{R}^n} \tilde{G}_n(\xi) e^{\frac{i \sum_{j=1}^n \varepsilon_j \xi_j^2}{2h}} d\xi \\ &\quad = \frac{e^{\frac{i\pi}{4} \sum_{j=1}^n \varepsilon_j}}{(2\pi)^{n/2}} \sum_{k=0}^{\beta n - 1} \left(\frac{-ih}{2}\right)^k \frac{1}{k!} \int_{\mathbb{R}^n} (\sum_{j=1}^n \varepsilon_j x_j^2)^k \mathcal{F}(\tilde{G}_n)(x) dx_1...dx_n \\ &\qquad + O\left(\left(\frac{h}{2}\right)^{\beta n} \frac{1}{(\beta n)!} \int_{\mathbb{R}^n} (\sum_{j=1}^n x_j^2)^{\beta n} |\mathcal{F}(\tilde{G}_n)(x)| dx_1...dx_n\right) \end{aligned}$$

For $k \leq \beta n - 1$, the term of order k is equal to

$$\frac{1}{k!} \left(\frac{ih}{2}\right)^k (\sum_{j=1}^n \varepsilon_j \frac{\partial^2}{\partial x_j^2})^k \tilde{G}_n(0),$$

and thus is bounded in modulus by

$$\frac{1}{k!} \left(\frac{h}{2}\right)^k n^k \max_{j=0}^{2k} ||D^j \tilde{G}_n||_0.$$

The remainder term is

$$O\left(\left(\frac{h}{2}\right)^{\beta n} \frac{1}{(\beta n)!} \int_{\mathbb{R}^n} (\sum_{j=1}^n x_j^2)^{\beta n} |\mathcal{F}(\tilde{G}_n)(x)| dx_1...dx_n\right)$$

and can be bounded, as in Lemma 4.1.2, by

$$2U_n \frac{1}{(\beta n)!}\left(\frac{h}{2}\right)^{\beta n}\Big(||(\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2})^{\beta n}\tilde{G}_n||_{L^2(\mathbb{R}^n,dx)} + ||(\sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2})^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\tilde{G}_n||_{L^2(\mathbb{R}^n,dx)}\Big).$$

If $\tilde{G}_n$ is supported in $B(0,\tilde{\eta})^n$, this is bounded by

$$4U_n Vol(B(0,\tilde{\eta}))^n \frac{1}{(\beta n)!}\left(\frac{h}{2}\right)^{\beta n} n^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\max_{j=0}^{(2\beta+1)n+3}||D^j\tilde{G}_n||_0.$$

The interest of this bounds depends, of course, of the *a priori* estimates on the derivatives of $\tilde{G}_n$.

In our case (see (4.0.2)), the function $\tilde{G}_n$ will be a product

$$\tilde{G}_n(\xi) = A_1(\xi)...A_n(\xi)$$

where the A_is are in the Gevrey class, and, more precisely, satisfy

$$||D^k A_i||_0 \leq J_i G_s(A)^k (k!)^s$$

for all i, for all k, for some common Gevrey constant $G_s(A)$, and some real numbers J_i.

Lemma 4.1.4. *Let $\tilde{G}_n(\xi) = A_1(\xi)...A_n(\xi)$, where the A_is satisfy*

$$||D^k A_i||_0 \leq J_i G_s(A)^k (k!)^s$$

for all i, for all k, for some common Gevrey constant $G_s(A)$, and some real numbers J_i. Then the following estimate holds for the derivatives of $\tilde{G}_n$:

$$||D^k\tilde{G}_n||_0 \leq J_1...J_n G_s(A)^k n^k k^{sk}$$

for every $k \geq 0$.

In particular, if $k \leq \beta n$,

$$||D^k\tilde{G}_n||_0 \leq J_1...J_n G_s(A)^k n^{k(s+1)}\beta^{sk}$$

Proof.

$$\begin{aligned}|D^k\tilde{G}_n(\xi)| &= |\sum_{l_1+...+l_n=k}\frac{k!}{l_1!...l_n!}D^{l_1}A_1(\xi)...D^{l_n}A_n(\xi)| \\ &\leq J_1...J_n G_s(A)^k \sum_{l_1+...+l_n=k}\frac{k!}{l_1!...l_n!}(l_1!)^s...(l_n!)^s \\ &\leq J_1...J_n G_s(A)^k \sum_{l_1+...+l_n=k}\frac{k!}{l_1!...l_n!}k^{sk} \leq J_1...J_n G_s(A)^k n^k k^{sk}\end{aligned}$$

□

As a consequence, we obtain the final estimates:

Proposition 4.1.5. *(1) For all n, for all $\tilde{G}_n$ that is compactly supported in $B(0,\tilde{\eta})^n$,*

$$|\frac{1}{(2\pi h)^{n/2}}\int_{\mathbb{R}^n}\tilde{G}_n(\xi)e^{\frac{i\sum_{j=1}^n\varepsilon_j\xi_j^2}{2h}}d\xi - e^{\frac{i\pi}{4}\sum_{j=1}^n\varepsilon_j}\tilde{G}_n(0)| \leq \sum_{k=1}^{\beta n-1}\frac{h^k}{2^k k!}n^k\max_{l=0}^{2k}||D^l\tilde{G}_n||_0$$

$$+ 4U_nVol(B(0,\tilde{\eta}))^n\frac{h^{\beta n}}{2^{\beta n}(\beta n)!}n^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\max_{l=0}^{(2\beta+1)n+3}||D^l\tilde{G}_n||_0$$

(2) Let $\tilde{G}_n(\xi) = A_1(\xi)...A_n(\xi)$, where the A_is satisfy

$$||D^kA_i||_0 \leq J_iG_s(A)^k(k!)^s$$

for all i, for all k, for some common Gevrey constant $G_s(A)$, and some real numbers J_i. Assume that, for all n, $\tilde{G}_n$ is compactly supported in $B(0,\tilde{\eta})^n$. Then, for all n,

$$|\frac{1}{(2\pi h)^{n/2}}\int_{\mathbb{R}^n}\tilde{G}_n(\xi)e^{\frac{i\sum_{j=1}^n\varepsilon_j\xi_j^2}{2h}}d\xi - e^{\frac{i\pi}{4}\sum_{j=1}^n\varepsilon_j}\tilde{G}_n(0)|$$

$$\leq J_1...J_n\Big[\sum_{k=1}^{\beta n-1}\frac{h^k}{2^k k!}n^k\beta^{2sk}G_s(A)^{2k}n^{2k(s+1)} + 4U_nVol(B(0,\tilde{\eta}))^n\frac{h^{\beta n}}{2^{\beta n}(\beta n)!}.$$

$$n^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\beta^{s((2\beta+1)n+3)}G_s(A)^{(2\beta+1)n+3}n^{((2\beta+1)n+3)(s+1)}\Big]$$

and

$$|\frac{1}{(2\pi h)^{n/2}}\int_{\mathbb{R}^n}\tilde{G}_n(\xi)e^{\frac{i\sum_{j=1}^n\varepsilon_j\xi_j^2}{2h}}d\xi|$$

$$\leq J_1...J_n\Big[\sum_{k=0}^{\beta n-1}\frac{h^k}{2^k k!}n^k\beta^{2sk}G_s(A)^{2k}n^{2k(s+1)} + 4U_nVol(B(0,\tilde{\eta}))^n\frac{h^{\beta n}}{2^{\beta n}(\beta n)!}.$$

$$n^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\beta^{s((2\beta+1)n+3)}G_s(A)^{(2\beta+1)n+3}n^{((2\beta+1)n+3)(s+1)}\Big]$$

$$\leq J_1...J_n\Big[\exp(\beta^{2s}hn^{2s+3}G_s(A)^2) + 4U_nVol(B(0,\tilde{\eta}))^n\frac{h^{\beta n}}{2^{\beta n}(\beta n)!}.$$

$$n^{\beta n+\lfloor\frac{n+1}{4}\rfloor+1}\beta^{s((2\beta+1)n+3)}G_s(A)^{(2\beta+1)n+3}n^{((2\beta+1)n+3)(s+1)}\Big]$$

The estimate is not interesting for all values of n, but only at most for

$$Vol(B(0,\tilde{\eta}))h^{\beta}G_s(A)^{(2\beta+1)}n^{(2\beta+1)(s+1)+1} < 1;$$

(so that the remainder term has a chance to be negligible). Note also that this estimate is not necessarily adapted for all purposes, in particular if the product $J_1...J_n$ grows too fast. In fact, the result of Proposition 4.1.3 is particularly well adapted for studying the quantity

$$\Big(|\frac{1}{(2\pi h)^{n/2}}\int_{\mathbb{R}^n}\tilde{G}_n(\xi)e^{\frac{i\sum_{j=1}^n\varepsilon_j\xi_j^2}{2h}}d\xi|\Big)^{1/n},$$

which is the one we will be concerned with in Section 4, when discussing entropy.

Application to integral (4.0.2)**:** We will now apply the discussion of paragraph 4.1 to study the behaviour of integral (4.0.2):

$$\frac{1}{(2\pi h)^{(n-1)d/2}}\int A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1})e^{\frac{i}{2h}\sum_{k=0}^{n-1}d^2(\xi_k,\xi_{k+1})}$$
$$a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n)dVol(\xi_1)...dVol(\xi_{n-1}).$$

We work in local coordinates: denoting $\Omega = B(0,\eta)\in\mathbb{R}^d$ the open ball centered at 0 and of radius η, let us choose a family of charts $\Phi_k : B(0,\eta)\longrightarrow M$, "adapted" to the geodesic ξ^c as defined in paragraph 3.2. With this choice of coordinates, we can treat integral (4.0.2) as an integral over $B(0,\eta)^{n-1}\subset\mathbb{R}^{(n-1)d}$. The volume element on M, $d\,Vol(\xi_k)$, is equivalent to the Lebesgue measure in the local coordinates:

$$d\,Vol(\xi_k) = Jac[\Phi_k](\Phi_k^{-1}\xi_k)d(\Phi_k^{-1}\xi_k).$$

We use the notation $Jac[\Phi](x)$ to denote the jacobian of a map Φ at a point x.

We will denote j_k the function $j_k(x) = Jac[\Phi_k](x)$, defined on $B(0,\eta)\subset\mathbb{R}^d$.

In local coordinates, the integral takes the form:

$$\frac{1}{(2\pi h)^{(n-1)d/2}}\int_{B(0,\eta)^{n-1}\subset(\mathbb{R}^d)^{n-1}} d\xi_1...d\xi_{n-1}e^{\frac{i}{2h}\sum_{k=0}^{n-1}d^2(\Phi_k(\xi_k),\Phi_{k+1}(\xi_{k+1}))}$$
$$A_{\alpha_1}(\Phi_1(\xi_1))...A_{\alpha_{n-1}}(\Phi_{n-1}(\xi_{n-1}))a^{(h)}(\Phi_0(\xi_0),\Phi_1(\xi_1))...a^{(h)}(\Phi_{n-1}(\xi_{n-1}),\Phi_n(\xi_n))j_1(\xi_1)...j_k(\xi_k);$$

Remark 4.1.6. Since the charts Φ_k are analytic, Assumption (I) implies the existence of R and C, independent of k and on h, such that, for all l,

$$||D^l\big(a^{(h)}\circ(\Phi_k,\Phi_{k+1})\big)||\leq CR^l l! \tag{4.1.7}$$
$$||D^l\big(d\circ(\Phi_k,\Phi_{k+1})\big)||\leq CR^l l!$$
$$||D^l j_k||\leq CR^l l!.$$

Recall also that $||D^l(A_\alpha\circ\Phi_k)||\leq G_s(A)^l(l!)^s$ for all l.

From now on, all the calculations are going to be performed in local coordinates; but to keep the notations reasonable, the charts Φ_k will no longer appear explicitly, and we will simply write $d(x,y)$, $a^{(h)}(x,y)$... instead of $d(\Phi_k(x),\Phi_{k+1}(y))$, $a^{(h)}(\Phi_k(x),\Phi_{k+1}(y))$, etc...

In these local charts, $\mathbb{R}^d$, as well as $(\mathbb{R}^d)^{n-1}$, are endowed with their usual scalar product and the associated norm, denoted respectively $\langle.,.\rangle$ and $||.||_2$ independently of n, in accordance with the notations introduced in Remark 3.3.2.

In the local coordinates, the phase function

$$F_{n-1}((\xi_1,...,\xi_{n-1})) = \frac{1}{2}\sum_{k=0}^{n-1}d^2(\xi_k,\xi_{k+1})$$

has a unique (and non-degenerate) critical point $(0,...,0)\in(\mathbb{R}^d)^{n-1}$.

In order to apply the results of paragraph 4.1 to integral (4.0.2), we must first put it in the form (4.1.2); we have to choose coordinates in which F_{n-1} is quadratic. This can be done, as usual, thanks to the Morse lemma, but we will need to control *uniformly in n* the size of the domain where the lemma applies, and the growth of the derivatives of the change of coordinates:

4.2. **The Morse Lemma.**

Proposition 4.2.1. *If η is small enough, there exists, for all n, a diffeomorphism w from $B(0,\eta)^{n-1}$ onto its image, mapping $0=(0,..,0)$ to itself, and tangent to identity at this point, such that*

$$F_{n-1}(\xi_1,...,\xi_{n-1}) - F_{n-1}(0,...,0) = \frac{1}{2}\langle \bar{H}_n(0).w(\xi_1,...,\xi_{n-1}), w(\xi_1,...,\xi_{n-1})\rangle,$$

where $\bar{H}_n(0)$ is the hessian matrix of F_{n-1} at 0.

Of course the diffeomorphism w depends on n, although this does not appear in the notations.

Keeping the notations of Section 2, we decompose $(\mathbb{R}^d)^{n-1}$ into $V^{n-1}\oplus V_\perp^{n-1}$, where $V=\{0\}\times\mathbb{R}^{d-1}\subset\mathbb{R}^d$, and $V_\perp=\mathbb{R}\times\{0\}^{d-1}$ is its orthogonal. The space V corresponds to the orthogonal hyperplane to the geodesic ξ^c, and $V_\perp$, to the direction of the geodesic.

The matrix $\bar{H}_n(0)$ is made up, on the one hand, of the block $H_n(0)$ studied in Paragraph 3.3, corresponding to variations in the direction V^{n-1}, on the other hand, of a block of size $(n-1)\times(n-1)$ corresponding to variations in the direction of the geodesics, in other words, along the space $V_\perp^{n-1}$. This last block takes the form:

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}$$

To prove Proposition 4.2.1 we follow the steps of a classical proof of the Morse lemma, but we keep under control the size of the domain of definition of the diffeomorphism w, conjugating F_{n-1} to its hessian at 0. We show that the diameter of this domain does not depend on the length n.

Proof. For $\xi=(\xi_1,...,\xi_{n-1})\in(\mathbb{R}^d)^{n-1}$, let us denote $\xi_{V^{n-1}}$ (respectively $\xi_{V_\perp^{n-1}}$) the projection of ξ to the space V^{n-1} (respectively, to its orthogonal $V_\perp^{n-1}$) in $(\mathbb{R}^d)^{n-1}$. Thanks to our choice of coordinates, $\xi_{V_\perp^{n-1}}$ is a critical point of F_{n-1} restricted to $\xi+V^{n-1}$, and we can write:

$$\begin{aligned} F_{n-1}(\xi) &= F_{n-1}(\xi_{V_\perp^{n-1}}) + \int_0^1\int_0^1 D^2F_{n-1}(\xi_{V_\perp^{n-1}} + st\xi_{V^{n-1}}).t\xi_{V^{n-1}}.\xi_{V^{n-1}}dsdt \\ &= F_{n-1}(\xi_{V_\perp^{n-1}}) + \frac{1}{2}\langle H_n(\xi)\xi_{V^{n-1}},\xi_{V^{n-1}}\rangle \end{aligned}$$

where $H_n(\xi)$ is the tridiagonal matrix of size $(n-1)(d-1)\times(n-1)(d-1)$, representing the bilinear form $2\int_0^1\int_0^1 D^2F_{n-1}(\xi_{V_\perp^{n-1}}+st\xi_{V^{n-1}})tdsdt$ on V^{n-1}, with respect to the scalar product $\langle .,.\rangle$.

Note that

$$\langle H_n(0)\xi_{V^{n-1}},\xi_{V^{n-1}}\rangle = D^2F_{n-1}(0).\xi_{V^{n-1}}^2.$$

For $\xi\in B(0,\eta)^{n-1}$, we will define

$$w(\xi) = \left(\xi_{V_\perp^{n-1}},\, W(\xi).\xi_{V^{n-1}}\right) \in V_\perp^{n-1}\times V^{n-1} = (\mathbb{R}^d)^{n-1}, \tag{4.2.1}$$

where $W(\xi)$ is a square root (to be determined below) of the matrix $H_n(0)^{-1}H_n(\xi)$. Note that w preserves the decomposition $(\mathbb{R}^d)^{n-1} = V^{n-1} \oplus V_\perp^{n-1}$. Then w will be a local diffeomorphism in a neighbourhood of 0, tangent to identity at 0, and such that

$$F_{n-1}(\xi) = F_{n-1}(\xi_{V_\perp^{n-1}}) + \frac{1}{2}\langle H_n(0).w(\xi)_{V^{n-1}}, w(\xi)_{V^{n-1}}\rangle. \tag{4.2.2}$$

Besides, F_{n-1} is already quadratic on $V_\perp^{n-1}$, since it is nothing but the variation of energy along the geodesic. This way, F_{n-1} will be conjugated to its hessian at 0 by the diffeomorphism w.

Let us make this more explicit. We have seen in paragraph 3.3 that $H_n(0)$ is invertible, and that there exists $K > 0$ such that $\| H_n(0)^{-1} \|_\infty \leq K$, for all n.

Note that there exists $C > 0$ such that, for all n, for all $\xi \in B(0,\eta)^{n-1}$,

$$\| H_n(\xi) - H_n(0) \|_\infty \leq C \| \xi \|_\infty \leq C\eta.$$

This simply follows from the fact that the matrix $H_n(\xi)$ is tridiagonal, and that its coefficients have bounded derivatives. Hence, $||I - H_n(0)^{-1}H_n(\xi)||_\infty \leq KC\eta$.

So, if η is small enough, $W(\xi)$ may be defined as the convergent series :

$$W(\xi) = \sum_{k=0}^{+\infty} c_k \left(I - H_n(0)^{-1}H_n(\xi)\right)^k, \tag{4.2.3}$$

the c_ks being the coefficients of the expansion of $(1-x)^{1/2}$ near $x = 0$. By definition, $W(\xi)^2 = H_n(0)^{-1}H_n(\xi)$; note also that $W^*(\xi) = H_n(0)W(\xi)H_n(0)^{-1}$, so that $W^*(\xi)H_n(0)W(\xi) = H_n(\xi)$. Thus, if we define the diffeomorphism w as in (4.2.1), we have indeed (4.2.2).

There remains to control the size of the region on which w acts as a diffeomorphism: we will show that w is a diffeomorphism from $B(0,\eta)^{n-1}$ onto its image, that is to say, it is injective and its derivative is invertible at every point – for η small enough, but independent of n. We prove the second point, the proof of the injectivity of w goes along similar lines.

The differential of $\xi \mapsto W(\xi).\xi_{V^{n-1}}$ at ξ is the map

$$\zeta \in (\mathbb{R}^d)^{n-1} \mapsto W(\xi).\zeta_{V^{n-1}} + DW(\xi).\zeta_{V^{n-1}}.\xi_{V^{n-1}}.$$

There exists $C > 0$, independent of n, such that, for all $\xi \in B(0,\eta)^{n-1}$, $\| W(\xi).\zeta_{V^{n-1}} - \zeta_{V^{n-1}} \|_\infty \leq C\eta||\zeta||_\infty$, and

$$\begin{aligned}
&\| DW(\xi).\zeta_{V^{n-1}}.\xi_{V^{n-1}} \|_\infty \\
&\leq \| H_n(0)^{-1}.DH_n(\xi).\zeta_{V^{n-1}} \|_\infty \left(\sum_{k=1}^{+\infty}(k-1) \mid c_k \mid \| I - H_n(0)^{-1}H_n(\xi) \|_\infty^{k-1}\right) ||\xi_{V^{n-1}}||_\infty \\
&\qquad \leq C \| \xi \|_\infty ||\zeta||_\infty \leq C\eta||\zeta||_\infty
\end{aligned}$$

Thus, there exists C such that

$$||Dw(\xi)_{|V^{n-1}} - I||_\infty \leq C\eta, \tag{4.2.4}$$

therefore Dw is invertible if η is small enough.

It follows, in particular, that the image of w is included in $B(0,C\eta)^n$, for some constant C independent of n. □

Remark 4.2.2. For later purposes, we will also need to estimate the norms of $\bar{H}_n(0)^{1/2}$ and $\bar{H}_n(0)^{-1/2}$ in l^∞. Remember that $\bar{H}_n(0)$ is made up of two blocks, namely $H_n(0)$ and the matrix A defined by (4.2).

For $\lambda = (\lambda_1, ..., \lambda_{n-1}) \in \mathbb{R}^{n-1}$, $\langle A\lambda, \lambda\rangle = \lambda_1^2 + \sum_{j=1}^{n-2}(\lambda_{j+1} - \lambda_j)^2 + \lambda_{n-1}^2$. Thus, using the Cauchy-Schwarz inequality and the fact that $||\lambda||_\infty \leq ||\lambda||_2 \leq \sqrt{n}||\lambda||_\infty$,

$$||\lambda||_2^2 \leq n||\lambda||_\infty^2 \leq n\Big(|\lambda_1| + \sum_{j=1}^{n-2}|\lambda_{j+1} - \lambda_j|\Big)^2 \leq n^2\langle A\lambda, \lambda\rangle,$$

which implies that $||A^{-1/2}||_2 \leq n$ and that $||A^{-1/2}||_\infty \leq n^{3/2}$. We know that the inverse of the block $H_n(0)$ is bounded, uniformly in n: it follows that the inverse of $\bar{H}_n(0)^{-1/2}$ is bounded in l^∞ by $n^{3/2}$.

In addition, since $\bar{H}_n(0)$ is positive and uniformly bounded in l^2-norm, $\bar{H}_n(0)^{1/2}$ is also uniformly bounded in l^2-norm, and hence $\bar{H}_n(0)^{1/2}$ is bounded, in l^∞-norm, by $\sqrt{n}$.

To finish this section, we investigate the growth of the derivatives of w. We prove that the diffeomorphisms w and w^{-1} are in the set $\mathcal{G}_1(R, C)$ with respect to the $||.||_\infty$ norm, for some R, C not depending on n. In the three following lemmas, we recall some basic properties of the class of Gevrey functions (see [Ho], Proposition 8.4.1 for 4.2.3 and 4.2.4, and [Ca], Chapter 1, for 4.2.5).

Of course we assume that the functions f, g are such that their product (Lemma 4.2.3) or composition (Lemma 4.2.4) is well defined.

Lemma 4.2.3. *For all $s \geq 1$, there exists c such that, if f and g take values in a same normed algebra, and if $f \in \mathcal{G}_1(1,1)$ and $g \in \mathcal{G}_s(1,1)$, then the product $f.g$ is in $\mathcal{G}_s(c, c)$.*

If $f \in \mathcal{G}_1(R, C)$ and $g \in \mathcal{G}_s(R, C)$, the lemma applied to $x \mapsto \frac{1}{C^2}f(\frac{x}{R})g(\frac{x}{R})$ yields that $f.g$ is in $\mathcal{G}_s(cR, cC^2)$.

Lemma 4.2.4. *For all $s \geq 1$, for there exists c such that, if $f \in \mathcal{G}_1(1,1)$ and $g \in \mathcal{G}_s(1,1)$, then the composition $g \circ f$ is in $\mathcal{G}_s(c, c)$.*

If $f \in \mathcal{G}_1(C, C)$ and $g \in \mathcal{G}_s(R, C)$ for $R, C \geq 1$, the lemma applied to $x \mapsto \frac{1}{C}g(\frac{1}{R}.Rf(x/RC^2))$ yields that $g \circ f$ is in $\mathcal{G}_s(cC^2R, cC)$.

Lemma 4.2.5. *For all $R, C \geq 0$, there exists R', C' such that, for all n, for every norm $||.||$ on $\mathbb{R}^n$:*

if f is a diffeomorphism between two open subsets of $\mathbb{R}^n$, if $f \in \mathcal{G}_1(R, C)$, and if $||(Df)^{-1}|| \leq C$ then its inverse f^{-1} is in $\mathcal{G}_1(R', C')$ with respect to the norm $||.||$.

Corollary 4.2.6. *There exists C such that, for all n, $w \in \mathcal{G}_1(C, C)$ with respect to the $||.||_\infty$-norm on $(\mathbb{R}^d)^{n-1}$.*

Corollary 4.2.7. *There exists C such that, for all n, $w^{-1} \in \mathcal{G}_1(R, C)$ with respect to the $||.||_\infty$-norm on $(\mathbb{R}^d)^{n-1}$.*

Proof. (of the corollaries). It follows from the analyticity of the metric and from the definition of H_n (paragraph 4.2) that $H_n \in \mathcal{G}_1(C, C)$ for some C independent of n. Since $W = (H_n(0)^{-1}H_n)^{1/2}$, Corollary 4.2.6 follows from Lemma 4.2.4, and the formula (4.2.1) defining w. Corollary 4.2.7 follows from Lemma 4.2.5, Corollary 4.2.6 and the estimate (4.2.4). □

4.3. **Proof of Theorem 4.0.1.** We now start applying the stationary phase method to (4.0.2), following the scheme described in paragraph 4.1. We perform the change of variable $\zeta = \bar{H}_n(0)^{1/2} w(\xi)$ that makes the phase function F_{n-1} quadratic:

$$\begin{aligned}(4.3.1)\quad & \frac{1}{(2\pi h)^{(n-1)d/2}} \int dVol(\xi_1)...dVol(\xi_{n-1}) e^{\frac{i}{2h}\sum_{k=0}^{n-1} d^2(\xi_k,\xi_{k+1})} \\ & a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n) A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1}) \\ &= \frac{1}{(2\pi h)^{(n-1)d/2} det\bar{H}_n(0)^{1/2}} \int d\zeta_1...d\zeta_{n-1} e^{\frac{i}{2h}\langle\zeta,\zeta\rangle} \\ & b^{(h)}(w^{-1}\bar{H}_n(0)^{-1/2}\zeta) Jac[w^{-1}](\bar{H}_n(0)^{-1/2}\zeta)\end{aligned}$$

We have introduced the notation

$$(4.3.2)\quad b^{(h)}(\xi) = A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1}) a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n) j_1(\xi_1)...j_{n-1}(\xi_{n-1}).$$

Remark 4.2.2 shows that the new integral takes place in $B(0,\eta\sqrt{n})^{n-1}$.

Applying Proposition 4.1.3 to the second term of (4.3.1), we obtain:

Proposition 4.3.1. *For any $\beta > 0$,*

$$\begin{aligned} & \frac{1}{(2\pi h)^{(n-1)d/2}} \int dVol(\xi_1)...dVol(\xi_{n-1}) e^{\frac{i}{2h}\sum_{k=0}^{n-1} d^2(\xi_k,\xi_{k+1})} \\ & a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n) A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1}) = \frac{e^{\frac{i\pi}{4}(n-1)d}}{det\bar{H}_n(0)^{1/2}} \times \\ & \Big[\sum_{k=0}^{\beta n-1} \frac{1}{k!} \left(\frac{ih}{2}\right)^k \big(\sum_{j=1}^{n-1}\sum_{i=1}^{d} \frac{\partial^2}{\partial {x_j^i}^2}\big)^k \big(b^{(h)} \circ w^{-1} \circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}] \circ \bar{H}_n(0)^{-1/2}\big)(0) \\ & + O\Big(4U_{(n-1)d} Vol(B(0,\eta\sqrt{n}))^{n-1} \frac{1}{(\beta n)!} \big(\frac{h}{2}\big)^{\beta n} \\ & ((n-1)d)^{\beta n + \lfloor \frac{(n-1)d+1}{4} \rfloor + 1} \max_{j=0}^{(2\beta+d)n+3} ||(b^{(h)} \circ w^{-1} \circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}] \circ \bar{H}_n(0)^{-1/2})||_0\Big) \Big].\end{aligned}$$

The notation $x_j^i \in \mathbb{R}$ ($i = 1,...,d$) stands for the coordinates of the vector $x_j \in \mathbb{R}^d$.

For a function g on $(\mathbb{R}^d)^{n-1}$, we have denoted $||g||_0 = \sup_\zeta |g(\zeta)|$, and

$$\begin{aligned}||D^l g||_0 &= \sup_\zeta ||D^l g(\zeta)||_\infty \\ &= \sup_\zeta \sup \Big\{ \frac{|D^l g(\zeta).(v^{(1)},...,v^{(l)})|}{||v^{(1)}||_\infty ... ||v^{(l)}||_\infty},\ (v^{(1)},...,v^{(l)}) \in (\mathbb{R}^{d(n-1)} \setminus \{0\})^l \Big\}.\end{aligned}$$

In Proposition 4.3.1, the coefficient of h^0 is

(4.3.3)
$$\frac{e^{\frac{i\pi}{4}(n-1)d}}{det\bar{H}_n(0)^{1/2}} b^{(h)}\circ w^{-1}(0,...,0).Jac[w^{-1}](0,...,0) = \frac{e^{\frac{i\pi}{4}(n-1)d}}{det\bar{H}_n(0)^{1/2}} b^{(h)}(0,...,0)$$
$$= \frac{e^{\frac{i\pi}{4}(n-1)d}}{det\bar{H}_n(0)^{1/2}} A_{\alpha_1}(\xi_1^c)...A_{\alpha_{n-1}}(\xi_{n-1}^c)a^{(h)}(\xi_0^c,\xi_1^c)...a^{(h)}(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)$$

(coming back to M in the last line).

This is bounded in modulus by

$$\frac{1}{det\bar{H}_n(0)^{1/2}}|a_0(\xi_0^c,\xi_1^c)...a_0(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)|(1+Ch)^{n-1},$$

since $|A_\alpha|\leq 1$ and, on the set $\{(x,y)\in M^2, 1-2\varepsilon\leq d(x,y)\leq 1+2\varepsilon\}$, one has $|a^{(h)}(x,y)|\leq |a_0(x,y)|(1+Ch)$ for some constant C.

The calculation also shows that, for fixed n,

$$\frac{e^{\frac{i\pi}{4}(n-1)d}}{det\,\bar{H}_n(0)^{1/2}} a_0(\xi_0^c,\xi_1^c)...a_0(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)$$

is the leading term in the asymptotic expansion in powers of h of the integral

$$\frac{1}{(2\pi h)^{(n-1)d/2}}\int d\xi_1...d\xi_{n-1}\chi(\xi_1,...\xi_{n-1})e^{\frac{i}{2h}\sum_{k=0}^{n-1}d^2(\xi_k,\xi_{k+1})}a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n)$$

whenever χ is a C^∞ function, with compact support, such that $\chi(\xi_1^c,...\xi_{n-1}^c)=1$. The Van Vleck formula says that this leading term is $e^{-\frac{i\pi}{4}d}Jac[\exp^n](v_\alpha(\xi_0,\xi_n))^{-1/2}$; where $v_\alpha(\xi_0,\xi_n)$ is the vector in T_xM that generates a geodesic joining ξ_0 to ξ_n in time n, homotopic to any piecewise geodesic path going throught the supports of A_{α_1},..., $A_{\alpha_{n-1}}$.

In other words,

$$\frac{e^{\frac{i\pi}{4}(n-1)d}}{det\,\bar{H}_n(0)^{1/2}} a_0(\xi_0^c,\xi_1^c)...a_0(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c) = e^{-\frac{i\pi}{4}d}Jac[\exp^n](x,y)^{-1/2},$$

and thus we can bound the coefficient of h^0 by

(4.3.4)
$$\frac{1}{det\,\bar{H}_n(0)^{1/2}}|a^{(h)}(\xi_0^c,\xi_1^c)...a^{(h)}(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)| \leq Jac[\exp^n](v_\alpha(\xi_0,\xi_n))^{-1/2}(1+Ch)^n.$$

In order to bound the other terms of the expansion, we have to bound

$$||D^l\left(b^{(h)}\circ w^{-1}\circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}]\circ \bar{H}_n(0)^{-1/2}\right)||_0$$

for all l.

We begin with $l=0$.

On the one hand, for all $\xi\in B(0,\eta)^{n-1}$ we have

$$|b^{(h)}(\xi)|\leq (1+C\eta)^n|a^{(h)}(\xi_0^c,\xi_1^c)|...|a^{(h)}(\xi_{n-1}^c,\xi_n^c)|\,j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)$$

for some C: this comes from the expression (4.3.2) of $b^{(h)}$, and the fact that the functions $|a^{(h)}|$ and j_k are positive, with derivatives bounded uniformly in h.

In order to bound $|Jac[w^{-1}](\xi)|$, we need a simple lemma that will serve several times:

Lemma 4.3.2. *For any n, for any $n \times n$ matrix $M = (M_{i,j})_{1\leq i,j\leq n}$,*

$$|detM| \leq \prod_{i=1}^{n}(\sum_{j=1}^{n}|M_{i,j}|).$$

This is proved by induction on n, using the development of the determinant with respect to one line of the matrix.

We know that $||(Dw)^{-1}(\xi) - I||_\infty \leq C\eta$. The norm of a matrix acting on $l^\infty([1, n-1], \mathbb{R}^d)$ is given by the max of the l^1-norms of its lines:

$$||M||_\infty = \max_i \sum_j |M_{i,j}| = \max_i ||M_i||_1,$$

where M_i stands for the i-th line of M. Thus, the lines of the matrix $(Dw)^{-1}(\xi)$ are bounded, in l^1-norm, by $1 + C\eta$, for some C that does not depend on n. By Lemma 4.3.2,

$$|Jac[w^{-1}](\xi)| \leq (1 + C\eta)^n.$$

So, there exists a constant C (always independent of n), such that

$$||b^{(h)} \circ w^{-1} \circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}] \circ \bar{H}_n(0)^{-1/2}||_0$$
$$\leq (1 + C\eta)^n|a^{(h)}(\xi_0^c, \xi_1^c)|...|a^{(h)}(\xi_{n-1}^c, \xi_n^c)|\, j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)$$

on $(H_n(0)^{1/2} \circ w)B(0, \eta)^{n-1}$; the last term can be estimated by (4.3.4).

We now bound the higher order derivatives

$$||D^l\left(b^{(h)} \circ w^{-1} \circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}] \circ \bar{H}_n(0)^{-1/2}\right)||_0.$$

First write

$$||D^l\left(b^{(h)} \circ w^{-1} \circ \bar{H}_n(0)^{-1/2}.Jac[w^{-1}] \circ \bar{H}_n(0)^{-1/2}\right)||_0$$
$$\leq ||D^l\left(b^{(h)} \circ w^{-1}.Jac[w^{-1}]\right)||_0.||\bar{H}_n(0)^{-1/2}||_\infty^l$$
$$\leq ||D^l\left(b^{(h)} \circ w^{-1}.Jac[w^{-1}]\right)||\, n^{3l/2},$$

the last inequality comes from Remark 4.2.2.

The analyticity properties of w and w^{-1} were described in Corollaries 4.2.6 and 4.2.7.

Note that $b^{(h)}$ is in the product form

$$b^{(h)}(\xi) = A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1})a^{(h)}(\xi_0, \xi_1)...a^{(h)}(\xi_{n-1}, \xi_n)j_1(\xi_1)...j_{n-1}(\xi_{n-1}),$$

so that its derivatives can be estimated by Lemma 4.1.4. Due to the initial change of variable, we also have to deal with the term $Jac[w^{-1}]$, which is not exactly in the product form treated in Lemma 4.1.4. However, the determinant of a square matrix is nothing but the exterior product of its line-vectors; Lemma 4.1.4 applies with a slight technical modification:

Proposition 4.3.3. *There exists $C > 0$ such that, for all $n \in \mathbb{N}$, for all k,*

$$||D^k \left(b^{(h)} \circ w^{-1}.Jac[w^{-1}]\right)||_0$$
$$\leq (1 + C\eta)^{n-1} \, |a^{(h)}(\xi_0^c, \xi_1^c)...a^{(h)}(\xi_{n-1}^c, \xi_n^c)| j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c) \, n^k k^{ks}(CG_s(A))^k,$$

where $G_s(A)$ denotes a common Gevrey constant for all the A_α.

Proof. First write

$$b^{(h)} \circ w^{-1}(\zeta).Jac[w^{-1}](\zeta) = det[L_1(\zeta), ..., L_{n-1}(\zeta)],$$

where, for all ζ and for all $l = 1, ..., n-1$, $L_l(\zeta)$ is the $d \times d(n-1)$ matrix

$$L_l(\zeta) = A_{\alpha_l}\big((w^{-1})_l(\zeta)\big) \, j_l\big((w^{-1})_l(\zeta)\big) \, a^{(h)}\big((w^{-1})_{l-1}(\zeta), (w^{-1})_l(\zeta)\big) \, (Dw^{-1})_l(\zeta).$$

We have denoted $(w^{-1})_l(\zeta)$ the l-th component, in $(\mathbb{R}^d)^{n-1}$, of $(w^{-1})(\zeta)$, and $(Dw^{-1})_l(\zeta)$ the l-th $d \times d(n-1)$ submatrix of $(Dw^{-1})(\zeta)$ (that is, the l-th "line" of the matrix $(Dw^{-1})(\zeta)$, decomposed into $d \times d$ blocks). From the estimates (4.1.7) and Lemmas 4.2.3– 4.2.7, we know that there exists C such that, for all l, the function

$$L_l : ((\mathbb{R}^d)^{n-1}, ||.||_\infty) \longrightarrow (\mathbb{R}^{d \times d(n-1)}, ||.||_1)$$

is in $\mathcal{G}_s(CG_s(A), C)$; the norm on the initial space $(\mathbb{R}^d)^{n-1}$ is still the l^∞ norm, whereas the norm on the target space $\mathbb{R}^{d \times d(n-1)}$ is the l^1 norm:

$$||M||_1 = \max_{i=1,...,d} \sum_{j=1}^{d(n-1)} |M_{i,j}|,$$

which is the l^∞ operator norm.

Moreover, we have already seen that there exists $C > 0$ such that

$$||Dw^{-1} - I||_\infty \leq C\eta$$

and

$$|A_{\alpha_l}\big((w^{-1})_l(\zeta)\big) \, a^{(h)}\big((w^{-1})_{l-1}(\zeta), (w^{-1})_l(\zeta)\big) \, j_l(\zeta_l)| \leq (1+C\eta)|a^{(h)}(\xi_{l-1}^c, \xi_l^c)| \, j_l(\xi_l^c),$$

for all n and for all $\zeta \in w(B(0,\eta)^{n-1})$. Thus, there exists $C > 0$ such that

$$||L_l(\zeta)||_1 \leq (1 + C\eta)|a^{(h)}(\xi_{l-1}^c, \xi_l^c)| \, j_l(\xi_l^c).$$

Possibly adjusting C, we then have

$$||D^k L_l||_{l^\infty \longrightarrow l^1} \leq (1 + C\eta)|a^{(h)}(\xi_{l-1}^c, \xi_l^c)| \, j_l(\xi_l^c) \, (CG_s(A))^k (k!)^s,$$

for all k.

Finally, applying Lemma 4.3.2,

$$|D^k \left(det[L_1(\zeta), ..., L_{n-1}(\zeta)]\right)| = |\sum_{l_1+...+l_{n-1}=k} \frac{k!}{l_1!...l_{n-1}!} det[D^{l_1}L_1(\zeta), ...D^{l_{n-1}}L_{n-1}(\zeta)]|$$
$$\leq (1 + C\eta)^{n-1}|a^{(h)}(\xi_0^c, \xi_1^c)...a^{(h)}(\xi_{n-1}^c, \xi_n^c)| \, j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)$$
$$\sum_{l_1+...+l_{n-1}=k} \frac{k!}{l_1!...l_{n-1}!}(l_1!)^s...(l_{n-1}!)^s (CG_s(A))^{l_1}...(CG_s(A))^{l_{n-1}}$$
$$\leq (1 + C\eta)^{n-1}|a^{(h)}(\xi_0^c, \xi_1^c)...a^{(h)}(\xi_{n-1}^c, \xi_n^c)| \, j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c) n^k k^{ks}(CG_s(A))^k$$

□

And again, the last term can be estimated by (4.3.4):

$$\frac{1}{det\,\bar{H}_n(0)^{1/2}}|a^{(h)}(\xi_0^c,\xi_1^c)...a^{(h)}(\xi_{n-1}^c,\xi_n^c)j_1(\xi_1^c)...j_{n-1}(\xi_{n-1}^c)| \leq Jac[\exp^n](x,y)^{-1/2}(1+Ch)^n.$$

Injecting these estimates in Proposition 4.3.1, we finally obtain Theorem 4.0.1:

$$(4.3.5)\quad |\frac{1}{(2\pi h)^{(n-1)d/2}}\int dVol(\xi_1)...dVol(\xi_{n-1})e^{\frac{i}{2h}\sum_{k=0}^{n-1}d^2(\xi_k,\xi_{k+1})}$$
$$a^{(h)}(\xi_0,\xi_1)...a^{(h)}(\xi_{n-1},\xi_n)A_{\alpha_1}(\xi_1)...A_{\alpha_{n-1}}(\xi_{n-1})| \leq \frac{1}{det\bar{H}_n(0)^{1/2}}\times$$
$$\Big[\sum_{k=0}^{\beta n-1}\frac{1}{k!}\Big(\frac{h}{2}\Big)^k((n-1)d)^k\max_{j=0}^{2k}||D^j\big(b^{(h)}\circ w^{-1}\circ\bar{H}_n(0)^{-1/2}.Jac[w^{-1}]\circ\bar{H}_n(0)^{-1/2}\big)||_0$$
$$+4U_{(n-1)d}Vol(B(0,\eta\sqrt{n}))^{n-1}\frac{1}{(\beta n)!}\big(\frac{h}{2}\big)^{\beta n}((n-1)d)^{\beta n+\lfloor\frac{(n-1)d+1}{4}\rfloor+1}.$$
$$\max_{j=0}^{(2\beta+d)n+3}||D^j\big(b^{(h)}\circ w^{-1}\circ\bar{H}_n(0)^{-1/2}.Jac[w^{-1}]\circ\bar{H}_n(0)^{-1/2}\big)||_0\Big]$$

$$\leq (1+Ch)^{n-1}(1+C\eta)^{n-1}Jac[\exp^n](\xi_0,\xi_n)^{-1/2}\times$$
$$\Big[\sum_{k=0}^{\beta n-1}\frac{1}{k!}\Big(\frac{h}{2}\Big)^k(nd)^kn^{2k}(2k)^{2ks}(CG_s(A))^{2k}+4U_{(n-1)d}Vol(B(0,\eta\sqrt{n}))^{n-1}\frac{1}{(\beta n)!}\Big(\frac{h}{2}\Big)^{\beta n}.$$
$$(nd)^{\beta n+\lfloor\frac{(n-1)d+1}{4}\rfloor+1}((2\beta+d)n+3)^{((2\beta+d)n+3)s}(CG_s(A))^{(2\beta+d)n+3}\Big]$$

$$\leq (1+Ch)^{n-1}(1+C\eta)^{n-1}Jac[\exp^n](\xi_0,\xi_n)^{-1/2}\times$$
$$\Big[\exp\big(Chn^{2s+3}G_s(A)^2\big)+C^n\eta^{(n-1)d}n^{(n-1)d/2}h^{\beta n}n^{((2\beta+d)n+5)s}G_s(A)^{(2\beta+d)n+3}\Big]$$

for some C depending on d and β, but not on h nor on n. In the last line, we used the inequality $n!\geq C^nn^n$ to bound $U_{(n-1)d}$ and $\frac{1}{(\beta n)!}$.

4.4. **Jacobians.** To prove Theorem 1.3.3, there remains to express $Jac[\exp^n](v_\alpha(\xi_0,\xi_n))$ in terms of the unstable jacobian.

Let us introduce the functions

$$J^{(k)}(v)=\frac{Jac[\exp^k]\big(g^{-k}v\big)}{Jac[\exp^{k+1}]\big(g^{-k}v\big)},$$

defined on TM; one can write

$$(4.4.1)\quad \log Jac[\exp^n](v_\alpha(\xi_0,\xi_n))=\sum_{k=0}^{n-1}\log\frac{Jac[\exp^{k+1}](v_\alpha(\xi_0,\xi_{k+1}^c))}{Jac[\exp^k](v_{\alpha(\xi_0,\xi_k^c)})}$$

$$(4.4.2)\quad =-\sum_{k=0}^{n-1}\log J^{(k)}(\dot{\xi}_k^c)$$

if ξ^c is the geodesic joining ξ_0 to ξ_n in time n.

Lemma 4.4.1. *There exists a continuous, positive function φ on $TM \setminus \{0\}$, such that, on every compact subset of $TM \setminus \{0\}$, the sequence $J^{(k)}$ converges uniformly to $J^u.\frac{\varphi}{\varphi \circ g^1}$, where J^u is the unstable jacobian.*

Proof. Let $x \in M$ and $v \in T_xM$. Assume, for instance, that v has norm 1. Let us denote π the natural projection $T_M \longrightarrow M$. The jacobian of $\exp^k_{\pi(g^{-k}v)}$ at $g^{-k}v$, in the direction orthogonal to the energy layer $S^1_{\pi(g^{-k}v)}M$ in $T_{\pi(g^{-k}v)}M$, is exactly k. Thus, we may write

$$J^{(k)}(v) \quad = \quad \frac{k}{k+1} \frac{Jac_{S^1_{\pi(g^{-k}v)}M}[\exp^k](g^{-k}v)}{Jac_{S^1_{\pi(g^{-k}v)}M}[\exp^{k+1}](g^{-k}v)},$$

and since $\exp_k = \pi \circ g^k$, this is equal to

$$\frac{k}{k+1} \frac{Jac_{S^1_{\pi(g^{-k}v)}M}[g^k](g^{-k}v)}{Jac_{S^1_{\pi(g^{-k}v)}M}[g^{k+1}](g^{-k}v)} \times \frac{Jac_{g^k S^1_{\pi(g^{-k}v)}M}[\pi](v)}{Jac_{g^{k+1} S^1_{\pi(g^{-k}v)}M}[\pi](g^1 v)}$$

$$= \quad \frac{k}{k+1} Jac_{g^{k+1} S^1_{\pi(g^{-k}v)}M}[g^{-1}](g^1 v) \times \frac{Jac_{g^k S^1_{\pi(g^{-k}v)}M}[\pi](v)}{Jac_{g^{k+1} S^1_{\pi(g^{-k}v)}M}[\pi](g^1 v)}.$$

(We refer to Remark 4.0.2 for the notations concerning Jacobians).

But the spheres S^1_xM are transverse, in TM, to the weak stable foliation. This implies that the tangent space to $g^k S^1_{\pi(g^{-k}v)}M$, in S^1M, converges (uniformly in v) to the strong unstable space at v. Recall that the unstable jacobian $J^u(v)$ is defined by

$$J^u(v) = Jac_{W^u(g.v)}[g^{-1}](g^1 v),$$

where $W^u(v)$ denotes the unstable leaf at v. Consequently, the sequence $J^{(k)}(v)$ converges, uniformly in v, to

$$J^u(v) \times \frac{Jac_{W^u(v)}[\pi](v)}{Jac_{W^u(g.v)}[\pi](g^v)}$$

Since the unstable leaves are transverse to the spheres in S^1M, the projection $\pi : S^1M \longrightarrow M$, restricted to the unstable leaves, is an immersion.

The function φ we are looking for is

$$\varphi(v) = Jac_{W^u(v)}\pi(v)$$

□

As a consequence of Lemma 4.4.1 and the expression (4.4.2), given $\eta > 0$, there exists N_1 such that we have

$$\frac{1}{n} |\log Jac[\exp^n](v_\alpha(\xi_0, \xi_n)) + \sum_{k=0}^{n} \log J^u(\dot{\xi}^c_k)| \leq \eta$$

for all $n \geq N_1$, and for all $\xi_0, \xi_n \in M$ such that $d(\xi_0, \xi_n) \in [n(1-\eta), n(1+\eta)]$.

Thus,

$$Jac[\exp^n](v_\alpha(\xi_0, \xi_n)) \geq e^{-n\eta - \sum_{k=0}^{n-1} \log J^u(\dot{\xi}^c_k)} \tag{4.4.3}$$

if $n \geq N_1$.

By continuity, if ε is small, we also have

$$Jac[\exp^n](\xi_0,\xi_n) \geq e^{-2n\eta-\sum_{k=0}^{n-1}\log J^u(v_k)} \tag{4.4.4}$$

for every sequence $(v_0,...,v_{n-1}) \in (S^{[1-\varepsilon,1+\varepsilon]}M)^n$ such that $A_{\alpha_k}(v_k)A_{\alpha_{k+1}}(g^1 v_k) > 0$ for all k. And finally, denoting $\chi = -\sup_{v\in S^1M}\log J^u(v)$, one can also choose ε so that

$$Jac[\exp^n](\xi_0,\xi_n) \geq e^{n(\chi-2\eta)}. \tag{4.4.5}$$

4.5. **Proof of Theorem 1.3.3.** The explicit integral expression for μ_h is

$$\begin{aligned}\mu_h\big([\alpha_0,...,\alpha_n]\big) = \\ e^{\frac{-in}{2h}}\int_{M^{n+1}} dVol(\xi_0)dVol(\xi_1)...dVol(\xi_n)\bar{\psi}_h(\xi_n)A_{\alpha_0}(\xi_0)....A_{\alpha_n}(\xi_n) \\ e^{(h)}(\xi_n,\xi_{n-1},1)e^{(h)}(\xi_{n-1},\xi_{n-2},1)...e^{(h)}(\xi_1,\xi_0,1)\psi_h(\xi_0),\end{aligned} \tag{4.5.1}$$

where $e^{(h)}(.,.,1)$ is the kernel of the Fourier Integral Operator $\exp\left(\frac{ih\Delta}{2}\right)$ localized near the layer of energy 1.

Theorem 1.3.3 follows directly from Theorem 4.0.1 and the estimate (4.4.4), together with the classical inequality:

$$\int \phi(z)K(z,u)\bar{\phi}(u)dzdu \leq ||\phi||_{L^2}^2 \sup_z\Big(\int |K(z,u)|du, \int |K(u,z)|du\Big).$$

In Theorem 1.3.3 the coefficient $1+R(n,h)$ has the explicit expression:

(4.5.2)

$$1+R(n,h) = \Big[\exp\big(Chn^{2s+3}G_s(A)^2\big) + C^n\eta^{(n-1)d}n^{(n-1)d/2}h^{\beta n}n^{((2\beta+d)n+5)s}G_s(A)^{(2\beta+d)n+3}\Big].$$

If $hG_s(A)^2$ goes to 0 like a power of h, we see that $R(n,h)$ goes to 0 as $h \longrightarrow 0$, uniformly for $n \leq \mathcal{K}|logh|$ (for any arbitrary $\mathcal{K}$).

4.6. **Checking Assumption (I) for a surface of constant negative curvature.** If the injectivity radius of M is much larger than 1, and if $d_M(x,y) \leq 1+2\varepsilon$, only one geodesic joining x to y will contribute to the asymptotic development of $e^{(h)}(x,y,1)$; thus we may use for $e^{(h)}(x,y,1)$ the expression for the propagator in the universal cover, if we know it.

In the hyperbolic upper half-plane, the paper [McK72] provides an explicit expression of the propagator $e^{(h)}(x,y,1)$, as a function of $D = d(x,y)$:

$$\begin{aligned} e^{(h)}(x,y,1) &= \frac{e^{-ih/8}\sqrt{2}}{(2i\pi h)^{3/2}}\int_D^{+\infty}\frac{be^{ib^2/2h}db}{\sqrt{\cosh b-\cosh D}} \\ &= \frac{e^{iD^2/2h}e^{-ih/8}\sqrt{2}}{(2i\pi h)^{3/2}}\int_0^{+\infty}\frac{ue^{iu^2/2h}du}{\sqrt{\cosh(\sqrt{D^2+u^2})-\cosh D}}\end{aligned}$$

With the notations of the previous sections, we have

$$a^{(h)}(x,y) = \frac{-ie^{-ih/8}\sqrt{2}}{(2i\pi h)^{1/2}}\int_0^{+\infty}\frac{ue^{iu^2/2h}du}{\sqrt{\cosh(\sqrt{D^2+u^2})-\cosh D}}$$

This defines a family of analytic functions of $D = d(x,y)$, indexed by h. It admits a continuation to the complex domain $\{D \in \mathbb{C}, |D-1| \leq 1/2\}$, which is uniformly bounded for $h \in (0,1]$, as can be seen by an application of the stationary phase method in the limit $h \longrightarrow 0$.

Since the derivatives of an analytic function on a ball are controlled by its supremum norm, the fact that the family $(a^{(h)})$ continued to $\{D \in \mathbb{C}, |D-1| \leq 1/2\}$ is uniformly bounded implies that it is uniformly analytic: Assumption (I) is satisfied.

5. Appendix A1: Small scale differential calculus

Usual pseudo-differential calculus uses symbols whose derivatives behave nicely as $h \longrightarrow 0$. However, as is well known to anyone having worked out the details of the stationary phase formula, it is still valid if the derivatives of the symbols do not explode too fast:

Lemma 5.0.1. *Let $(a^{(h)})_{h\in(0,1]}$ be a family of C^∞ functions on $\mathbb{R}^d \times \mathbb{R}^d$, with a fixed compact support, and satisfying the following estimates on the derivatives:*

$$||D^n a^{(h)}||_0 \leq C_n h^{-n\kappa}$$

for all $n \in \mathbb{N}$, for some $\kappa \in [0,1/2)$ and some sequence of real numbers (C_n).Then the integral $\int_{\mathbb{R}^d\times\mathbb{R}^d} a^{(h)}(x,\xi) e^{\frac{i\langle\xi,x\rangle}{2h}} dx d\xi$ obeys the following asymptotics as $h \longrightarrow 0$:

$$\frac{1}{(2\pi h)^d}\int_{\mathbb{R}^d\times\mathbb{R}^d} a^{(h)}(x,\xi) e^{\frac{i\langle\xi,x\rangle}{2h}} dx d\xi = a^{(h)}(0,0) + O(h^{1-2\kappa}).$$

It follows that the main results of pseudo-differential calculus still hold if the derivatives of the symbols do not explode faster than powers of $h^{-\kappa}$ ($\kappa < 1/2$). For instance:

Theorem 5.0.2. *(Calderon-Vaillancourt Theorem)*
On a d-dimensional compact manifold, that there exists an integer K_d such that, for all $a \in C_c^\infty(TM)$,

$$||Op_h(a)||_{L^2(M)} \leq K_d(||a||_0 + h^{1/2}||Da||_0 + ... + h^{K_d/2}||D^{K_d}a||_0).$$

In particular, if $a^{(h)}$ depends on h in a way that

$$||D^n a^{(h)}||_0 \leq C_n h^{-n\kappa}$$

for all $n \in \mathbb{N}$, for some $\kappa \in [0,1/2)$ and some sequence of real numbers (C_n), then the operators $Op_h(a^{(h)})$ are uniformly bounded in $L^2(M)$.

One can then show:

Theorem 5.0.3. *Let $(a^{(h)})$ and $(b^{(h)})$ be two families of C^∞ functions on TM, with a common compact support, and satisfying estimates of the form*

$$||D^n a^{(h)}||_0 \leq C_n h^{-n\kappa}$$

and

$$||D^n b^{(h)}||_0 \leq C_n h^{-n\kappa}.$$

Then
(i)

$$||Op_h(a^{(h)})Op_h(b^{(h)})] - Op_h(a^{(h)}b^{(h)})||_{L^2(M)} = O(h^{1-2\kappa}).$$

(ii)

$$||\,[Op_h(a^{(h)}), Op_h(b^{(h)})]\,||_{L^2(M)} = O(h^{1-2\kappa}).$$

(iii) (Egorov Theorem) For any given t,

$$||Op_h(a^{(h)})(t) - Op_h(a^{(h)} \circ g^t)||_{L^2(M)} = O(h^{1-2\kappa}).$$

Remember the notation: $A(t) = e^{\frac{-ith\Delta}{2}} A e^{\frac{ith\Delta}{2}}$, for any operator A.
We will also need a result about the range of validity of the Egorov theorem.

Theorem 5.0.4. (Ehrenfest time for the evolution of observables, from [BR02]) *For every $\kappa \in [0, 1/2)$, there exists $\bar{\kappa} > 0$ such that, if $(a^{(h)})$ is a family of C^∞ functions on T^*M, with a common compact support, satisfying estimates of the form*

$$||D^n a^{(h)}||_0 \le C_n h^{-n\kappa},$$

then

$$\sup_{|t| \le \bar{\kappa} |\log h|} ||Op_h(a^{(h)})(t) - Op_h(a^{(h)} \circ g^t)||_{L^2(M)} = O(h^{\bar{\kappa}}),$$

for all $h \in (0, 1]$.

This follows almost directly from the arguments in [BR02]; the assumptions that the symbol $a^{(h)}$ and its derivatives are bounded can be relaxed to $||D^n a^{(h)}||_0 \le C_n h^{-n\kappa}$. Of course, the larger κ, the smaller $\bar{\kappa}$.
Gathering the results of Theorem 5.0.3 and Theorem 5.0.4, we obtain:

Corollary 5.0.5. *For every $\kappa \in [0, 1/2)$, there exists $\bar{\kappa} > 0$ such that: If $(a^{(h)})$, $(b^{(h)})$ are families of C^∞ functions on TM, with a common compact support, and satisfying estimates of the form*

$$||D^n a^{(h)}||_0 \le C_n h^{-n\kappa},$$

$$||D^n b^{(h)}||_0 \le C_n h^{-n\kappa},$$

then there exists a constant C such that

$$||\,[Op_h(a^{(h)})(t), Op_h(b^{(h)})]\,||_{L^2(M)} \le C h^{\bar{\kappa}}$$

for all $|t| \le \bar{\kappa} |\log h|$.

We can prove Lemma 2.3.3:

Corollary 5.0.6. *Let χ_h be a pseudo-differential operator, whose symbol is an energy cut-off, supported in a neighbourhood of the energy layer $||v|| = 1$. There exists $\bar{\kappa} > 0$ such that, for every $N \le 2\bar{\kappa} |\log h|$, for every permutation τ of $\{0, ..., N\}$, for every sequence $t_0, ..., t_N$ such that $|t_i| \le \bar{\kappa} |\log h|$, for every sequence $\alpha_0, ..., \alpha_N$,*

$$||A_{\alpha_N}(t_N).....A_{\alpha_1}(t_1) A_{\alpha_0}(t_0)\chi_h - A_{\alpha_{\tau N}}(t_{\tau N}).....A_{\alpha_{\tau 1}}(t_{\tau 1}) A_{\alpha_{\tau 0}}(t_{\tau 0})\chi_h||_{L^2(M)} = O(h^{\bar{\kappa}})$$

Proof. In the case when τ is a transposition of two consecutive integers, the proposition follows directly from Corollary 5.0.5, since the functions A_α satisfy $||D^n A_\alpha||_0 \le (n!)^s h^{-\kappa n}$.
Otherwise, the result can be proved noting that one can write any permutation of $\{0, ..., N\}$ as the product of at most $(N + 1)^2$ such transpositions. □

As a corollary we can prove Lemma 2.2.2:

Corollary 5.0.7. *Let χ_h be a pseudo-differential operator, whose symbol is an energy cut-off, supported in a neighbourhood of the energy layer $||v|| = 1$. There exist κ and $\alpha > 0$ such that, for all $n \leq \kappa|\log h|$, for every subset $W \subset \Sigma_n$,*

$$||\chi_h^* \sum_{\mathcal{C}\in W} \hat{\mathcal{C}}_h \chi_h||_{L^2(M)} \leq 1 + O(h^\alpha).$$

Proof. Define $B_i = \sqrt{A_i}$. By Corollary 5.0.6, we have

$$\sum_{\mathcal{C}\in W} \langle \hat{\mathcal{C}}\chi_h\psi, \chi_h\psi\rangle = \sum_{[\alpha_0,\ldots,\alpha_n]\in W} ||B_{\alpha_n}(n)\ldots B_{\alpha_0}\chi_h\psi||^2_{L^2(M)} + \sharp W.O(h^{\bar{\kappa}}).$$

Since $\sharp W$ grows exponentially with n, we may choose κ small enough so that $\sharp W.O(h^{\bar{\kappa}}) = O(h^\alpha)$.

□

6. Appendix A2: construction of the partition of unity (A_i^h).

The purpose of this Appendix is to show how to construct the A_i so as to satisfy the requirements of paragraph 2.1.

Of course, this holds if we have the property: There exists $p > 0$ such that

$$\int_B |\psi_h(x)|^2 dVol(x) = O(h^p).$$

where B is the tubular neighbourhood of size h^κ of the boundary of the partition P. Thus, one may try to modify the partition P so that its boundary is piecewise smooth, and the smooth hypersurfaces $(S_k)_{k=1,\ldots,L}$ forming the boundary satisfy

$$\int_{V_k(h^\kappa)} |\psi_h(x)|^2 dVol(x) = O(h^p) \tag{6.0.1}$$

where $V_k(h^\kappa)$ is a tubular neighbourhood of S_k of size h^κ.

We show here how to do so; starting with an initial partition $P(0) = P$ whose boundary consists of a finite number of smooth hypersurfaces $(S^k(0))_{k=1,\ldots,L}$, we will deform it slightly to a partition $P(h)$, with boundary components $(S^k(h))_{k=1,\ldots,L}$ that satisfy (6.0.1). The new partition will depend on h, but in a way that does not affect the proof of Theorem 1.1.1: in our construction the boundary component $(S^k(h))_{k=1,\ldots,L}$ will converge to the original $(S^k(0))_{k=1,\ldots,L}$.

We start with a simple remark. Consider an open subset $U \subset M$ equipped with a chart $\Phi : U \longrightarrow \mathbb{R}^d$ that sends U to the cube $(-2,2)^d$. Let $\tilde{S} \subset [-1,1]^{d-1}$, $\tilde{S}(0) = \tilde{S} \times \{0\} \subset (-2,2)^d$, and $S(0) = \Phi^{-1}(\tilde{S})$. And more generally, given $0 < \varepsilon < 1$ and $0 < s < 1/4$, we define

$$\tilde{S}_\varepsilon = \{x, d(x, \tilde{B}) \leq \varepsilon\} \subset (-2,2)^{d-1}$$

$$\tilde{S}_\varepsilon(m,h) = \tilde{S}_\varepsilon \times \{mh^{1/2-s}\}$$

$$\tilde{V}_\varepsilon(m,h) = \tilde{S}_\varepsilon \times [(m-1/2)h^{1/2-s}, (m+1/2)h^{1/2-s}$$

and, finally,

$$S_\varepsilon(m,h) = \Phi^{-1}(\tilde{S}_\varepsilon(m,h))$$

$$V_\varepsilon(m,h) = \Phi^{-1}(\tilde{V}_\varepsilon(m,h))$$

(the latter is a tubular neighbourhood of size $h^{1/2-s}$ of the former); m is an integer in $[-h^{-1/2+2s}, h^{-1/2+2s}]$. Since

$$\sum_{m\in[-h^{-1/2+2s},h^{-1/2+2s}]} \int_{V_\varepsilon(m,h)} |\psi_h(x)|^2 dVol(x) \leq 1$$

there must exist an $m_0 \in [-h^{-1/2+2s}, h^{-1/2+2s}]$ (depending on h) such that

$$\int_{V_\varepsilon(m_0,h)} |\psi_h(x)|^2 dVol(x) \leq h^{1/2-2s}.$$

This means that $S_\varepsilon(m_0,h)$ satisfies (6.0.1) with $\kappa = 1/2-s$ and $p = 1/2-2s$ (which is even better than what we need). Besides, $S_\varepsilon(m_0,h)$ is at distance h^s from $S_\varepsilon(0)$.

We conclude that, even if $S(0)$ did not satisfy (6.0.1), there is a hypersurface h^s-close to it that satisfies it.

Let us now consider a partition $P(0)$, with boundary components $(S^k(0))_{k=1,...,L}$. For every k, we know that there exists a hypersurface $S^k_\varepsilon(h)$ h^s-close to $S^k_\varepsilon(0)$ thatsatisfies (6.0.1) with $p = 1/2 - 2s$. The problem is to show, in addition, that for each k, there exists $S^k(h) \in S^k_\varepsilon(h)$ such that the $S_k(h)$s form the boundary of a new partition.

Although this is probably always true for general partitions with piecewise smooth boundary, we will avoid a tedious combinatorial argument by considering only special "cubic" partitions, that we describe below:

In the universal cover M, consider a polyhedral fundamental domain $D(0)$ for the action of $\Gamma = \pi_1(M)$, whose boundary is piecewise smooth; consider also an open, relatively compact subset $U \subset M$, containing $D(0)$, and equipped with a chart $\Phi : U \longrightarrow \mathbb{R}^d$ that sends U to the cube $(-2,2)^d$. Given $\alpha > 0$, one has a partition of $(-2,2)^d$ into cubes of size ε, delimited by the hypersurfaces $\tilde{S}^{k,m}(0) = \{x_k = m\alpha\}$ ($k = 1,...,d$, $m \in \mathbb{Z}$, $|m| \leq 2/\alpha$). This partition gives a partition of U which, restricted to the fundamental domain $D(0)$, gives our partition $P(0)$ of M. More precisely, the boundary of $P(0)$ is formed by the image in M of

– parts of the $S^{k,m}(0) = \Phi^{-1}(\tilde{S}^{k,m}(0)$;

– the boundary of $D(0)$.

Most elements of $P(0)$ are sent to cubes by the chart Φ, except for those intersecting the boundary of the fundamental domain.

The boundary of the "polyhedra" $D(0)$ consists in a finite number of smooth hypersurfaces $S^k(0)$; applying the previous procedure, we can find some $S^k_\varepsilon(h)$ satisfying (6.0.1) and such that

– for each k, we can find a subset $S^k(h) \subset S^k_\varepsilon(h)$ such that the $S^k(h)$s form the boundary of a new fundamental domain $D(h)$.

– $S^k(h)$ is at distance h^s from $S^k(0)$.

In the cube $(-2,2)^d$, always by the same procedure, we can move the $\tilde{S}^{k,m}(0)$s to

$$\tilde{S}^{k,m}(h) = \{x_k = m\alpha + m_0(k,m)h^{1/2-s}\}$$

($m_0(k,m) \in [-h^{-1/2+2s}, h^{-1/2+2s}]$ as previously) so that

$$S^{k,m}(h) := \Phi^{-1}(\tilde{S}^{k,m}(h))$$

satisfies (6.0.1), for every k,m. Besides, the $\tilde{S}^{k,m}(h)$ still delimit a partition of $(-2,2)^d$ into cubes (parallelepipedes) and thus the $S^{k,m}(h)$ delimit a partition of the open set $U \in M$.

This partition of U, restricted to the fundamental domain $D(h)$, gives our partition $P(h)$ of M. More precisely, the boundary of $P(h)$ is formed by the image in M of

– parts of the $S^{k,m}(h) = \Phi^{-1}(\tilde{S}^{k,m}(0)$;

– the boundary of $D(h)$.

The boundary of the new partition $P(h)$ satisfies (6.0.1) and converges to the boundary of $P(0)$. The characteristic function of $P_i(h)$ converges to the characteristic function of $P_i(0)$, uniformly on every compact subset of the interior of $P_i(0)$ (for every $i = 1, ..., l$).

We construct A_i^h by applying the convolution (2.1.1) to $P_i(h)$ instead of P_i.

References

[AF94] R. ALICKI, M. FANNES, *Defining quantum dynamical entropy*, Lett. Math. Phys. **32** (1994), no. 1, 75–82.

[AN05] N. ANANTHARAMAN, S. NONNENMACHER, *Semi-classical entropy of the Walsh-quantized baker's map,* in preparation.

[An85] D.V. ANOSOV, *Geodesic flows that satisfy the U-condition (Y) (Russian)*, Current problems in mathematics. Mathematical analysis, algebra, topology. Trudy Mat. Inst. Steklov. 167 (1985), 3–24, 276.

[AMB92] S. AUBRY, R.S. McKAY, C. BAESENS, *Equivalence of uniform hyperbolicity for symplectic twist maps and phonon gap for Frenkel-Kontorova models*, Physica D **56** (1992), 123–134.

[BDB03] F. BONECHI, S. DE BIEVRE, *Controlling strong scarring for quantized ergodic toral automorphisms*, Duke Math. J. **117** (2003), no. 3, 571–587.

[BR02] A. BOUZOUINA, D. ROBERT, *Uniform semiclassical estimates for the propagation of quantum observables*, Duke Math. J. **111** (2002), no. 2, 223–252.

[BK83] M. BRIN, A. KATOK, *On local entropy*, Geometric dynamics (Rio de Janeiro, 1981), 30–38, Lecture Notes in Math., 1007, Springer, Berlin, 1983.

[Ca] H. CARTAN, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961.

[Co85] Y. COLIN DE VERDIERE, *Ergodicité et fonctions propres du laplacien*, Comm. Math. Phys **102** (1985), no. 3, 497–502.

[CP94] Y. COLIN DE VERDIERE, B. PARISSE, *Equilibre instable en régime semi-classique I : concentration microlocale*, Comm. Partial Differential Equations **19** (1994), no. 9-10, 1535–1563.

[CNT87] A. CONNES, H. NARNHOFER, W. THIRRING, *Dynamical entropy of C^* algebras and von Neumann algebras*, Comm. Math. Phys. **112** (1987), no. 4, 691–719.

[DH72] J.J. DUISTERMAAT, L. HÖRMANDER, *Fourier integral operators. II*, Acta Math. 128 (1972), no. 3-4, 183–269.

[Eb73] P. EBERLEIN, *When is a geodesic flow of Anosov type ? I,II.*, J. Differential Geometry **8** (1973), 437–463; ibid. **8** (1973), 565-577.

[FNDB03] F. FAURE, N, NONNENMACHER, S. DE BIEVRE, *Scarred eigenstates for quantum cat maps of minimal periods*, Comm. Math. Phys. **239** (2003), no. 3, 449–492.

[HM87] B. HELFFER, A. MARTINEZ, *Ergodicité et limite semi-classique*, Comm. Math. Phys. **109** (1987), no. 2, 313–326.

[He] S. HELGASON, *Topics in harmonic analysis on homogeneous spaces. Progress in Mathematics*, 13. Birkhäuser, Boston, Mass., 1981.

[Ho] L. HÖRMANDER, *The analysis of linear partial differential operators I*, Springer Verlal, seconde édition.

[Ho71] L. HÖRMANDER, *Fourier integral operators. I*, Acta Math. 127 (1971), no. 1-2, 79–183.

[KH] A.B. KATOK, B. HASSELBLATT, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Mathematics and its applications **54** (1995), Cambridge University Press.

[Kl74] W. KLINBENBERG, *Riemannian manifolds with geodesic flows of Anosov type*, Ann. of Math. (2) **99** (1974), 1–13.

[LY85] F. LEDRAPPIER, L.-S. YOUNG, *The metric entropy of diffeomorphisms. I. Characterization of measures satisfying Pesin's entropy formula*, Ann. of Math. (2) **122** (1985), no. 3, 509–539.

[LGS96] E. LEICHTNAM, F. GOLSE, M. STENZEL, *Intrinsic microlocal analysis and inversion formulae for the heat equation on compact real-analytic Riemannian manifolds*, Ann. Sci. Ecole Norm. Sup. (4) **29** (1996), no. 6, 669–736.

[Li03] E. LINDENSTRAUSS, *Invariant measures and arithmetic quantum unique ergodicity*, preprint (2003).

[McK72] H.P. McKEAN, *Selberg's trace formula as applied to a compact Riemann surface*, Comm. Pure Appl. Math. **25** (1972), 225–246.

[Ot98] J-P. OTAL, *Sur les fonctions propres du laplacien du disque hyperbolique*, C. R. Acad. Sci. Paris Sér. I Math. **327** (1998), no. 2, 161–166.

[RS94] Z. RUDNICK, P. SARNAK, *The behaviour of eigenstates of arithmetic hyperbolic manifolds*, Comm. Math. Phys **161** (1994), no. 1, 195–213.

[Ru91] R.O. RUGGIERO, *On the creation of conjugate points*, Math. Z. **208** (1991), no. 1, 41–55.

[Sa95] P. SARNAK, *Arithmetic quantum chaos*, The Schur lectures (1992) (Tel Aviv), 183–236, Israel Math. Conf. Proc., 8, Bar-Ilan Univ., Ramat Gan, 1995.

[Sa03] P. SARNAK, *Spectra of hyperbolic surfaces*, Bull. A.M.S. **40** (2003), no. 4, 441–478.

[Sn74] A. I. SNIRELMAN, *Ergodic properties of eigenfunctions (Russe)*, Uspehi Mat. Nauk. **29** (1974), no. 6(180), 181–182.

[So88] C. SOGGE, *Concerning the L^p norm of spectral clusters for second order elliptic operators o compact manifolds*, J. Func. An. **77** (1988), 123–138.

[Vo95] D. VOICULESCU, *Dynamical approximation entropies and topological entropy in operator algebras*, Comm. Math. Phys **170** (1995), 249–281.

[Ze86] S. ZELDITCH, *Pseudodifferential analysis on hyperbolic surfaces*, J. Funct. Anal. **68** (1986), no. 1, 72–105.

[Ze87] S. ZELDITCH, *Uniform distribution of eigenfunctions on compact hyperbolic surfaces*, Duke Math. J. **55** (1987), no. 4, 919–941.

Unité de Mathématiques Pures et Appliquées, Ecole Normale Supérieure, 6, allée d'Italie, 69364 LYON Cedex 07, FRANCE

E-mail address: nanantha@umpa.ens-lyon.fr