INTERTWINING THE GEODESIC FLOW AND THE SCHRÖDINGER GROUP ON HYPERBOLIC SURFACES

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Abstract. We construct an explicit intertwining operator $\mathcal{L}$ between the Schrödinger group $e^{it\frac{\triangle}{2}}$ and the geodesic flow on certain Hilbert spaces of symbols on the cotangent bundle $T^*\mathbf{X}_\Gamma$ of a compact hyperbolic surface $\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D}$. We also define Γ-invariant eigendistributions of the geodesic flow $PS_{j,k,\nu_j,-\nu_k}$ (Patterson-Sullivan distributions) out of pairs of $\triangle$-eigenfunctions, generalizing the diagonal case $j = k$ treated in [AZ]. The operator $\mathcal{L}$ maps $PS_{j,k,\nu_j,-\nu_k}$ to the Wigner distribution $W^\Gamma_{j,k}$ studied in quantum chaos. We define Hilbert spaces $\mathcal{H}_{PS}$ (whose dual is spanned by $\{PS_{j,k,\nu_j,-\nu_k}\}$), resp. $\mathcal{H}_W$ (whose dual is spanned by $\{W^\Gamma_{j,k}\}$), and show that $\mathcal{L}$ is a unitary isomorphism from $\mathcal{H}_W \to \mathcal{H}_{PS}$.

1. Introduction

On a hyperbolic surface $\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D}$, there is an intimate relation between the spectral properties of the laplacian $\triangle$ and those of the geodesic flow g^t on the unit tangent bundle $S\mathbf{X}_\Gamma$. The Selberg trace formula gives an exact formula for the trace of the Schrödinger flow $e^{it\frac{\triangle}{2}}$ as a sum over closed geodesics, and it may be interpreted as a twisted trace of the pull-back operator by g^t [G]. Equivalently, eigenvalues of $\triangle$ are (re-parameterizations of) the resonances of g^t (see [B] for background); see also [Bis, Po2, M] for some of the many different perspectives on this relation. In this article, we give a yet stronger relation between the two flows : we construct an explicit intertwining operator $\mathcal{L}$ (Definition 2) between the Schrödinger flow and the geodesic flow, which induces a similar intertwining operator $\mathcal{L}_\Gamma$ on the quotient. Our main result, Theorem 3, is that there exist Hilbert spaces of symbols on which $\mathcal{L}_\Gamma$ is a unitary intertwining operator between the classical and quantum flow. Much of the problem is to construct the appropriate Hilbert spaces, which we denote by $\mathcal{H}_W, \mathcal{H}_{PS}$ (see Definitions 6.1-6.2). In fact, the definition is quite flexible and one can construct many weighted Hilbert spaces for which $\mathcal{L}_\Gamma$ is unitary (as well as some Banach spaces). These Hilbert spaces cannot be the standard Hilbert spaces, $L^2(S\mathbf{X}_\Gamma)$ for g^t, resp. Hilbert-Schmidt operators for $e^{it\frac{\triangle}{2}}$, since the spectrum of g^t is continuous, while that of $e^{it\frac{\triangle}{2}}$ is discrete; and both the Hilbert and Banach spaces we define are also quite different from the Banach spaces constructed in [BT, BKL, BL, FRS, GL] in the theory of resonances of g^t.

The construction of $\mathcal{L}_\Gamma, \mathcal{H}_W, \mathcal{H}_{PS}$ and the proof of the intertwining property grew out of our previous work [AZ], where we introduced and studied a family of distributions (that we called Patterson-Sullivan distributions) on the unit tangent bundle of a hyperbolic surface. These distributions are invariant under the geodesic flow, and we showed that they are closely related to the *Wigner distributions* appearing in the theory of quantum ergodicity. The Patterson-Sullivan distributions are naturally constructed from the family of eigenfunctions

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of the laplacian, and we showed that they also arise as residues of dynamical zeta-functions at the poles located on the critical line.

In this paper, we introduce the family of *off-diagonal* Patterson-Sullivan distributions, and show how they are related to the off-diagonal Wigner distributions (appearing in the study of quantum mixing). This construction is a rather straightforward generalization of the work done in [AZ]. More importantly, we show that these formulae directly lead to an operator intertwining the geodesic flow and the Schrödinger flow on the hyperbolic plane (or a compact quotient $\mathbf{X}_\Gamma$). Roughly speaking, the dual Hilbert spaces $\mathcal{H}_W^*$, resp. $\mathcal{H}_{PS}^*$, intertwined by $\mathcal{L}$ are spanned by the Wigner, resp. Patterson-Sullivan, distributions. The main goal of this paper is to construct explicitly this intertwining operator, first on the hyperbolic plane, then on a compact quotient, and to investigate some of its properties. The existence of this operator opens the way to the construction of a quantization procedure $a \mapsto \mathrm{Op}(\mathcal{L}_\Gamma^{-1} a)$ satisfying the Egorov theorem in its exact form (without remainder term). We refer to §5.7 (see especially (5.27)) for further discussion.

We have to explain in what sense one can find an intertwining operator between the geodesic flow and the Schrödinger flow. The former acts on functions on the (co)tangent bundle $T\mathbf{X}_\Gamma$ whereas the latter acts on functions on the base manifold $\mathbf{X}_\Gamma$. In fact, we let the Schrödinger group act on the space of operators, by conjugation (as in the Heisenberg picture of quantum mechanics). Operators have a Schwartz kernel, which is a distribution on the product $\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma$. By taking the local Fourier transform of the kernel with respect to the second component, we get a distribution on the cotangent bundle $T^*\mathbf{X}_\Gamma$, called the *symbol* of the operator. This way, we see that the Schrödinger group acts naturally on the space of distributions on $T^*\mathbf{X}_\Gamma$ (in the paper we will always identify the tangent and the cotangent bundles by means of the riemannian metric). With this formulation, the Schrödinger flow acts on the same space as the geodesic flow, and it is in this sense that we shall intertwine their actions.

1.1. **Notation.** To state our results, we need to introduce some notation (see §2 for more details). We will denote $G = PSU(1,1) \simeq PSL(2,\mathbf{R})$, $K = PSO(2,\mathbf{R})$ a maximal compact subgroup, and G/K the corresponding symmetric space, for which we will in general use the picture of the hyperbolic disc $\mathbf{D} = \{z \in \mathbf{C}, |z| < 1\}$, endowed with the riemannian metric

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}.$$

This is a standard normalization in hyperbolic geometry, but we caution that it differs by a constant factor from the normalization used by Helgason [He]; for us, the L^2-spectrum of the laplacian on $\mathbf{D}$ is $(-\infty, -\frac{1}{4}]$. Hence some discrepancies between some of our formulae and Helgason's.

It is well-known that G can be identified with the unit tangent bundle $S\mathbf{D}$ of the hyperbolic disc $\mathbf{D}$ (when using the theory of pseudodifferential operators, it is more natural to work on the cotangent bundle, but we will always identify both). We will be particularly interested in the geodesic flow, which acts on G by right multiplication as follows : for all $g \in PSL(2,\mathbf{R})$, for all $t \in \mathbf{R}$, $g^t(g) = ga_t$ where $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix} \in SL(2,\mathbf{R})$. We will also use the action of the horocycle flow $(h^u)_{u\in\mathbf{R}}$, acting by $h^u(g) = gn_u$ where $n_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \in SL(2,\mathbf{R})$.

We will work with two special parameterizations of the unit tangent bundle, identified with G. The first one is obtained by writing $G \sim (G/K) \times K \sim \mathbf{D} \times B$ where B is the boundary at infinity of $\mathbf{D}$, identified with the unit circle S^1 in the Poincaré disc model. The group K and the boundary at infinity S^1 are identified by the map

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \mapsto e^{2i\theta}.$$

This way, a point in G can be parameterized by the coordinates (z, b), where $z \in \mathbf{D}$ and $b \in B$. In geometric terms, if (z, b) is identified with a unit tangent vector in $S\mathbf{D}$, then b represents the (forward) limit point of the geodesic generated by (z, b). The action of G on itself by left-multiplication yields an action of G on B (Section 3).

We shall also use the following parameterization: denote $B^{(2)} = \{(b', b) \in B \times B, b \neq b'\}$ the set of pairs of distinct points in the boundary. Each oriented geodesic in $\mathbf{D}$ is completely determined by its (unique) forward limit point b in B and its (unique) backward limit point $b' \neq b$ in B : we will denote $\gamma_{b',b}$ the geodesic going from $b' \in B$ to $b \in B$. Thus, $B^{(2)}$ can be naturally identified with the set of oriented geodesics of $\mathbf{D}$. The elements of G can be parameterized by (b', b, τ) with $(b', b) \in B^{(2)}$ and $\tau \in \mathbf{R}$. We identify (b', b, τ) with the point $(z, b) \in \mathbf{D} \times B$, where z is on the geodesic $\gamma_{b',b}$, situated τ units from the point $z_{b,b'} \in \gamma_{b,b'}$ closest to the origin $o \stackrel{\text{def}}{=} eK \in \mathbf{D}$.

Our final goal is to obtain formulae that are valid on a compact quotient of $\mathbf{D}$; that is, we consider a co-compact discrete subgroup $\Gamma \subset G$. We assume it has no torsion[1] and contains only hyperbolic elements. Then the quotient $\mathbf{X}_\Gamma = \Gamma\backslash\mathbf{D}$ is a compact hyperbolic surface.

1.2. **Quantization, Wigner distributions and Patterson-Sullivan distributions.** A quantization procedure adapted to the hyperbolic disc was defined in [Z3], using Helgason's version of the Fourier transform [He]. For $(z, b) \in \mathbf{D} \times B$, define the Busemann function $\langle z, b\rangle$ as the signed distance to o of the horocycle going through the points $z \in \mathbf{D}, b \in B$. The family of functions $z \mapsto e^{(\frac{1}{2}+ir)\langle z,b\rangle}$ $(r > 0, b \in B)$ forms a basis of generalized eigenfunctions of the laplacian on $L^2(\mathbf{D})$ [He]. The hyperbolic pseudodifferential operators introduced by [Z3] are defined by

$$\mathrm{Op}(a)e^{(\frac{1}{2}+ir)\langle\bullet,b\rangle} = a(\bullet, b, r)e^{(\frac{1}{2}+ir)\langle\bullet,b\rangle}, \tag{1.1}$$

if $a = a(z, b, r)$ is a function on $\mathbf{D} \times B \times \mathbf{R} \simeq S\mathbf{D} \times \mathbf{R}$ which must have "reasonable" decay and smoothness properties (section 4). The function a is called the symbol of the operator. We note that by choosing $r < 0$ instead of $r > 0$ we obtain another basis of generalized

[1]This assumption is probably not necessary.

eigenfunctions of the laplacian. We also note that the Schwartz-kernel of the operator is formally given by

$$K_a(z,w) = \int_B \int_{r\in\mathbf{R}_+} a(z,b',r) e^{(\frac{1}{2}+ir)\langle z,b'\rangle} e^{(\frac{1}{2}-ir)\langle w,b'\rangle} dp(r)db', \tag{1.2}$$

where $dp(r)$ is the Plancherel measure defined in §3.1. Paley-Wiener type theorems relating the decay and regularity of a and those of K_a will be recalled in §4.2. In most formulae we assume that $K_a(z,w)$ decays sufficiently fast away from the diagonal $\{z=w\}$. This implies in particular that the corresponding symbol a has a holomorphic extension to $r\in\mathbf{C}$. In [Z3], it is often assumed that $\mathrm{Op}(a)$ is a *properly supported* pseudo-differential operator, i.e. that $K_a(z,w)$ is supported in a fixed tube $d(z,w)<R$ around the diagonal (where d is the hyperbolic distance between points).

Let $\mathbf{X}_\Gamma$ be a compact quotient of $\mathbf{D}$ as above, and fix an orthonormal basis (ϕ_k) of $L^2(\mathbf{X}_\Gamma)$ formed of eigenfunctions of the laplacian. We use the standard notations in hyperbolic spectral theory : the eigenfunctions ϕ_k satisfy

$$\triangle\phi_k = -\left(\frac{1}{4}-\nu_k^2\right)\phi_k = -\left(\frac{1}{4}+r_k^2\right)\phi_k,$$

where $\nu_k = ir_k \in \mathbf{R}\cup i\mathbf{R}$ is called the spectral parameter (on a compact surface, only a finite number of r_ks are imaginary). For each eigenvalue there are two possible choices for the spectral parameter.

The "Wigner distributions[2]" $W_{j,k}$ are defined on $S\mathbf{D}\times\mathbf{R}\simeq G\times\mathbf{R}$ by the formula

$$\int_{S\mathbf{D}\times\mathbf{R}} a\, dW_{j,k} \stackrel{\mathrm{def}}{=} \langle \mathrm{Op}(a)\phi_j, \phi_k\rangle_{\mathbf{D}}, \tag{1.3}$$

for a a function on $S\mathbf{D}\times\mathbf{R}$, with appropriate growth and smoothness properties with respect to $(z,b)\in S\mathbf{D}$. For instance, the matrix element is well-defined if a is smooth, and compactly supported with respect to $(z,b)\in S\mathbf{D}$; or if a belongs to the space $\mathcal{S}_0^0$ defined in §4.2. The distribution $W_{j,k}$ is invariant by the action of Γ on $S\mathbf{D}$, and thus can be used to define a distribution $W_{j,k}^\Gamma$ on the quotient $S\mathbf{X}_\Gamma\times\mathbf{R}\simeq\Gamma\backslash G\times\mathbf{R}$: if a is a smooth function on $\Gamma\backslash G\times\mathbf{R}$, in other words a Γ-invariant function on $G\times\mathbf{R}$, we define

$$\int_{S\mathbf{X}_\Gamma\times\mathbf{R}} a\, dW_{j,k}^\Gamma = \int_{S\mathbf{D}\times\mathbf{R}} \chi a\, dW_{j,k},$$

where χ is a *smooth fundamental cut-off function* for the action of Γ (see §3.3 for the definition). It can easily be seen (§3.3) that this definition does not depend on the choice of χ. To rephrase this definition, if a is a Γ-invariant symbol, and $\mathrm{Op}(a)$ is a properly supported pseudo-differential operator, it was shown in [Z3, Z1] that $\mathrm{Op}(a)$ preserves the space of Γ-invariant functions. We denote by $\mathrm{Op}_\Gamma(a)$ the operator $\mathrm{Op}(a)$ acting on Γ-invariant functions. Then

$$\int_{S\mathbf{X}_\Gamma\times\mathbf{R}} a\, dW_{j,k}^\Gamma = \langle \mathrm{Op}_\Gamma(a)\phi_j, \phi_k\rangle_{L^2(\mathbf{X}_\Gamma)}, \tag{1.4}$$

[2]This terminology is normally used in a euclidean context, but here we use it in also for non-euclidean geometries; we use quotation marks since the terminology is somewhat non-standard in the setting of hyperbolic geometry.

Growth and smoothness properties of a will be discussed in detail in Section 4.

The Wigner distribution $W^{\Gamma}_{j,k}$ may also be expressed in terms of the boundary values of the eigenfunctions ϕ_j, ϕ_k. The boundary values T_{k,ν_k} of ϕ_k is a distribution on the boundary B, with the property that

$$\phi_k(z) = \int_B e^{(\frac{1}{2}+\nu_k)\langle z,b\rangle} T_{k,\nu_k}(db),$$

for all $z \in \mathbf{D}$. It depends on the choice of a spectral parameter ν_k, and is unique if we pick ν_k such that $\frac{1}{2}+\nu_k \neq 0, -1, -2, \cdots$ ([He, Theorems 4.3 and 4.29]; see also [He2]). With a slight abuse of notation we sometimes denote T_{k,ν_k} more simply by T_{ν_k}. Using the definition of Op, we have

$$\int_{S\mathbf{X}_\Gamma \times \mathbf{R}} a\, dW^{\Gamma}_{j,k} = \int_{\mathbf{D}\times B} \chi(z,b) a(z,b,r_j) \overline{\phi_k(z)} e^{(\frac{1}{2}+\nu_j)\langle z,b\rangle} T_{\nu_j}(db)\, \mathrm{Vol}(dz)$$

where as above χ is a smooth fundamental cut-off (see §3.3).

It follows from its definition that $W^{\Gamma}_{j,k}$ is an eigendistribution of the quantum evolution. Define

$$\alpha^t(\mathrm{Op}(a)) = e^{-it\frac{\triangle}{2}}\, \mathrm{Op}(a) e^{it\frac{\triangle}{2}}. \tag{1.5}$$

We then have

$$\langle \alpha^t(\mathrm{Op}_\Gamma(a))\phi_j, \phi_k\rangle = e^{it\frac{(\nu_j^2-\overline{\nu_k}^2)}{2}} \langle \mathrm{Op}_\Gamma(a)\phi_j, \phi_k\rangle = e^{it\frac{(r_k^2-r_j^2)}{2}} \langle \mathrm{Op}_\Gamma(a)\phi_j, \phi_k\rangle \tag{1.6}$$

where the last identity holds only for real values of r_j, r_k.

We henceforth denote by V^t the operator on symbols, defined formally by

$$\alpha^t(\mathrm{Op}(a)) = \mathrm{Op}(V^t(a)). \tag{1.7}$$

As we shall see, $V^t a$ is well defined when a is smooth and compactly supported on $\mathbf{D}$, and then $V^t a$ belongs to the space $\mathcal{S}^0_0$ defined in §4.2. Besides, V^t extends to a unitary operator on the Hilbert space $L^2_W(G\times\mathbf{R}, dg\times dp(r))$ of Weyl-group invariant L^2 functions defined in §4.1. On the quotient, V^t induces an operator V^t_Γ acting on $\Pi\mathcal{S}^0_0$, defined in Definition 4.1 as the space of Γ-invariant symbols obtained by periodizing elements of $\mathcal{S}^0_0$. The operator V^t_Γ is unitary on $L^2_W(\Gamma\backslash G\times\mathbf{R}, dg\times dp(r))$ (see Definition 4.2).

Remark 1.1. The quantization of the geodesic flow (i.e. the quantum evolution) should be a Fourier integral operator, and it may be objected that $e^{it\frac{\triangle}{2}}$ is not a Fourier integral operator. To make it one, it suffices to rescale in time $t \to \hbar t$ to obtain the semi-classical Schrödinger group $e^{it\hbar\frac{\triangle}{2}}$. We do not do this because our intertwining relations are exact and apply with no essential difference to either group.

However, in §5.7 we discuss the re-scaling to $e^{it\hbar\frac{\triangle}{2}}$ and the relation between the semi-classical Egorov theorem and the exact intertwining. For background on the semi-classical Schrödinger group we refer to [AN].

The homogeneous quantization of the geodesic flow is the wave group $e^{it\sqrt{\triangle}}$. Our methods can be modified to intertwine this group as well with the geodesic flow but that is technically somewhat more complicated and we prefer to use $e^{it\frac{\triangle}{2}}$; we discuss this further in §5.4.

The Wigner distributions have been studied a lot in the theory of quantum ergodicity and quantum mixing. In the context of a homogeneous space $\Gamma\backslash G$, the Wigner distributions of this article are studied in [Z1, Z3, W, SV, SV2] as well as in [AZ]. In this paper we introduce the family of (off-diagonal) Patterson-Sullivan distributions. They are also constructed from pairs of eigenfunctions ϕ_j, ϕ_k, after taking their boundary values $T_{j,\nu_j}(db)$ and $T_{k,-\nu_k}(db)$.

DEFINITION 1. *$PS_{j,k,\nu_j,-\nu_k}(db', db, d\tau)$ is the Γ-invariant distribution on $B^{(2)}\times\mathbf{R} \sim G$ defined by*

$$PS_{(j,\nu_j),(k,-\nu_k)}(db', db, d\tau) = \frac{T_{j,\nu_j}(db)\overline{T_{k,-\nu_k}}(db')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} e^{(\nu_j+\overline{\nu_k})\tau} d\tau.$$

We note that the Patterson-Sullivan distributions depend on the eigenfunctions ϕ_j, ϕ_k, but also on the choice of the spectral parameters $\nu_j, -\nu_k$ (in contrast with the Wigner distributions, which depend only on the eigenfunctions); hence the notation $PS_{(j,\nu_j),(k,-\nu_k)}$. It is crucial to take opposite sign conventions for the choice of the spectral parameters associated with ϕ_j, ϕ_k (ν_j vs. $-\nu_k$) if we want the Patterson-Sullivan to be invariant under the geodesic flow in the diagonal principal series case, $\nu_j = \nu_k \in i\mathbf{R}$. This remark was generalized to higher-rank Lie groups by M. Schröder in his dissertation [SchDiss]. Hilgert and Schröder have extended the definition and properties of off-diagonal Patterson-Sullivan distributions to more general symmetric spaces [HilSc, SchDiss].

In the sequel, we will in general use the shorter notation $PS_{\nu_j,-\nu_k}$, although it is slightly abusive. In Proposition 6.1, we check that the distributions $PS_{\nu_j,-\nu_k}$ are (right)-Γ-invariant distributions on $B^{(2)}\times\mathbf{R}\sim G$. Besides, since the geodesic flow reads

$$g^t(b', b, \tau) = (b', b, \tau + t),$$

they are eigendistributions for the geodesic flow in the sense that

$$g^t_\sharp PS_{\nu_j,-\nu_k} = e^{-t(\nu_j+\overline{\nu_k})} PS_{\nu_j,-\nu_k} = e^{it(r_k-r_j)} PS_{\nu_j,-\nu_k} \tag{1.8}$$

(the last identity holds only for real values of r_j, r_k). As a result, $PS_{\nu_j,-\nu_k}$ induces an eigendistribution $PS^{\Gamma}_{\nu_j,-\nu_k}$ of the geodesic flow on $\Gamma\backslash G = S\mathbf{X}_\Gamma$, defined by

$$\int_{\Gamma\backslash G} a\, dPS^{\Gamma}_{\nu_j,-\nu_k} = \int_G (\chi a)\, dPS_{\nu_j,-\nu_k}, \tag{1.9}$$

for every smooth Γ-invariant function a. Once again χ is a smooth fundamental domain cutoff, see §3.3.

It was pointed out in [Z1] that the distribution

$$\epsilon_{\nu_j}(z,b) e^{\langle z,b\rangle} \operatorname{Vol}(dz) db \overset{\text{def}}{=} e^{(\frac{1}{2}+\nu_j)\langle z,b\rangle} T_{\nu_j}(db) \operatorname{Vol}(dz) \tag{1.10}$$

on $S\mathbf{X}_\Gamma$ is a joint eigendistribution of the horocycle and geodesic flows, contained in the (spherical) irreducible representation of G generated by ϕ_j. The PS-distributions are new objects, and it is illuminating to express them in terms of these more familiar ones.

Proposition 1.1. *The Patterson-Sullivan distributions are given by the (well-defined) products $\left(\epsilon_{\nu_j} \cdot \iota\overline{\epsilon_{-\nu_k}}\right)$:*

$$PS_{\nu_j,-\nu_k}(db', db, d\tau) = \frac{2^{-(\nu_j-\overline{\nu_k})}}{2\pi} \left(\epsilon_{\nu_j} \cdot \iota\overline{\epsilon_{-\nu_k}}\right)(z,b)\, e^{\langle z,b\rangle} \operatorname{Vol}(dz) db,$$

with $(z,b) \simeq (b',b,\tau)$ *and where* ι *is the involution* $(b',b,\tau) \mapsto (b,b',-\tau)$*, corresponding to* $(x,\xi) \mapsto (x,-\xi)$ *on* $S\mathbf{D}$*, or to the action of the non-trivial element of the Weyl group,* $g \mapsto gw$ *on* G*, with* $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Remark 1.2. We note that the Wigner distributions are naturally defined on $S\mathbf{X}_\Gamma \times \mathbf{R}$, whereas the Patterson-Sullivan distributions were only defined on $S\mathbf{X}_\Gamma$. In order to relate both, we need to extend the latter to $S\mathbf{X}_\Gamma \times \mathbf{R}$. We do so by identifying $S\mathbf{X}_\Gamma$ with $S\mathbf{X}_\Gamma \times \{\frac{r_j+r_k}{2}\}$. In other words, we extend the PS-distributions to $S\mathbf{X}_\Gamma \times \mathbf{R}$ by considering

$$PS^\Gamma_{\nu_j,-\nu_k} \otimes \delta_{\frac{r_j+r_k}{2}} \tag{1.11}$$

for real values of r_j, r_k. We must pay special attention, and treat separately, the case of low laplacian eigenvalues, when r is imaginary. When r_j or r_k is imaginary, the formula above will be generalized to

$$PS^\Gamma_{\nu_j,-\nu_k} \otimes \delta_{\frac{\nu_j-\overline{\nu_k}}{2i}}, \tag{1.12}$$

and our results will have to be restricted to functions $a(z,b,r)$ that have a holomorphic extension to $r \in \mathbf{C}$, so that it makes sense to pair them with a functional such as (1.12). The Paley-Wiener theorems of §4.2 show that this is the case if the kernel $K_a(z,w)$ is smooth and rapidly decaying away from the diagonal. The functional is (1.12), strictly speaking, no longer a distribution on $S\mathbf{X}_\Gamma \times \mathbf{R}$.

The extension (1.12) of formula (1.11) is somewhat arbitrary; however, in view of the sesquilinearity of the scalar product on L^2, it seemed rather natural to use an extension which is holomorphic w.r.t. ν_j and antiholomorphic w.r.t. ν_k.

1.3. **Definition of the intertwining operator.**

Definition **2.** *The intertwining operator*

$$\mathcal{L} : C_c^\infty(G \times \mathbf{R}) \to C(G \times \mathbf{R})$$

is defined by

$$\mathcal{L}a(g,R) = \int\int\int (1+u^2)^{-(\frac{1}{2}+iR)} a(ga_{\tau - \frac{\log(1+u^2)}{2}} n_u, r) e^{-2i(R-r)\tau} dr du d\tau. \tag{1.13}$$

Here, $n_u \in N$ is the one-parameter unipotent subgroup whose right-orbits define the horocycle flow and $a_t \in A$ is the one parameter subgroup whose right-orbits define the geodesic flow (see §2.1).

Extend the geodesic flow to $G \times \mathbf{R}$ by the formula

$$G^t(g,r) = (ga_{rt}, r). \tag{1.14}$$

We denote by G^t_Γ the induced flow on $\Gamma\backslash G \times \mathbf{R}$.

We will also consider the geodesic flow as an operator acting on functions, by composition : for a function a on G, we denote $g^t a \stackrel{\text{def}}{=} a \circ g^t$, and for a function on $G \times \mathbf{R}$, $G^t a \stackrel{\text{def}}{=} a \circ G^t$.

1.4. **Statement of results.** The main result of this paper is the intertwining relation

$$\mathcal{L} \circ V^t = G^t \circ \mathcal{L}. \tag{1.15}$$

We prove (1.15) in several levels. In Section 5, we work on the universal cover, and prove that

$$\mathcal{L} \circ V^t a = G^t \circ \mathcal{L} a$$

for a belonging to a space of smooth and rapidly decaying symbols on $S\mathbf{D} \times \mathbf{R}$, denoted by $\mathcal{S}_0^0$ and defined in §4.2. Theorem 7 (i) says that both sides are continuous functions and the equality holds pointwise. It is then natural to ask for an extension of $\mathcal{L}$ to the Hilbert space of square integrable symbols $a \in L^2(G \times \mathbf{R}_+, dg \times dp(r))$, which is in bijection with the space $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ of §4.1, and isometric to the space of Hilbert-Schmidt operators (Proposition 4.1). However, $\mathcal{L}$ is not bounded on $L^2_W(G \times \mathbf{R}, dg \times dp(r))$. Theorem 7 (ii) characterizes the image $\mathcal{L}(L^2_W(G \times \mathbf{R}, dg \times dp(r)))$, it is a Hilbert space called $\mathcal{H}_{PS}(\mathbf{D})$.

We then pursue the study of $\mathcal{L}$ on the quotient $\mathbf{X}_\Gamma$, following the same steps as on the universal cover. In Section 6.1, we define Hilbert spaces $\mathcal{H}_W = \mathcal{H}_W(\mathbf{X}_\Gamma)$ and $\mathcal{H}_{PS} = \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ of Γ-invariant symbols (in other words, symbols on the quotient $\mathbf{X}_\Gamma$) and their dual spaces, $\mathcal{H}_W^*$ and $\mathcal{H}_{PS}^*$. The Wigner distributions $W^\Gamma_{j,k}$ (1.4) form an orthonormal basis of $\mathcal{H}_W^*$, while the Patterson-Sullivan distributions (1.12) form an orthonormal basis of $\mathcal{H}_{PS}^*$. Both are modelled on the Hilbert space of Hilbert-Schmidt pseudo-differential operators. As mentioned above, the definitions are flexible and allow for choices of weights in the r variable; by choosing the weight appropriately, it suffices that $(I - \triangle)^{-s} A$ is a Hilbert-Schmidt operator for some s. The geodesic flow G^t_Γ induces an isometry of $\mathcal{H}_{PS}$ and V^t_Γ an isometry of $\mathcal{H}_W$. The intertwining operator $\mathcal{L}$ induces an operator

$$\mathcal{L}_{\Gamma\sharp} : \mathcal{H}^*_{PS}(\mathbf{X}_\Gamma) \to \mathcal{H}^*_W(\mathbf{X}_\Gamma), \quad \mathcal{L}_{\Gamma\sharp} PS_{\nu_j, -\nu_k} = W_{j,k}$$

on Γ-invariant distributions. We then define

$$\mathcal{L}_\Gamma : \mathcal{H}_W(\mathbf{X}_\Gamma) \to \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$$

by duality (6.10).

THEOREM **3.** *The operator $\mathcal{L}_{\Gamma\sharp} : \mathcal{H}^*_{PS} \to \mathcal{H}^*_W$ is an isometric isomorphism, and $\mathcal{L}_{\Gamma\sharp}$ sends $PS_{\nu_j,-\nu_k}$ to $W_{j,k}$. Dually, we have*

$$\mathcal{L}_\Gamma \circ V^t_\Gamma = G^t_\Gamma \circ \mathcal{L}_\Gamma,$$

as an equality between operators from $\mathcal{H}_W$ to $\mathcal{H}_{PS}$.

The Hilbert spaces $\mathcal{H}_{PS}, \mathcal{H}_W$ are defined *ad hoc* so that the theorem holds. Nevertheless, in Section 7 we show that they contain large classes of symbols, characterized by their smoothness and growth in the r-variable (Propositions 7.3 and 7.5).

Remark 1.3. It might seem more natural to first define $\mathcal{L}_\Gamma$ on symbols, but in fact this is a subtle issue that is clarified by first defining the adjoint on the distributions. One might try at first to define $\mathcal{L}_\Gamma$ directly on Γ-invariant symbols. But one finds that this is problematic due to the fact that the kernel of $\mathcal{L}$ is not smooth and does not decay fast enough away from the diagonal. By comparison, the adjoint intertwining operator is naturally defined on $PS_{\nu_j,-\nu_k}$. We then define $\mathcal{L}_\Gamma$ by duality and find that the dual intertwining operator only uses the cutoff of a Γ-invariant symbol to a fundamental domain and does not require a definition of $\mathcal{L}$ on automorphic symbols.

The intertwining relation comes from an exact relation between the Wigner and Patterson-Sullivan families of distributions. If a is a function on G, decaying fast enough, and $\nu \in \mathbf{C}$, define

$$L_\nu a(g) = \int_{\mathbf{R}} (1+u^2)^{-(\frac{1}{2}+\nu)} a(gn_u)du.$$

If a is a function on $G \times \mathbf{C}$, and $\nu \in \mathbf{C}$, define the function a_ν on G by $a_\nu(g) = a(g, \nu)$.

THEOREM **4.** *Let $a = a(z, b, r)$ be a Γ-invariant function, with*

$$a(z, b, r) = \sum_{\gamma \in \Gamma} \tilde{a}(\gamma \cdot z, \gamma \cdot b, r),$$

with $\tilde{a}$ satisfying adequate decay and smoothness properties (i.e. that $\tilde{a} \in \mathcal{S}_0^0$; see §6.4 for the definition). Then we have

$$W^\Gamma_{j,k}(a) = 2^{1+\nu_j - \overline{\nu_k}} PS_{\nu_j, -\nu_k}(L_{-\overline{\nu_k}} \tilde{a}_{\nu_j}) = PS_{\nu_j, -\nu_k}(\mathcal{L}\tilde{a}).$$

1.5. **Asymptotic equivalence of Wigner and Patterson-Sullivan distributions.** In [AZ], it is proved that, after suitable normalization, the diagonal Wigner distributions and Patterson-Sullivan distributions are asymptotically the same in the semi-classical limit. The same is true for the off-diagonal elements:

THEOREM **5.** *Let $a \in C^\infty(\Gamma \backslash G)$. Given a sequence of pairs (ν_{j_n}, ν_{k_n}) of spectral parameters with $-i\nu_{j_n} \to +\infty$ and $|\nu_{j_n} - \nu_{k_n}| \leq \tau_0$ for some $\tau_0 \geq 0$, we have the asymptotic formula*

$$\int_{S\mathbf{X}_\Gamma} a(g) W^\Gamma_{j_n,k_n}(dg) = 2^{1+\nu_{j_n} - \overline{\nu_{k_n}}} \left(\frac{\pi}{r_{k_n}}\right)^{1/2} e^{-\frac{i\pi}{4}} \int_{S\mathbf{X}_\Gamma} a(g) PS^\Gamma_{\nu_{j_n}, -\nu_{k_n}}(dg) + O(\nu_{k_n}^{-1}).$$

The proof is very similar to that in the diagonal case in [AZ], starting from Theorem 4. Hence we only sketch the key points in §8. This result has been extended to more general symmetric spaces by Hilgert and Schröder [HilSc].

In [Z2], it is shown that the off-diagonal Wigner distributions W_{j_n,k_n} with $j_n \neq k_n$ and with a limiting spectral gap $r_{j_n} - r_{k_n} \longrightarrow \tau_0$ tend to zero when the geodesic flow is mixing, at least after the removal of a subsequence of spectral density zero. It then follows from Theorem 5 that:

COROLLARY **6.** *Take a sequence of pairs (j_n, k_n), with $j_n \neq k_n$ and $r_{j_n} - r_{k_n} \longrightarrow \tau_0$. Assume that this sequence has positive density, in the sense that*

$$\liminf_{\lambda \to +\infty} \frac{\sharp\{n, |r_{j_n}| \leq \lambda\}}{\sharp\{j, |r_j| \leq \lambda\}} > 0.$$

Then there exists a subsequence of full density such that $r_{k_n}^{-1/2} PS_{\nu_{j_n}, -\nu_{k_n}} \longrightarrow 0$.

1.6. **Relations to other work.** The existence of the intertwining operator is rather unexpected from the viewpoint of microlocal analysis and quantum chaos, but is quite natural from the viewpoint of automorphic distributions and invariant triple products [BR, BR2, MS, D, SV, SV2], where it may be interpreted as intertwining the family of Wigner triple products $\ell^W(a, \phi_j, \phi_k) = \langle \mathrm{Op}(a)\phi_j, \phi_k\rangle$ and the family of Patterson-Sullivan triple products $\ell^{PS}(a, \phi_j, \phi_k) = \langle a, dPS_{\nu_j, -\nu_k}\rangle$. It follows from general principles that there exist constants C_{r_j, r_k} $\ell^{PS}(a, \phi_j, \phi_k) = C_{r_j, r_k} \ell^W(a, \phi_j, \phi_k)$ and essentially $\mathcal{L}$ is an integral operator with matrix elements C_{r_j, r_k}. This will be explained in detail in [Z4] and explicit formulae relating

Wigner and Patterson-Sullivan distributions on generating symbols are given there and in [AZ]. $\mathcal{L}$ might have an independent interest in representation theory as the intertwining operator between these two families of triple products.

The relations between Wigner and Patterson-Sullivan distributions, and the exact formulae relating them in [AZ, Z4], shed some light on the limit formula for quantum variances of Wigner distributions of Hecke eigenfunctions, proved by Luo-Sarnak [LS], Zhao [Zh] and Sarnak-Zhao [SZ]. The quantum variance for a zeroth order pseudo-differential operator A is defined as

$$V_A(\lambda) \stackrel{\text{def}}{=} \frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} |\langle A\phi_j, \phi_j\rangle - \int a_0 d\omega|^2, \qquad (N(\lambda) = \#\{j : \lambda_j \leq \lambda\}) \tag{1.16}$$

where $\int a_0 d\omega$ is the Liouville average of the principal symbol a_0 of A. It was suggested by Feingold and Peres [FP] that the quantum variance should tend to 0 the following way :

$$V_A(\lambda) \sim \frac{B(a_0, a_0)}{\sqrt{\lambda}}.$$

The bilinear form B should be proportional to $\hat{\rho}_{a_0,a_0}(0)$, where

$$\rho_{\phi,\psi}(t) = \int_{S\mathbf{X}_\Gamma} \phi(x)\psi(g^t x) d\omega(x) - \int \phi d\omega \int \psi d\omega \tag{1.17}$$

is the "dynamical correlation function", and $\hat{\rho}_{\phi,\psi}(\tau) = \int_{-\infty}^{\infty} e^{-i\tau t}\rho_{\phi,\psi}(t)dt$ is its Fourier transform. In [LS, Zh, SZ] a version of this conjecture (with additional arithmetic factors) was proved for the basis of Hecke eigenfunctions.

Since Patterson-Sullivan distributions are defined independently on the choice of a quantization procedure, it is natural to consider the classical variances for the diagonal PS-distributions:

$$PS_a(\lambda) \stackrel{\text{def}}{=} \frac{1}{N(\lambda)} \sum_{j:\lambda_j \leq \lambda} |\langle a, \widehat{PS}_{\nu_j,-\nu_j}\rangle - \int a d\omega|^2. \tag{1.18}$$

Here, $\widehat{PS}_{\nu_j,-\nu_j} \stackrel{\text{def}}{=} \frac{1}{\langle \mathbf{1}, PS_{\nu_j,-\nu_j}\rangle_{S\mathbf{X}_\Gamma}} PS_{\nu_j,-\nu_j}$ are normalized PS-distributions (see [AZ]) so that the statement is correct for constant functions.

By Theorem 5 (or the explicit relations in [AZ, Z4]), we have

$$V_{Op(a)}(\lambda) \simeq PS_a(\lambda),$$

hence the Luo-Sarnak-Zhao results also imply that

$$PS_a(\lambda) \sim \frac{\text{Const. } \hat{\rho}_{a_0,a_0}(0)}{\sqrt{\lambda}}, \tag{1.19}$$

for a certain constant Const. Since Patterson-Sullivan variances are sums of g^t-invariant bilinear forms, they should have a closer relation to dynamical correlation functions than variances for Wigner distributions. Hence it seems reasonable to ask whether (1.19) holds for any basis of eigenfunctions on any compact hyperbolic surface.

Finally, we point out a possibly tenuous relation of our intertwining problem to the one studied by Bismut on locally symmetric spaces of non-compact type in Chapter 10 of [Bis]. On the infinitesimal level, we are intertwining the generator of the geodesic flow to the operator P taking a symbol a to the symbol of $[\mathrm{Op}(a), \triangle]$. As discussed in [Z3] the latter operator P has the form $\frac{H^2}{4} + X_+^2 + irH$, where $\frac{H^2}{4} + X_+^2$ is elliptic along the stable foliation and irH is the semi-classical operator of order 2 where H generates the geodesic flow. By comparison, Bismut's hypoelliptic laplacian L_b^X is essentially the weighted sum of the harmonic oscillator on the fiber of $T^*\mathbf{X}_\Gamma$ and bH. We note that the H terms are identical if we set $b = r$, while the other terms are in a sense orthogonal (Bismut's is vertical while ours is horizontal). But both have the essential property that as the semi-classical parameter $b = r \to \infty$, the operators converge to the generator of the geodesic flow. There is a possible parallel of our conjugation problem to the conjugation between the hypoelliptic laplacian and a certain elliptic operator in [Bis] (Chapter 10). We encounter similar problems in defining the domain of $\mathcal{L}$ and its inverse.

Notational issue : In what follows we have to face the issue that we are sometimes using the *sesquilinear* pairing between two L^2 functions (or more generally, two elements of a complex Hilbert space), and sometimes the *bilinear* pairing between a distribution and a test function (more generally, an element of a vector space and a linear form). We will try to keep distinct notations to avoid confusion, denoting by $\langle a, b\rangle$ the scalar product of two L^2 functions (linear w.r.t. a, antilinear w.r.t. b), and by $T(a)$ the pairing between a distribution T and a test function a. More generally, we shall restrict the bracket notation $\langle ., .\rangle$ to *sesquilinear* pairings [3].

If L is a linear operator on a space endowed with a sesquilinear form, we shall denote by $L^\dagger$ its adjoint in the hermitian sense, that is, $\langle a, Lb\rangle = \langle L^\dagger a, b\rangle$.

We will denote $L_\sharp$ the adjoint of L in the usual sense of duality : if T is a linear form, then $L_\sharp T$ is the linear form defined by $(L_\sharp T)(a) \stackrel{\mathrm{def}}{=} T(La)$. Thus, $L \mapsto L_\sharp$ is linear whereas $L \mapsto L^\dagger$ is antilinear.

If T is a distribution and Φ a diffeomorphism, we shall also denote by $\Phi_\sharp T$ the pushforward of T by Φ : $(\Phi_\sharp T)(a) \stackrel{\mathrm{def}}{=} T(a \circ \Phi)$. The two notations should not interfere.

2. Coordinates on $S\mathbf{D}$.

2.1. **Dynamics and group theory of $G = PSL(2, \mathbf{R})$.** We recall that $PSL(2, \mathbf{R})$ acts on the upper half plane $\mathbf{H} = \{z \in \mathbf{C}, \Im m(z) > 0\}$ by $g(z) = \frac{az+b}{cz+d}$, if $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We recall the expression of the hyperbolic distance in the upper half plane model :

$$\cosh d(z, w) = 1 + \frac{|z - w|^2}{2\Im m(z)\Im m(w)}. \tag{2.1}$$

The action of $PSL(2, \mathbf{R})$ is by isometries, and $PSL(2, \mathbf{R})$ can be identified with the unit tangent bundle $S\mathbf{H}$ via the map $g \mapsto (g(i), ig'(i))$ $(i = \sqrt{-1})$.

The upper half plane is isometric to the hyperbolic disc via the map

$$z \mapsto \frac{z - i}{z + i}.$$

[3] We also use the $\langle ., .\rangle$ notation for the Busemann function, but that should not cause confusion

To define the Patterson-Sullivan and Wigner distributions it will be more convenient to work with the disc model. However, some computations are easier with the upper half plane model.

A set of generators of the Lie algebra $sl(2,\mathbf{R})$ is given by

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.2}$$

The subgroups they generate are denoted by A, N, K respectively. We also put $X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and denote the associated subgroup by $\overline{N}$. In the identification $S\mathbf{D} \equiv PSL(2,\mathbf{R})$ the geodesic flow $(g^t)_{t\in\mathbf{R}}$ is given by the right action of the group A of diagonal matrices with positive entries : $g \mapsto ga_t$ where $a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$. The action of the horocycle flow $(h^u)_{u\in\mathbf{R}}$ is defined by the right action of N, in other words by $g \mapsto gn_u$ where $n_u = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$. We shall also denote $\bar{n}_u = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$.

2.2. **Adapted coordinates.** As explained in the introduction, the identification $G \sim (G/K) \times (G/NA) \sim (G/K) \times K$ leads to the coordinates (z, b) (where $z \in \mathbf{D}$, $b \in G/NA \sim K \sim S^1$) to parameterize points in $G \sim S\mathbf{D}$. The identification $G \sim (G/K) \times (G/NA)$ is G-equivariant, and thus the action of g on itself by left-multiplication reads $g\cdot(z, b) = (g\cdot z, g\cdot b)$, where on the first component G acts by isometry on the symmetric space G/K, and on the second coordinate G acts on the boundary $G/NA \sim K$.

We denote by o the origin eK in G/K, and by $\langle z, b\rangle$ the signed distance to o of the horocycle through the points $z \in \mathbf{D}, b \in B$. This notation follows [He], but we warn again that our normalization of the metric differs by a factor 2 from Helgason's.

We also use the identification $G \equiv S\mathbf{D} \equiv B^{(2)} \times \mathbf{R}$. It is based on the identification of $B^{(2)} == \{(b', b) \in B \times B, b \neq b'\}$ with the space of oriented geodesics of $\mathbf{D}$. To $(b', b) \in B^{(2)}$ there corresponds a geodesic $\gamma_{b',b}$ whose forward endpoint at infinity equals b and whose backward endpoint equals b'. The choice of time parameter is defined so that $(b', b, 0)$ is the closest point $z_{b',b}$ to the origin o on $\gamma_{b',b}$, and (b', b, t) denotes the point t units from $(b', b, 0)$ in signed distance towards b. We note that $\langle z_{b,b'}, b\rangle = \langle z_{b,b'}, b'\rangle$. We define $g(b', b) \in PSU(1, 1)$ to be the unique element satisfying

$$\begin{aligned} &g(b', b) \cdot 1 = b, \\ &g(b', b) \cdot (-1) = b', \\ &g(b', b) \cdot o = z_{b',b}, \end{aligned}$$

where $1, -1, b, b'$ are points of the boundary $B = S^1$, seen as the unit circle in $\mathbf{C}$ in the disc model. We thus identify $B^{(2)} \times \mathbf{R} \simeq G$ by $(b', b, t) \mapsto g(b', b)a_t$. In these coordinates, the action of $g \in G$ by left-multiplication is expressed by

$$g \cdot (b', b, t) = \left(g \cdot b', g \cdot b, t + \frac{\langle g \cdot o, g \cdot b\rangle - \langle g \cdot o, g \cdot b'\rangle}{2} \right). \tag{2.3}$$

We will need the following formula :

Lemma 2.1.

$$\log \frac{|b-b'|}{2} + \langle g(b',b)a_\tau n_u \cdot o, b\rangle = \tau,$$

where b, b' are seen as elements of $S^1 \subset \mathbf{C}$, and $|b-b'|$ is their usual distance in $\mathbf{C}$.

Proof. To prove this, we use the identity

$$\langle g a_t n_u \cdot o, g\cdot 1\rangle = \langle a_t n_u \cdot o, 1\rangle + \langle g\cdot o, g\cdot 1\rangle = t + \langle g\cdot o, g\cdot 1\rangle$$

to reduce the lemma to the claim that

$$\langle g(b',b)\cdot o, g(b',b)\cdot 1\rangle = -\log\frac{|b-b'|}{2}. \tag{2.4}$$

However, a basic identity gives

$$|g\cdot\beta - g\cdot\beta'|^2 e^{\langle g\cdot o, g\cdot\beta\rangle + \langle g\cdot o, g\cdot\beta'\rangle} = |\beta-\beta'|^2. \tag{2.5}$$

If we let $\beta = 1, \beta' = -1$ (so that $g(b',b)\cdot 1 = b, g(b',b)\cdot(-1) = b'$) and recall that $\langle g(b',b)\cdot o, b\rangle = \langle g(b',b)\cdot o, b'\rangle$, then (2.5) implies

$$4 = |b-b'|^2 e^{[\langle z_{b',b}, b\rangle + \langle z_{b,b'}, b'\rangle]} = |b-b'|^2 e^{2\langle z_{b',b}, b\rangle}, \tag{2.6}$$

which completes the proof of (2.4) and hence of the lemma. □

2.3. **Time reversal.** Time reversal is the map $\iota : (x,\xi) \to (x,-\xi)$ on the tangent bundle. In the coordinates (b',b,t) it takes the form,

$$\iota(b',b,t) = (b,b',-t). \tag{2.7}$$

That is, it reverses the endpoints of the oriented geodesic $\gamma_{b,b'}$ and preserves the point $z_{b,b'}$ closest to o. In the group theoretic picture, time reversal is given by the action of the non-trivial element of the Weyl group, $g \mapsto gw$ on G, where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

2.4. **A coordinate change.** The formulae below are useful at several places in the paper.

$$n_u = k_u a_{-\log(1+u^2)} \bar{n}_{f(u)}, \tag{2.8}$$

where $f(u) = \frac{u}{1+u^2}$ and where

$$k_u = \begin{pmatrix} \frac{1}{\sqrt{1+u^2}} & \frac{u}{\sqrt{1+u^2}} \\ -\frac{u}{\sqrt{1+u^2}} & \frac{1}{\sqrt{1+u^2}} \end{pmatrix}.$$

This comes from the explicit calculation,

$$\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{1+u^2}} & \frac{u}{\sqrt{1+u^2}} \\ -\frac{u}{\sqrt{1+u^2}} & \frac{1}{\sqrt{1+u^2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1+u^2}} & 0 \\ 0 & \sqrt{1+u^2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{u}{1+u^2} & 1 \end{pmatrix}. \tag{2.9}$$

This formula implies that the element $g = n_u a_t \in G$ corresponds to the endpoints $b = 1$, $b' = k_u w \in K$, in other words, using the S^1 model,

$$b' = e^{2i\theta}, \qquad \text{where } e^{i\theta} = \frac{u}{\sqrt{1+u^2}} + \frac{i}{\sqrt{1+u^2}}.$$

By calculation, we find $|b' - b|^2 = \frac{4}{1+u^2}$ and $db' = \frac{1}{\pi}\frac{du}{1+u^2}$. These calculations also show that

$$\langle n_u a_t, 1\rangle = t$$

and

$$\langle n_u a_t, b'\rangle = -t + \log(1+u^2).$$

For $t = \frac{\log(1+u^2)}{2}$, we see that $\langle n_u a_t, 1\rangle = \langle n_u a_t, b'\rangle$, and thus $n_u a_{\frac{\log(1+u^2)}{2}} = g_{b',b}$ (with $b = 1, b' = e^{2i\theta}$ as above). It follows that $g = n_u a_t \in G$ has the coordinates $(b', b, \tau) = (e^{2i\theta}, 1, t - \frac{\log(1+u^2)}{2})$.

3. Harmonic analysis on the hyperbolic disc and its compact quotients

3.1. **Poisson 1-form, Haar measure and Plancherel measure.** We shall denote by db the normalized Haar measure on K, identified with the boundary B or with S^1. The Poisson 1-form is defined by

$$P(z,b)db = e^{\langle z,b\rangle}db. \tag{3.1}$$

Using the identities

$$\langle g\cdot z, g\cdot b\rangle = \langle z,b\rangle + \langle g\cdot o, g\cdot b\rangle, \tag{3.2}$$

and

$$\frac{d}{db} g\cdot b = e^{-\langle g\cdot o, g\cdot b\rangle}, \tag{3.3}$$

it follows that

$$P(g\cdot z, g\cdot b)d(g\cdot b) = P(z,b)db. \tag{3.4}$$

Haar measure on G is denoted dg. In terms of z, b coordinates it is given by

$$dg = P(z,b)\,\mathrm{Vol}(dz)db, \tag{3.5}$$

where $\mathrm{Vol}(dz)$ is the hyperbolic area form. Under the identification $G \sim S\mathbf{D}$, the Haar measure on G is the same as Liouville measure on $S\mathbf{D}$. In the (b', b, t) coordinates, Haar measure reads as follows:

Lemma 3.1. *Under the identifications* $(b, b', t) \simeq g = g(b, b')a_t \simeq (z, b)$, *we have*

$$dg = P(z,b)\,\mathrm{Vol}(dz)db = 4\pi\frac{db \otimes db'}{|b-b'|^2} \otimes dt.$$

The fact that the measure $\frac{db\otimes db'}{|b-b'|^2} \otimes dt$ is invariant under the action of G follows from formulae (2.3), (2.5), (3.3). We leave aside the calculation of the normalization factor, which can be done thanks to the formulae in §2.4, but does not play a very important role.

Non-euclidean Fourier analysis is based on the family of non-euclidean "plane waves"

$$e_{\nu,b}(z) \stackrel{\text{def}}{=} e^{(\frac{1}{2}+\nu)\langle z,b\rangle},$$

$\nu \in \mathbf{C}, b \in B$. They are complex-valued eigenfunctions of the laplacian :

$$\triangle e_{\nu,b} = -\left(\frac{1}{4} - \nu^2\right) e_{\nu,b}.$$

The L^2 spectral decomposition of the laplacian on $\mathbf{D}$ only requires the tempered spectrum, that is, the case $\nu = ir$ where $r \in \mathbf{R}$ (corresponding to a laplacian eigenvalue $\frac{1}{4} - \nu^2 \geq \frac{1}{4}$). The Helgason-Fourier transform of a function f on $\mathbf{D}$ is defined by

$$\mathcal{F}f(b,r) = \int_{\mathbf{D}} e^{(\frac{1}{2}-ir)\langle z,b\rangle} f(z)\,\mathrm{Vol}(dz),$$

$b \in B, r \in \mathbf{R}$. The Fourier transform automatically has the following symmetry :

$$\int_B \mathcal{F}f(b,r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}db = \int_B \mathcal{F}f(b,-r)e^{(\frac{1}{2}-ir)\langle z,b\rangle}db, \tag{3.6}$$

for all $z \in \mathbf{D}$ and $r \in \mathbf{R}$. Plancherel measure is the measure on $\mathbf{R}$ defined by

$$dp(r) = \frac{1}{2\pi} r \tanh(\pi r) dr, \tag{3.7}$$

and the Fourier inversion formula reads

$$f(z) = \frac{1}{2}\int_{\mathbf{R}}\int_B \mathcal{F}f(b,r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}dp(r)db = \int_{\mathbf{R}_+}\int_B \mathcal{F}f(b,r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}dp(r)db,$$

see [He]. We have the Plancherel formula for $f \in L^2(\mathbf{D})$, $\|f\|_{L^2(\mathbf{D},\mathrm{Vol})} = \|\mathcal{F}f\|_{L^2(B\times\mathbf{R}_+, db\times dp(r))}$.

3.2. **Integral representation of eigenfunctions.** We now consider Fourier analysis on the quotient $\mathbf{X}_\Gamma$ of $\mathbf{D}$ by a discrete co-compact subgroup $\Gamma \subset G$.

Theorem 3.2. *(*[He]*, Theorems 4.3 and 4.29; see also* [He2]*) Let ϕ be an eigenfunction with exponential growth, for the eigenvalue $\lambda = -\left(\frac{1}{4} - \nu^2\right) \in \mathbf{C}$. Then there exists a distribution $T_{\nu,\phi} \in \mathcal{D}'(B)$ such that*

$$\phi(z) = \int_B e^{(\frac{1}{2}+\nu)\langle z,b\rangle} T_{\nu,\phi}(db),$$

for all $z \in \mathbf{D}$. The distribution is unique if $\frac{1}{2} + \nu \neq 0, -1, -2, \cdots$.

The distribution $T_{\nu,\phi}$ is usually called the *boundary values* of ϕ (for the spectral parameter ν), in analogy with the theory of boundary values of harmonic functions. This theorem applies, in particular, to a Γ-invariant eigenfunction of the laplacian, since such a function is bounded. By uniqueness of $T_{\nu,\phi}$, we see that ϕ being Γ-invariant is equivalent to

$$\gamma_\sharp^{-1} T_{\nu,\phi}(db) = e^{-(\frac{1}{2}+\nu)\langle \gamma\cdot o, \gamma\cdot b\rangle} T_{\nu,\phi}(db) \tag{3.8}$$

for $\gamma \in \Gamma$ and $b \in B$.

3.3. **Fundamental domains and cutoffs for $\Gamma\backslash G$.** We denote by $\mathcal{D}$ a fundamental domain for the action of Γ on $\mathbf{D} = G/K$. We use the same notation for the fundamental domain $\mathcal{D}$ lifted to G.

When dealing with integrals against irregular distributions, it is convenient to replace the characteristic function of a fundamental domain by a smooth (compactly supported) cutoff χ on G satisfying

$$\Pi\chi = 1,$$

where we define the periodization operator Π by

$$\Pi\chi(g) = \sum_{\gamma\in\Gamma} \chi(\gamma g). \tag{3.9}$$

Existence of such functions χ is obvious. We will call such a function χ a *smooth fundamental cutoff* for the action of Γ on G. When needed, we may assume that χ is a right-K-invariant function, that is, $\chi(z, b) = \chi(z)$.

Let χ, χ' be two smooth fundamental cutoffs. We will use repeatedly the following : if T is a Γ-invariant distribution on G, then $T(\chi f) = T(\chi' f)$, for any $f \in C^\infty(\Gamma\backslash G)$ (seen as a Γ-invariant function on G). To see this, write

$$T(\chi f) = T(\chi f.(\Pi\chi')) = T(\Pi(\chi f).\chi') = T(f\chi'). \tag{3.10}$$

4. Pseudo-differential calculus on the Poincaré disc

Throughout this article, we use a special hyperbolic calculus of pseudodifferential operators introduced in [Z3]. In the hyperbolic calculus, a complete symbol $a(z, b, r)$ ($(z, b) \in \mathbf{D}\times B, r \in \mathbf{R}$) is quantized by the operator $\mathrm{Op}(a)$ on $\mathbf{D}$ defined by

$$\mathrm{Op}(a)e_{ir,b}(z) = a(z, b, r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}$$

for $z \in \mathbf{D}, b \in B$ and $r \in \mathbf{R}_+$. By the non-euclidean Fourier inversion formula, we define $\mathrm{Op}(a)$ on $C_c^\infty(\mathbf{D})$:

$$\mathrm{Op}(a)u(z) = \int_B \int_{\mathbf{R}_+} a(z, b, r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}\mathcal{F}u(b, r)dp(r)db. \tag{4.1}$$

We recall that the measure $dp(r) = \frac{1}{2\pi} r \tanh(\pi r)dr$ is the Plancherel measure for G (3.7). At the formal level, the kernel of $\mathrm{Op}(a)$ is thus given by

$$K_a(z, w) = \int_B \int_{\mathbf{R}_+} a(z, b, r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}e^{(\frac{1}{2}-ir)\langle w,b\rangle}dp(r)db. \tag{4.2}$$

Both integrals are well-defined, and (4.2) holds, as long as a is smooth and compactly supported w.r.t. (b, r) (with a support possibly depending on z). See §4.2 for more general conditions on a.

Now assume that a has the following symmetry w.r.t. the transformation $(z, b, r) \mapsto (z, b, -r)$:

$$\int a(z, b, r)e^{(1/2+ir)\langle z,b\rangle}e^{(1/2-ir)\langle w,b\rangle}db = \int a(z, b, -r)e^{(1/2-ir)\langle z,b\rangle}e^{(1/2+ir)\langle w,b\rangle}db \tag{4.3}$$

for all $z, w \in \mathbf{D}$ and $r \in \mathbf{R}$. It then follows from the Plancherel formula for the hyperbolic Fourier transform that we can recover the symbol from the kernel by

$$a(z,b,r) = e^{-(\frac{1}{2}+ir)\langle z,b\rangle} \int_{\mathbf{D}} K_a(z,w) e^{(\frac{1}{2}+ir)\langle w,b\rangle} \operatorname{Vol}(dw) \tag{4.4}$$

for all $r \in \mathbf{R}$. In this case, formula (4.2) holds with $\int_{\mathbf{R}_+}$ replaced by $\int_{\mathbf{R}_-}$. We now discuss several particular classes of symbols a.

4.1. **Hilbert-Schmidt operators on D and L^2 symbols.**

We recall that if $\mathcal{H}$ is a Hilbert space, the algebra of Hilbert-Schmidt operators on $\mathcal{H}$ is the algebra of operators A for which the trace $\operatorname{Tr} AA^\dagger$ is finite; it is endowed with the inner product $\langle A, B\rangle_{\mathrm{HS}} \stackrel{\mathrm{def}}{=} \operatorname{Tr} AB^\dagger$. It is well known that the Hilbert-Schmidt operators on $\mathcal{H} = L^2(M, d\nu)$ for any measure space form a Hilbert space isomorphic to $L^2(M \times M, d\nu \times d\nu)$. In the case $M = \mathbf{D}$ and $\nu = \operatorname{Vol}$, we will denote $\mathrm{HS}(\mathbf{D})$ the space of Hilbert-Schmidt operators on $L^2(\mathbf{D}, \operatorname{Vol}(dz))$. We denote $L^2_W(G\times\mathbf{R}, dg\times dp(r))$ the space of functions in $L^2(G\times\mathbf{R}, dg\times dp(r))$ that have the symmetry (4.3) with respect to the Weyl group. We endow it with the norm

$$\begin{aligned} \|a\|^2_{L^2_W} = \frac{1}{2}\int_{\mathbf{D}}\int_B\int_{\mathbf{R}} |a(z,b,r)|^2 P(z,b) \operatorname{Vol}(dz) db dp(r) \\ = \int_{\mathbf{D}}\int_B\int_{\mathbf{R}_+} |a(z,b,r)|^2 P(z,b) \operatorname{Vol}(dz) db dp(r). \end{aligned}$$

The following is a consequence of the Plancherel formula :

Proposition 4.1. *The quantization map $a \mapsto \operatorname{Op}(a)$ defines a unitary equivalence*

$$L^2_W(G \times \mathbf{R}, dg \times dp) \simeq \mathrm{HS}(\mathbf{D}).$$

In other words

$$||\operatorname{Op}(a)||^2_{\mathrm{HS}(\mathbf{D})} = \frac{1}{2}\int_{\mathbf{D}}\int_B\int_{\mathbf{R}} |a(z,b,r)|^2 P(z,b) \operatorname{Vol}(dz) db dp(r).$$

Proof. It suffices to prove the identity for a which is smooth compactly supported w.r.t. (z,b,r). By (4.2),

$$\begin{aligned} ||\operatorname{Op}(a)||^2_{\mathrm{HS}(\mathbf{D})} &= \textstyle\int_{\mathbf{D}\times\mathbf{D}} |K_a(z,w)|^2 \operatorname{Vol}(dz) \operatorname{Vol}(dw) \\ &= \textstyle\int_{\mathbf{D}\times\mathbf{D}} \int_{B\times\mathbf{R}^+} \int_{B\times\mathbf{R}^+} a(z,b,r)\overline{a(z,b',r')} dp(r) db dp(r') db' \operatorname{Vol}(dz) \\ &\quad \left(\textstyle\int_{\mathbf{D}} e^{(\frac{1}{2}+ir)\langle z,b\rangle} e^{(\frac{1}{2}-ir)\langle w,b\rangle} e^{(\frac{1}{2}-ir')\langle z,b'\rangle} e^{(\frac{1}{2}+ir')\langle w,b'\rangle} \operatorname{Vol}(dw)\right) \\ &= \textstyle\int_{\mathbf{D}} \int_{B\times\mathbf{R}^+} |a(z,b,r)|^2 P(z,b) dp(r) db \operatorname{Vol}(dz) \end{aligned}$$

by the Fourier inversion formula. □

4.2. **Schwartz class and associated symbols.** Schwartz functions on G were first defined by Harish-Chandra [HC66]; the definition was extended to G/K by Eguchi and his collaborators [Eg74, Eg79]. Writing the hyperbolic disc as G/K, f belongs to the Schwartz space $\mathcal{C}^p(G/K)$ (for $0 < p \leq 2$) if and only if f is a smooth function on G which is right-K-invariant, and

$$\sup_{g\in G} \varphi_o(gK)^{-2/p}(1 + d(gK, o))^q |LRf(g)| < +\infty,$$

for any $q > 0$, and for any differential operators L, R on G which are respectively left- and right-invariant. Here φ_o stands for the spherical function on G/K, $\varphi_o(z) = \int e^{\frac{1}{2}\langle z,b\rangle} db$. It satisfies $\varphi_o(z) \asymp d(z, o)e^{-d(z,o)/2}$ as the hyperbolic distance $d(z, o) \longrightarrow +\infty$. Functions on $\mathcal{C}^p(G/K)$ are, in particular, in L^p (they are sometimes called Schwartz functions of L^p-type).

The Fourier transforms of Schwartz functions of L^p-type were characterized by Eguchi [Eg74, Eg79] : letting $\epsilon = \epsilon(p) = \frac{2}{p} - 1$, $\mathcal{F}(\mathcal{C}^p(G/K))$ coincides with the space $\mathcal{C}(B \times \mathbf{R}^\epsilon)_W$ of functions u on $B \times \mathbf{R}$ such that

– u extends holomorphically to the strip $\mathbf{R}^\epsilon = \{|\Im m(r)| < \frac{\epsilon}{2}\}$ (this condition is empty for $\epsilon = 0$);

– on this strip (or, in the case $\epsilon = 0$, on the real axis), we have a bound

$$\sup_{(b,r)} (1 + |r|)^q \left| \frac{\partial^\alpha}{\partial r^\alpha} Du(b, r) \right| < +\infty, \tag{4.5}$$

for all $q > 0$, every integer α, and every K-left-invariant differential operator D acting on B (here we use the identification $B \sim K$);

– besides, u must satisfy the symmetry (3.6) (this symmetry condition with respect to the Weyl group is indicated by the subscript $_W$).

We now define the space $\mathcal{K}^{p,q}(G/K \times G/K)$ (resp. $\mathcal{K}_{p,q}(G/K \times G/K)$, $\mathcal{K}^p_q(G/K \times G/K)$, $\mathcal{K}^q_p(G/K \times G/K)$) of kernels of operators sending $\mathcal{C}^p(G/K)$ continuously to $\mathcal{C}^q(G/K)$ (resp. $(\mathcal{C}^p(G/K))'$ to $(\mathcal{C}^q(G/K))'$, $\mathcal{C}^p(G/K)$ to $(\mathcal{C}^q(G/K))'$, $(\mathcal{C}^p(G/K))'$ to $\mathcal{C}^q(G/K)$). We denote the corresponding symbol classes by $\mathcal{S}^{p,q}(G/K \times B \times \mathbf{R})_W$, $\mathcal{S}_{p,q}(G/K \times B \times \mathbf{R})_W$, $\mathcal{S}^p_q(G/K \times B \times \mathbf{R})_W$ and so on. On each of these spaces, we can use formula (4.2) to relate the kernel to the Fourier transform of the symbol.

We will in particular consider the space $\mathcal{K}^0_0(G/K \times G/K)$ of "smoothing" operators, sending $(\bigcap \mathcal{C}^p(G/K))'$ to $\bigcap \mathcal{C}^p(G/K)$. A kernel $\mathcal{K}(z, w)$ is in $\mathcal{K}^0_0(G/K \times G/K)$ if and only if it is a smooth function on $G \times G$, satisfying $\mathcal{K}(gk, g'k') = \mathcal{K}(g, g')$ for all $g, g' \in G, k, k' \in K$, and

$$\sup_{g,g'\in G} \varphi_o(gK)^{-2/p}(1 + d(gK, o))^q \varphi_o(g'K)^{-2/p}(1 + d(g'K, o))^q |LR\mathcal{K}(g, g')| < +\infty, \tag{4.6}$$

for any $p, q > 0$, and for any differential operators L, R on $G \times G$ which are respectively left- and right-invariant.

Corresponding symbols $a(z, b, r)$ are characterized by the fact that $a(z, b, r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}$ belongs to $\bigcap_\epsilon \bigcap_p \mathcal{C}(B \times \mathbf{R}^\epsilon; \mathcal{C}^p(G/K))_W$ (i.e. functions $a(z, b, r)$ with the $\mathcal{C}(B \times \mathbf{R}^\epsilon)$-regularity in the (b, r) variables, taking values in $\mathcal{C}^p(G/K)$). We will denote this space of "smoothing" symbols by

$$\mathcal{S}^0_0 := \mathcal{S}^0_0(G/K \times B \times \mathbf{R})_W := \bigcap_\epsilon \bigcap_p \mathcal{C}(B \times \mathbf{R}^\epsilon; \mathcal{C}^p(G/K))_W. \tag{4.7}$$

For $a \in \mathcal{S}_0^0$ and $\nu \in \mathbf{C}$, it follows from the definition that

$$\mathrm{Op}(a)e^{(\frac{1}{2}+\nu)\langle \bullet, b\rangle}(z) = a(z, b, -i\nu)e^{(\frac{1}{2}+\nu)\langle z,b\rangle} \tag{4.8}$$

for any $\nu \in \mathbf{C}$.

4.3. $\mathrm{Op}(a)$ **and** $\mathrm{Op}_\Gamma(a)$**.** A key point of the non-euclidean pseudo-differential algebra is that it is automatically left invariant [Z3]. We say that a symbol a is Γ-invariant if $a(\gamma\cdot z, \gamma\cdot b, r) = a(z, b, r)$. Denote t_g the action of $g \in G$ on functions on G/K, defined by $t_g f(z) = f(g^{-1}z)$. We recall from [Z3] that a being Γ-invariant is equivalent to having $[t_\gamma, \mathrm{Op}(a)] = 0$ for all $\gamma \in \Gamma$. This commutation relation is also equivalent to the fact that $K_a(\gamma\cdot z, \gamma\cdot w) = K_a(z, w)$.

By the decay properties of the spherical function φ_o, we see that $L^2(\mathbf{X}_\Gamma)$ can be continuously embedded in $(\mathcal{C}^p(G/K))'$ if $p \leq 1$. As a result, if a kernel $K(z, w)$ is Γ-invariant, and is such that $\chi(z)K(z, w) \in \mathcal{K}_p^p(G/K \times G/K)$, then K defines naturally a bounded operator on the quotient, $K_\Gamma : L^2(\mathbf{X}_\Gamma) \longrightarrow L^2(\mathbf{X}_\Gamma)$: for $\phi \in L^2(\mathbf{X}_\Gamma)$, one can define $K_\Gamma\phi$ by the identity

$$\langle K_\Gamma \phi, \psi\rangle_{\mathbf{X}_\Gamma} \stackrel{\text{def}}{=} \langle \chi K \phi, \psi\rangle_{\mathbf{D}}, \tag{4.9}$$

for all $\psi \in L^2(\mathbf{X}_\Gamma)$, and where χ is our smooth fundamental cut-off of §3.3. Besides, this definition does not depend on the choice of the fundamental cut-off χ; this can be checked the same way as in §3.3.

We can rephrase this in terms of symbols. Assume that $a(z, b, r)$ is a Γ-invariant symbol, and that $\chi(z)a(z, b, r)$ belongs to $\mathcal{S}_p^p$ for $p \leq 1$. In accordance with (4.9) we define the bounded operator $\mathrm{Op}_\Gamma(a) : L^2(\mathbf{X}_\Gamma) \longrightarrow L^2(\mathbf{X}_\Gamma)$ by

$$\begin{aligned} \langle \mathrm{Op}_\Gamma(a)\phi, \psi\rangle_{\mathbf{X}_\Gamma} &\stackrel{\text{def}}{=} \langle \chi\, \mathrm{Op}(a)\phi, \psi\rangle_{\mathbf{D}} \\ &= \langle \mathrm{Op}(\chi a)\phi, \psi\rangle_{\mathbf{D}}. \end{aligned}$$

We extend the definition of the periodization operator (3.9) to functions of (z, b, r):

$$\Pi\tilde{a}(z, b, r) = \sum_{\gamma\in\Gamma} \tilde{a}(\gamma\cdot z, \gamma\cdot b, r).$$

Using the fact that the growth of Γ is exponential, and that elements of $\mathcal{S}_0^0$ decay superexponentially fast in z, we see that the series defining $\Pi\tilde{a}$ converges absolutely if $\tilde{a}$ is in $\mathcal{S}_0^0$. The class of symbols in Theorem 3 is given in:

Definition 4.1. We denote by $\Pi\mathcal{S}_0^0$ the image of $\mathcal{S}_0^0$ under Π, where $\mathcal{S}_0^0$ is defined in (4.7).

If $a(z, b, r)$ is Γ-invariant and if $a(z, b, r) = \sum_{\gamma\in\Gamma} \tilde{a}(\gamma\cdot z, \gamma\cdot b, r)$ where $\tilde{a} \in \mathcal{S}_0^0$, we have

$$\langle \mathrm{Op}_\Gamma(a)\phi, \psi\rangle_{\mathbf{X}_\Gamma} = \langle \mathrm{Op}(\tilde{a})\phi, \psi\rangle_{\mathbf{D}},$$

for $\phi, \psi \in L^2(\mathbf{X}_\Gamma)$ – and in particular this does not depend on the choice of $\tilde{a}$.

For $a \in \Pi\mathcal{S}_0^0$, we can use Helgason's integral representation theorem 3.2, together with (4.8), to have an alternative formula for the action of a pseudodifferential operator on the laplacian eigenfunctions ϕ_j :

$$\mathrm{Op}_\Gamma(a)\phi_j(z) = \int_B a(z, b, -i\nu_j)e^{(\frac{1}{2}+\nu_j)\langle z,b\rangle} T_{\nu_j}(db). \tag{4.10}$$

There is a standard relation between the Schwartz kernel $K_a(z,w)$ of $\mathrm{Op}(a)$ on $\mathbf{D}$ and the Schwartz kernel $K_a^\Gamma(z,w)$ of $\mathrm{Op}_\Gamma(a)$ on $\mathbf{X}_\Gamma$. If $K_a(z,\cdot)$ decays fast enough – say, $a \in \Pi\mathcal{S}_0^0$ – then the series

$$\sum_{\gamma\in\Gamma} K_a(z,\gamma w), \tag{4.11}$$

converges absolutely. Besides, the sum (4.11) yields K_a^Γ, which can be checked by noting that

$$\int_{\mathbf{X}_\Gamma} K_a^\Gamma(z,w) f(w)\,\mathrm{Vol}(dw) = \int_{\mathbf{D}} K_a(z,w) f(w)\,\mathrm{Vol}(dw) \tag{4.12}$$

for any f Γ-automorphic and locally L^2.

4.4. V^t **and** V_Γ^t**.** The laplacian induces a unitary 1-parameter group $(e^{it\frac{\triangle}{2}})$ on the Hilbert space $L^2(\mathbf{D})$. The Schwartz spaces $\mathcal{C}^p(G/K)$ are stable under $e^{it\frac{\triangle}{2}}$.

We define $V^t a$ by $e^{-it\frac{\triangle}{2}}\,\mathrm{Op}(a) e^{it\frac{\triangle}{2}} = \mathrm{Op}(V^t a)$. The Hilbert-Schmidt norm of operators is preserved under conjugacy by a unitary operator, and Proposition 4.1 implies that V^t defines a unitary 1-parameter group on $L^2_W(G\times\mathbf{R}, dg\times dp(r))$. Each of the symbol spaces defined in §4.2 : $\mathcal{S}^{p,q}(G/K\times B\times\mathbf{R})_W$, $\mathcal{S}_{p,q}(G/K\times B\times\mathbf{R})_W$, $\mathcal{S}^p_q(G/K\times B\times\mathbf{R})_W$... are also preserved by V^t.

The explicit expression of V^t is

(4.13)

$$V^t a(z,b,r) = e^{-(\frac{1}{2}+ir)\langle z,b\rangle} \int e^{(\frac{1}{2}+ir)\langle w,b\rangle} e^{(\frac{1}{2}-ir')\langle w,b'\rangle} e^{(\frac{1}{2}+ir')\langle z,b'\rangle} a(w,b,r) e^{\frac{it}{2}(r'^2-r^2)}\,\mathrm{Vol}(dw)db'dp(r').$$

The generator D^V of V^t is a skew-adjoint diffential operator on $L^2(G\times\mathbf{R}^+, dg\times dp(r))$, which satisfies $\mathrm{Op}(D^V a) = \left[\mathrm{Op}(a), \frac{i\triangle}{2}\right]$. An explicit computation yields

$$D^V = -\frac{i}{2}\left(\frac{H^2}{4} + X_+^2\right) + r\frac{H}{2} \tag{4.14}$$

where H and X_+ are the elements of the Lie algebra defined in §2.1. The differential operator D^V on $G\times\mathbf{R}$ is left-invariant; it thus defines a differential operator D_Γ^V on $\Gamma\backslash G\times\mathbf{R}$, which is skew-adjoint on $L^2(\Gamma\backslash G\times\mathbf{R}, dg\times dp(r))$. The definition of V^t goes to the quotient $\mathbf{X}_\Gamma$, as follows.

Definition 4.2. We denote by V_Γ^t the unitary flow on $L^2(\Gamma\backslash G\times\mathbf{R}, dg\times dp(r))$ generated by D_Γ^V (4.14).

Lemma 4.2. *On $\Pi\mathcal{S}_0^0$, V_Γ^t acts as follows : if $a = \Pi\tilde{a}$ with $\tilde{a}\in\mathcal{S}_0^0$, we have*

$$V_\Gamma^t a = \Pi V^t \tilde{a}.$$

The main thing to check is that this expression does not depend on the choice of $\tilde a$, by the following computation :

(4.15)

$$\begin{aligned}(\Pi V^t\tilde a)(z,b,r)e^{(\frac12+ir)\langle z,b\rangle} &= \sum_{\gamma\in\Gamma}\int K_{V^t\tilde a}(\gamma\cdot z,\gamma\cdot w)e^{(\frac12+ir)\langle w,b\rangle}\operatorname{Vol}(dw)\\ &= \sum_{\gamma\in\Gamma} e^{-it\frac{\triangle_z}{2}}e^{-\frac{it}{2}\left(\frac14+r^2\right)}\int K_{\tilde a}(\gamma\cdot z,\gamma\cdot w)e^{(\frac12+ir)\langle w,b\rangle}\operatorname{Vol}(dw)\\ &= e^{-it\frac{\triangle_z}{2}}e^{-\frac{it}{2}\left(\frac14+r^2\right)}\sum_{\gamma\in\Gamma}\int K_{\tilde a}(\gamma\cdot z,\gamma\cdot w)e^{(\frac12+ir)\langle w,b\rangle}\operatorname{Vol}(dw)\\ &= e^{-it\frac{\triangle_z}{2}}e^{-\frac{it}{2}\left(\frac14+r^2\right)}(\Pi\tilde a)(z,b,r)e^{(\frac12+ir)\langle z,b\rangle}.\end{aligned}$$

Since the kernels $K_{\tilde a}$ and $K_{V^t\tilde a}$ satisfy the decay condition (4.6), all these manipulations are legitimate, for all $r\in\mathbf{C}$. Once we know that $\Pi V^t\tilde a$ does not depend on the choice $\tilde a$, it is easily checked that the generator of this flow is $a\mapsto \Pi D^V\tilde a = D^V_\Gamma a$.

5. Intertwining the geodesic flow and the Schrödinger group on the universal cover

In this section, we prove the intertwining formula (1.15) on the universal cover $\mathbf{D}$. This is done by defining analogues of Wigner and Patterson-Sullivan distributions on $\mathbf{D}$ and by finding an explicit relation between both.

5.1. Wigner distributions on D.

Definition 5.1. For $b,b'\in B$ and $\nu,\nu'\in i\mathbf{R}$, the "Wigner distributions"[4] $W_{(\nu,b),(\nu',b')}\in\mathcal{D}'(S\mathbf{D}\times\mathbf{R})$ are defined formally by:

$$\int_{S\mathbf{D}\times\mathbf{R}} a(z,\tilde b,r)W_{(\nu,b),(\nu',b')}(dz,d\tilde b,dr)=\langle \operatorname{Op}(a)e_{\nu,b},e_{\nu',b'}\rangle$$

for a having the symmetry (4.3).

For $b\in B$, we denote $\delta_b(d\tilde b)$ the distribution density on B corresponding to the Dirac mass at b, defined by $\int_B f(\tilde b)\delta_b(d\tilde b)=f(b)$ for every smooth f.

Proposition 5.1. *We have:* $W_{(\nu,b),(\nu',b')}(dz,d\tilde b,dr)=e_{\nu,b}(z)\overline{e_{\nu',b'}(z)}\delta_b(d\tilde b)\delta_{-i\nu}(dr)\operatorname{Vol}(dz)$.

Proof. If $\nu=ir$, it is immediate from the definitions that

$$\langle \operatorname{Op}(a)e_{\nu,b},e_{\nu',b'}\rangle=\int_{\mathbf{D}} a(z,b,r)e_{ir,b}(z)\overline{e_{\nu',b'}(z)}\operatorname{Vol}(dz). \tag{5.1}$$

□

From this explicit formula (5.1), we see immediately that $\operatorname{supp}W_{(\nu,b),(\nu',b')}\subset S\mathbf{D}\times\{-i\nu\}$, so $W_{(\nu,b),(\nu',b')}$ can be extended to functions $a(z,b,r)$ depending continuously on r, with values in $C^\infty_c(S\mathbf{D})$ (in particular, functions a that do not depend on r and are C^∞_c with respect to (z,b)). The definition also extends, in a straightforward manner, to $\nu,\nu'\in\mathbf{C}$ and $a\in\mathcal{S}^0_0$.

[4]As in the introduction, the quotation marks indicate that the term ordinarily refers to the euclidean situation, and that our terminology is not completely standard.

We define the *Wigner transform* of a function $a \in C_c^\infty(G \times \mathbf{R})$ obeying the symmetry (4.3) by

$$\begin{aligned}\mathcal{W} : C_c^\infty(G \times \mathbf{R}) &\to L^2\left(B \times i\mathbf{R} \times B \times i\mathbf{R}, db \otimes p(dr) \otimes db' \otimes p(dr')\right),\\ \mathcal{W}a(\nu, b, \nu', b') &= W_{(\nu,b),(-\nu',b')}(a).\end{aligned}$$

Note the "minus" sign in front of ν'. The reason for this choice will appear when we introduce the Patterson-Sullivan distributions. The following proposition proves completeness of the Wigner distributions.

Proposition 5.2. *The Wigner transform extends to $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ as an isometry and satisfies the inversion formula,*

$$a(z, b, r) = \frac{1}{2} e^{-(\frac{1}{2}+ir)\langle z,b\rangle} \int_B \int_{\mathbf{R}} e^{(\frac{1}{2}-ir')\langle z,b'\rangle} \mathcal{W}a(ir, b, ir', b') db' dp(r').$$

Proof. For $r, r' \in \mathbf{R}$, the Wigner transform is given by

$$\mathcal{W}a(ir, b, ir', b') = \int_{\mathbf{D}} a(z, b, r) e^{(\frac{1}{2}+ir)\langle z,b\rangle} e^{(\frac{1}{2}+ir')\langle z,b'\rangle} \operatorname{Vol}(dz),$$

it is the Fourier transform of $a(z, b, r)e^{(\frac{1}{2}+ir)\langle z,b\rangle}$ with respect to z, evalutated at $(b', -r')$. The inversion formula and the isometry

$$\begin{aligned}(5.2)\qquad ||a||_{L^2_W(G\times\mathbf{R},dg\times dp(r))} &= ||\mathcal{W}a(ir, b, ir', b')||_{L^2(B\times i\mathbf{R}_+\times B\times i\mathbf{R}_+,db\otimes p(dr)\otimes db'\otimes p(dr'))}\\ (5.3)\qquad &= \frac{1}{2}||\mathcal{W}a(ir, b, ir', b')||_{L^2(B\times i\mathbf{R}\times B\times i\mathbf{R},db\otimes p(dr)\otimes db'\otimes p(dr'))}.\end{aligned}$$

follow from the Plancherel and inversion formulae for $\mathcal{F}$. □

5.2. Patterson-Sullivan distributions on D.

Definition 5.2. For $\nu, \nu' \in i\mathbf{R}$, the Patterson-Sullivan distribution $PS_{(\nu,b),(-\nu',b')} \stackrel{\text{def}}{=} PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ associated to the two eigenfunctions $e_{(\nu,b)}(z) = e^{(\frac{1}{2}+\nu)\langle z,b\rangle}$ and $e_{(-\nu',b')}(z) = e^{(\frac{1}{2}-\nu')\langle z,b'\rangle}$ is the distribution on $S\mathbf{D} = B^{(2)} \times \mathbf{R}$ defined by

$$(5.4)\qquad PS_{e_{(\nu,b)},e_{(-\nu',b')}}(d\tilde{b}, d\tilde{b}', d\tau) = \frac{\delta_b(d\tilde{b})\delta_{b'}(d\tilde{b}')}{|\tilde{b} - \tilde{b}'|^{1+\nu-\overline{\nu'}}} e^{(\nu+\overline{\nu'})\tau} d\tau.$$

We use the coordinates defined in §2.2. We note that $PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ is undefined if $b = b'$. We chose this somewhat awkward notation (writing $\overline{\nu'}$ instead of $-\nu'$) so that the definition can be straightforwardly extended to $\nu \in \mathbf{C}$ when needed. The choice of opposite sign conventions for ν and ν' is motivated by the fact that we want the Patterson-Sullivan distributions to be geodesic flow invariant in the "diagonal" case ($\nu = \nu' \in i\mathbf{R}$). For higher rank symmetric spaces, this formula was generalized by Schröder [SchDiss]. He pointed out the fact that ν' has to be replaced by $-w.\nu'$ (where w is the *longest element* of the Weyl group) if one wants the "diagonal" Patterson-Sullivan distributions ($\nu = \nu' \in i\mathfrak{a}^*$) to be A-invariant.

We now prove an analogue of Proposition 1.1 on the universal cover. Recall that $S\mathbf{D}$ is naturally endowed with the density $e^{\langle z,b\rangle} \operatorname{Vol}(dz)db$, corresponding to the Liouville measure on the unit tangent bundle, or to the Haar measure in the group theoretic picture $S\mathbf{D} = G$. Here we have to distinguish between *distributions* and *distribution densities* on a manifold, see [Ho I, Ch. VI][5]. On the boundary B (endowed with the density db) we will denote $\delta_{b_o}(b)$ the distribution defined by the Dirac mass at a point b_o, and $\delta_{b_o}(db) = \delta_{b_o}(b)db$ the corresponding distribution density, defining the linear form $f \mapsto f(b_o)$ on $C^\infty(B)$. We recall that distributions can be multiplied under certain assumptions on their wavefront sets [Ho I, Thm 8.2.10].

Proposition 5.3. *On $S\mathbf{D}$, define the distribution $\epsilon_{\nu,b}(z,\tilde{b}) = e^{(-\frac{1}{2}+\nu)\langle z,\tilde{b}\rangle}\delta_b(\tilde{b})$, corresponding to the distribution density $\epsilon_{\nu,b}(z,\tilde{b})e^{\langle z,\tilde{b}\rangle}\,\mathrm{Vol}(dz)d\tilde{b} = e^{(\frac{1}{2}+\nu)\langle z,\tilde{b}\rangle}\delta_b(\tilde{b})d\tilde{b}\,\mathrm{Vol}(dz)$.*
We have

$$\left(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}}\right)(z,\tilde{b})e^{\langle z,\tilde{b}\rangle}\,\mathrm{Vol}(dz)d\tilde{b} = 2\pi.2^{(\nu-\overline{\nu'})}PS_{e_{(\nu,b)},e_{(-\nu',b')}}(dz,d\tilde{b})$$

where ι denotes time-reversal. The product on the left-hand side is well-defined for $b \neq b'$.

Proof. Writing $\epsilon_{\nu,b}$ in (b,b',t) coordinates, we have

$$\epsilon_{\nu,b}(\tilde{b},\tilde{b}',t) = e^{(-\frac{1}{2}+\nu)\langle g(\tilde{b},\tilde{b}')a_t\cdot 0,\tilde{b}\rangle}\delta_b(\tilde{b}). \tag{5.5}$$

Its time reversal is thus

$$\iota\epsilon_{\nu,b}(\tilde{b},\tilde{b}',t) = e^{(-\frac{1}{2}+\nu)\langle g(\tilde{b}',\tilde{b})a_{-t}\cdot 0,\tilde{b}'\rangle}\delta_b(\tilde{b}'). \tag{5.6}$$

By the identity of Lemma 2.1 of §2.2, we have

$$\langle g(\tilde{b}',\tilde{b})a_t\cdot o,b\rangle = t - \log\frac{|\tilde{b}-\tilde{b}'|}{2}. \tag{5.7}$$

Multiplying the two distributions gives

$$\left(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}}\right)(\tilde{b},\tilde{b}',t) = 2^{-1+\nu-\overline{\nu'}}\frac{e^{t(\nu+\overline{\nu'})}}{|b-b'|^{-1+\nu-\overline{\nu'}}}\delta_b(\tilde{b})\delta_{b'}(\tilde{b}'). \tag{5.8}$$

Multiplying by $e^{\langle z,b\rangle}\,\mathrm{Vol}(dz)db = 4\pi\frac{db\otimes db'}{|b-b'|^2}dt$, we find

$$\left(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}}\right)(\tilde{b},\tilde{b}',t)4\pi\frac{db\otimes db'}{|b-b'|^2}dt = 2\pi.2^{\nu-\overline{\nu'}}\frac{e^{t(\nu+\overline{\nu'})}}{|b-b'|^{1+\nu-\overline{\nu'}}}\delta_b(\tilde{b})\delta_{b'}(\tilde{b}')db\,db'\,dt.$$

□

Remark 5.1. As in [AZ], we observe that Patterson-Sullivan distributions are eigendistributions of the geodesic flow. For a distribution density on $S\mathbf{D}$ we define $g^t_\sharp$ as the pushforward by g^t. For $g = (z,\tilde{b}) \in G$, we have $\epsilon_{\nu,b}(ga_t) = e^{(-\frac{1}{2}+\nu)t}\epsilon_{ir,b}(g)$. This implies that

$$g^t_\sharp(\epsilon_{\nu,b}(z,\tilde{b})e^{\langle z,\tilde{b}\rangle}\,\mathrm{Vol}(dz)d\tilde{b}) = e^{(\frac{1}{2}-\nu)t}\epsilon_{\nu,b}(z,\tilde{b})e^{\langle z,\tilde{b}\rangle}\,\mathrm{Vol}(dz)d\tilde{b}.$$

Proposition 5.4. *For $\nu,\nu' \in i\mathbf{R}$, we have:*

$$g^t_\sharp PS_{e_{(\nu,b)},e_{(-\nu',b')}} = e^{-t(\nu+\overline{\nu'})}PS_{e_{(\nu,b)},e_{(-\nu',b')}}.$$

[5] The choice of a preferred density allows to identify both, and this is the only place of paper where we have to be careful about this distinction. The Patterson-Sullivan and Wigner distributions are actually *distribution densities*.

The proof is immediate. If we extend $PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ to $S\mathbf{D}\times\mathbf{R}$ by taking $PS_{e_{(\nu,b)},e_{(-\nu',b')}}\otimes\delta_{\frac{\nu-\overline{\nu'}}{2i}}$, and if we extend the geodesic flow to $S\mathbf{D}\times\mathbf{R}$ by letting

$$G^t(z,b,r)=(g^{rt}(z,b),r),$$

we have

$$G^t_\sharp\left(PS_{e_{(\nu,b)},e_{(-\nu',b')}}\otimes\delta_{\frac{\nu-\overline{\nu'}}{2i}}\right)=e^{i\frac{(\nu^2-\overline{\nu'}^2)t}{2}}PS_{e_{(\nu,b)},e_{(-\nu',b')}}\otimes\delta_{\frac{\nu-\overline{\nu'}}{2i}}.$$

In other words, for $\nu=ir,\nu'=ir'$,

$$G^t_\sharp\left(PS_{e_{(ir,b)},e_{(-ir',b')}}\otimes\delta_{\frac{r+r'}{2}}\right)=e^{-i\frac{(r^2-r'^2)t}{2}}PS_{e_{(ir,b)},e_{(-ir',b')}}\otimes\delta_{\frac{r+r'}{2}}.$$

When working on compact quotients we will have to worry about a possible extension of these formulae to the case of complex r, and this is why we pay attention to write formulae that can be adapted in a straightforward manner to $r\in\mathbf{C}$.

One remarks that $PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ is a distribution on $S\mathbf{D}$ for any $\nu,\nu'\in\mathbf{C}$, and that Proposition 5.3 still holds. Also, $PS_{e_{(\nu,b)},e_{(-\nu',b')}}\otimes\delta_{\frac{\nu-\overline{\nu'}}{2i}}(a)$ is well-defined for a test function $a\in\mathcal{S}^0_0$.

5.3. **Radon-Fourier transform along geodesics.** PS-distributions are closely connected to the Radon transform along geodesics. As reviewed in §2.2, the unit tangent bundle $S\mathbf{D}$ can be identified with $B^{(2)}\times\mathbf{R}$: the set $B^{(2)}$ represents the set of oriented geodesics, and $\mathbf{R}$ gives the time parameter along geodesics. We denote by $\gamma_{b',b}$ the oriented geodesic with endpoints b',b.

Definition 5.3. The geodesic Fourier-Radon transform is defined by

$$\mathcal{R}:C_c(S\mathbf{D})\to C_c(B^{(2)}\times\mathbf{R}),\quad\text{by }\ \mathcal{R}f(b',b,r)=\int_{\mathbf{R}}f(g(b',b)a_t)e^{-irt}dt.$$

It is clear that $\mathcal{R}$ intertwines composition with g^t and multiplication by e^{irt}, i.e.

$$\mathcal{R}(f\circ g^t)(b',b,r)=e^{irt}\mathcal{R}f(b',b,r).\tag{5.9}$$

By the Fourier inversion formula,

$$f(g(b',b)a_t)=\frac{1}{2\pi}\int_{\mathbf{R}}\mathcal{R}f(b',b,r)e^{irt}dr.\tag{5.10}$$

We call "Patterson-Sullivan transform" the pairing of the family of PS-distributions with a test function.

Definition 5.4. The PS-transform is defined as follows:
$PS:C_c^\infty(G\times\mathbf{R})\to C^\infty(B^{(2)}\times i\mathbf{R}\times i\mathbf{R})$ on G by

$$PSa(\nu,b,\nu',b')\stackrel{\text{def}}{=}PS_{(\nu,b),(-\nu',b')}\left(a_{\frac{\nu-\overline{\nu'}}{2i}}\right)=\frac{1}{|b-b'|^{1+\nu-\overline{\nu'}}}\int_{\mathbf{R}}a\left(g(b',b)a_\tau,\frac{\nu-\overline{\nu'}}{2i}\right)e^{(\nu+\overline{\nu'})\tau}d\tau$$

The PS-transform is related to the Fourier-Radon transform as follows:

$$PSa(\nu,b,\nu',b')=\frac{1}{|b-b'|^{1+\nu-\overline{\nu'}}}\mathcal{R}a_{\frac{\nu-\overline{\nu'}}{2i}}(b',b,i(\nu+\overline{\nu'})).\tag{5.11}$$

Using the inversion formula for $\mathcal{R}$, one gets the inversion formula for the PS-transform :

Lemma 5.5. *The function a is determined from its PS-transform $PSa(ir, b, ir', b')$ by*

$$a(b', b, t, R) = \frac{1}{\pi} e^{2iRt} |b - b'|^{1+2iR} \int_{\mathbf{R}} PSa(ir, b, i(2R - r), b') e^{-2irt} dr.$$

Using the Parseval identity for the 1-dimensional Fourier transform, as well as Lemma 3.1 expressing the Haar measure in the (b', b, t)-coordinates, one gets

Lemma 5.6.

$$\|a\|^2_{L^2(G\times\mathbf{R}, dg\otimes dp)} = \frac{1}{\pi} \int_{b,b'\in B, r,r'\in\mathbf{R}} |PSa(ir, b, ir', b')|^2 db db' \left(\frac{r + r'}{2}\right) \tanh\left(\pi \frac{r + r'}{2}\right) dr dr'.$$

This formula can be compared to that obtained for the Wigner transform (5.2). Note, however, that the $drdr'$-density is different in the two formulae. We also stress the fact that we do not ask a to have the symmetry (4.3) here.

5.4. **Operator sending the Patterson-Sullivan distributions to the Wigner distributions.** If a is a function on $S\mathbf{D} \simeq G$, and $\nu \in \mathbf{C}$, we define the function $L_\nu a$ on G by

$$L_\nu a(g) = \int_{\mathbf{R}} a(gn_u)(1 + u^2)^{-(\frac{1}{2}+\nu)} du.$$

In this section, we prove the following :

Proposition 5.7. *Let $a \in C_c^\infty(G)$, $\nu, \nu' \in i\mathbf{R}$ and $(b', b) \in B^{(2)}$. Then $L_{-\overline{\nu'}}(a) \in C^\infty(S\mathbf{D})$. Although $L_{-\overline{\nu'}}(a)$ is not compactly supported, the pairing $PS_{(\nu,b)(-\nu',b')}\left(L_{-\overline{\nu'}}(a)\right)$ is well defined, and we have*

$$PS_{(\nu,b)(-\nu',b')}\left(L_{-\overline{\nu'}}(a)\right) = 2^{-(1+\nu-\overline{\nu'})} W_{(\nu,b)(-\nu',b')}(a).$$

The right side is well-defined by the remark following Proposition 5.1. A proof of Proposition 5.7 by direct computation was given in [AZ], in the "diagonal" case $\nu = \nu' \in i\mathbf{R}$. We now give two proofs of the Proposition, one by adpating the argument of [AZ] in the diagonal case $\nu = \nu' \in \mathbf{R}$, and a novel one based on Proposition 5.3 and on the invariance properties of the distributions $\epsilon_{\nu,b}$. We begin with the first proof:

Proof. By definition,

$$\begin{aligned}
(5.12)\quad W_{(\nu,b),(-\nu',b')}(a) &= \int_{\mathbf{D}} a(z, b) e^{(\frac{1}{2}+\nu)\langle z,b\rangle} e^{(\frac{1}{2}-\bar\nu')\langle z,b'\rangle} \operatorname{Vol}(dz) \\
&= \int_{\mathbf{D}} a(z, b) e^{(\frac{1}{2}-\bar\nu')\langle z,b\rangle} e^{(\frac{1}{2}-\bar\nu')\langle z,b'\rangle} e^{(\nu+\bar\nu')\langle z,b\rangle} \operatorname{Vol}(dz) \\
&= 2^{(1-2\bar\nu')} \left(\int_{\mathbf{D}} a(z, b) [\cosh s_{b',b}(z)]^{-(1-2\bar\nu')} e^{(\nu+\bar\nu')\langle z,b\rangle} \operatorname{Vol}(dz) \right) \frac{1}{|b - b'|^{1-2\bar\nu'}} \\
&= 2^{(1-2\bar\nu')} \left(\int_{\mathbf{D}} a(z, b) [\cosh s_{b',b}(z)]^{-(1-2\bar\nu')} \frac{e^{(\nu+\bar\nu')\langle z,b\rangle}}{|b - b'|^{-(\nu+\bar\nu')}} \operatorname{Vol}(dz) \right) \frac{1}{|b - b'|^{1+\nu-\bar\nu'}}
\end{aligned}$$

Here, $s_{b_1,b_2}(z)$ denotes the hyperbolic distance from z to the geodesic γ_{b_1,b_2} defined by (b_1, b_2), and we use the identity (see [AZ], §4)

$$(5.13)\qquad e^{\langle z,b\rangle} e^{\langle z,b'\rangle} = 4[\cosh s_{b',b}(z)]^{-2} |b - b'|^{-2}.$$

As in [AZ], we now write $(z,b) = g(b',b)a_\tau n_u$, $\mathrm{Vol}(dz) = d\tau du$ and $\cosh s_{b',b}(z) = \sqrt{1+u^2}$. We also use formula (2.1) to write $e^{\langle g(b',b)a_\tau n_u, b\rangle}|b-b'| = 2e^\tau$.

We thus obtain

$$W_{(\nu,b),(-\nu',b')}(a) = \frac{2^{(1+\nu-\bar{\nu}')}}{|b-b'|^{1+\nu-\bar{\nu}'}} \int_{\mathbf{R}\times\mathbf{R}} (1+u^2)^{-(\frac{1}{2}-\bar{\nu}')} a(g(b,b')a_\tau n_u) e^{(\nu+\bar{\nu}')\tau} du d\tau \tag{5.14}$$

$$= 2^{(1+\nu-\bar{\nu}')} PS_{(\nu,b)(-\nu',b')}\left(L_{-\overline{\nu'}}(a)\right).$$

□

We now give a second proof, using the action of $L_{-\overline{\nu'}}$ on the product $(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}})$ (we note that the Poisson density $e^{\langle z,\tilde{b}\rangle}\mathrm{Vol}(dz)d\tilde{b}$ is n_u-invariant, so can be taken out of the integral defining $L_{-\overline{\nu'}}$). We use that $a_t n_u = n_{ue^t} a_t$ and the KAN (Iwasawa) decomposition,

$$n_u = k_u a_{-\log(1+u^2)} \bar{n}_{f(u)}, \tag{5.15}$$

as in (2.8).

Proposition 5.8. (1) *$\epsilon_{\nu,b}$ is n_u-invariant;*
(2) *$L_\nu(\iota\epsilon_{\nu,b})(z,\tilde{b}) = \pi e_{(\nu,b)}(z)$ (it does not depend on $\tilde{b}$, in other words, it is a right-K-invariant function);*
(3) *we have $L_{-\overline{\nu'}}(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}}) = \pi\epsilon_{\nu,b}\overline{e_{(-\nu',b')}}$.*

Proof. (1) is obvious, and (3) follows from (2), so we just have to prove (2).

$$L_\nu(\iota\epsilon_{\nu,b})(g) \;=\; \int_{\mathbf{R}} (1+u^2)^{-(\frac{1}{2}+\nu)} \iota\epsilon_{r,b}(gn_u) du. \tag{5.16}$$

We rewrite n_u using (2.9). Then we:

- remove the right $\overline{N}$ factor since $\iota\epsilon_{\nu,b}$ is right-$\overline{N}$-invariant;
- replace the A factor inside $\iota\epsilon_{\nu,b}$ by a factor $e^{(-\frac{1}{2}+\nu)\log(1+u^2)}$ outside, since $\iota\epsilon_{\nu,b}$ is an A-eigendistribution of eigenvalue $=\frac{1}{2}-\nu$ which is evaluated for $a_{-\log(1+u^2)}$.
- then change variables to K with $\theta = \arctan u$.

We then have,

$$\begin{aligned} L_\nu(\iota\epsilon_{\nu,b})(g) &= \int_{\mathbf{R}} (1+u^2)^{-1}(\iota\epsilon_{\nu,b})(g)(gk_u)du = \pi\int_K (\iota\epsilon_{\nu,b})(gk)dk \\ &= \pi\int_K \epsilon_{\nu,b}(gk)dk = \pi\int \epsilon_{\nu,b}(z,b')e^{\langle z,b'\rangle}db' = \pi e^{(\frac{1}{2}+\nu)\langle z,b\rangle} \end{aligned} \tag{5.17}$$

for $g = (z,\tilde{b})$. □

Multiplying by the Poisson density, we finally get

$$L_{-\overline{\nu'}}(\epsilon_{\nu,b}.\iota\overline{\epsilon_{-\nu',b'}})\, e^{\langle z,\tilde{b}\rangle}\,\mathrm{Vol}(dz)d\tilde{b} = \pi e_{(\nu,b)(z)}\overline{e_{(-\nu',b')}}(z)\,\mathrm{Vol}(dz)\delta_b(\tilde{b})d\tilde{b}.$$

We recognize from Proposition 5.1 the expression of the Wigner distribution, for a test function a that does not depend on the r-parameter. Comparing with Proposition 5.3 completes the second proof of Proposition 5.7.

Proposition 5.7 was proven for a function a defined on $S\mathbf{D}$, in other words a function on $S\mathbf{D}\times\mathbf{R}$ that does not depend on the last variable. In the sequel, we will apply Proposition 5.7 to an arbitrary function on $S\mathbf{D}\times\mathbf{R}$, using it in the following form. If a is a function on $S\mathbf{D} \times \mathbf{R} \simeq G \times \mathbf{R}$, and $r \in \mathbf{R}$, we define the function a_r on $S\mathbf{D} \simeq G$ by $a_r(g) = a(g, r)$. Proposition 5.7 implies that

$$2^{-(1+\nu-\overline{\nu'})} W_{(\nu,b)(-\nu',b')}(a) = PS_{(\nu,b)(-\nu',b')}\left(L_{-\overline{\nu'}}(a_r)\right),$$

for $\nu = ir \in i\mathbf{R}$, and $\nu' \in i\mathbf{R}$.

Again, one sees that these formulae also hold for all $\nu, \nu' \in \mathbf{C}$ and test functions $a \in \mathcal{S}_0^0$.

5.5. **The operator $\mathcal{L}$.** Recall that we have extended the PS-distribution $PS_{(ir,b),(-ir',b')}$ ($r, r' \in \mathbf{R}$), originally defined on $S\mathbf{D}$, to $S\mathbf{D} \times \mathbf{R}$, by tensoring it by $\delta_{\frac{r+r'}{2}}$ on the $\mathbf{R}$-variable.

We now look for an operator $\mathcal{L}$ that acts on functions (distributions) defined on $S\mathbf{D} \times \mathbf{R}$, with the property that $PS\mathcal{L}a(ir, b, ir', b') = \mathcal{W}a(ir, b, ir', b')$ ($r, r' \in \mathbf{R}$). This means that we must have $PS\mathcal{L}a(ir, b, ir', b') = 2^{1+ir+ir'} PS_{(ir,b)(-ir',b')}\left(L_{ir'}(a_r)\right)$.

By the PS-inversion formula (Lemma 5.5), we have for all $(b', b) \in B^{(2)}$, $t \in \mathbf{R}$, $R \in \mathbf{R}$,

$$\begin{aligned}\mathcal{L}a(b', b, t, R) &= \frac{2^{1+2iR}}{\pi} e^{2iRt} |b - b'|^{1+2iR} \int_{\mathbf{R}} PS(\mathcal{L}a)(ir, b, i(2R - r), b') e^{-2irt} dr \\ &= \frac{2^{1+2iR}}{\pi} \int_{\mathbf{R}} (1 + u^2)^{-(\frac{1}{2}+iR)} a_r \circ h^u(b', b, \tau) e^{2i(R-r)(t-\tau-\frac{\log(1+u^2)}{2})} dr du d\tau\end{aligned} \tag{5.18}$$

In other words, letting $g = (b', b, t)$,

$$\mathcal{L}a(g, R) = \frac{2^{1+2iR}}{\pi} \int (1 + u^2)^{-(\frac{1}{2}+iR)} a(g a_{\tau-\frac{\log(1+u^2)}{2}} n_u, r) e^{2i(r-R)\tau} dr du d\tau. \tag{5.19}$$

5.6. **Intertwining.** In this section, we prove that the operator $\mathcal{L}$ intertwines V^t and G^t on $\mathbf{D}$. We recall the Hilbert space L^2_W in Proposition 5.2.

Theorem **7.** *(i) For $a \in \mathcal{S}_0^0$, $\mathcal{L}a$ is a continuous function, and we have the pointwise equality*

$$\mathcal{L} \circ V^t a = G^t \circ \mathcal{L}a.$$

(ii) The intertwining operator $\mathcal{L}$ extends to an isometry from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to the space $\mathcal{H}_{PS}(\mathbf{D})$ of functions such that

$$\frac{1}{4} \int |PSa(ir, b, ir', b')|^2 db db' p(dr) p(dr') < +\infty.$$

and we have

$$\mathcal{L} \circ V^t = G^t \circ \mathcal{L}, \tag{5.20}$$

where both sides are bounded operators from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to $\mathcal{H}_{PS}(\mathbf{D})$.

Proof. First, we consider the action of both sides on $a \in \mathcal{S}_0^0$. We know that the space $\mathcal{S}_0^0$ is preserved by V^t. We shall first check that $\mathcal{L}a$ is a continuous function when $a \in \mathcal{S}_0^0$, and that (5.20) then holds as a pointwise equality between $\mathcal{L}(V^t a)$ and $G^t(\mathcal{L}a)$.

For $g = (z, b)$, we see that

$$\begin{aligned}\mathcal{L}a(z, b, R) &= \tfrac{2^{1+2iR}}{\pi} \textstyle\int (1 + u^2)^{-(\frac{1}{2}+iR)} a \circ h^u \circ g^{\tau-\frac{\log(1+u^2)}{2}}(z, b, r) e^{2i(r-R)\tau} dr du d\tau \\ &= \tfrac{2^{1+2iR}}{\pi} \textstyle\int (1 + u^2)^{-(\frac{1}{2}+iR)} \hat{a} \circ h^u \circ g^{\tau-\frac{\log(1+u^2)}{2}}(z, b, 2\tau) e^{-2iR\tau} du d\tau,\end{aligned} \tag{5.21}$$

where we define $\hat{a}(z,b,\tau) = \int a(z,b,r)e^{ir\tau}dr$. It follows from the definition of $\mathcal{S}_0^0$ that $|\hat{a}(z,b,\tau)| \leq C_{N,M,x_0}e^{-N|\tau|}e^{-Md(z,x_o)}$ for any $N, M > 0$ and any given x_o.

If z stays in a fixed compact set, denoting $(\tilde{z},b) = h^u \circ g^{\tau - \frac{\log(1+u^2)}{2}}(z,b)$, one can check by hand that $\exp d(\tilde{z},x_o) \geq C(1+|u|)e^{|\tau|}$, with $C > 0$. For instance for $(z,b) = e \in G$, we compute explicitly

$$a_{\tau - \frac{\log(1+u^2)}{2}} n_u = \begin{pmatrix} \frac{e^{\tau/2}}{(1+u^2)^{1/4}} & \frac{ue^{\tau/2}}{(1+u^2)^{1/4}} \\ 0 & e^{-\tau/2}(1+u^2)^{1/4} \end{pmatrix}.$$

In the Poincaré upper half-plane model, identified with $PSL(2,\mathbf{R})/K$, this element represents a unit tangent vector based at

$$\tilde{z} = \frac{e^\tau}{(1+u^2)^{1/2}}i + \frac{ue^\tau}{(1+u^2)^{1/2}}.$$

Using (2.1), the hyperbolic distance of this point to the origin $x_o = i$ is given by

$$\cosh d(\tilde{z},i) = 1 + \frac{1}{2}\left[\left(\frac{e^\tau}{(1+u^2)^{1/2}} - 1\right)^2 + \frac{u^2e^{2\tau}}{(1+u^2)}\right](1+u^2)^{1/2}e^{-\tau} \geq \frac{1}{2}(1+u^2)^{1/2}e^{|\tau|}.$$

It follows that

$$|\hat{a}(\tilde{z},b,2\tau)| \leq C_{N,M}(1+u^2)^{-M/2}e^{-(N+M)|\tau|} \tag{5.22}$$

(with $C_{N,M}$ uniform as z stays in a compact set), so that the integral (5.21) does make sense, and defines a continuous function of the variables (z,b,R). To keep this paper reasonably short, we do not investigate the additional regularity properties of $\mathcal{L}a$.

To end the proof of the theorem, we keep the same notation and use the fact that

$$|\hat{a} \circ g^t(\tilde{z},b,2\tau)| \leq C_{N,M}(1+u^2)^{-M/2}e^{-N|\tau| - M|t+\tau|}$$

to see that, for fixed (z,b,R), we have $|(\mathcal{L}a) \circ g^t(z,b,R)| \leq C_{z,b,R,M}e^{-M|t|}$ (for $M > 0$ arbitrary). It follows that $PS\mathcal{L}a(ir,b,ir',b')$ is perfectly well defined for any $r, r' \in \mathbf{C}$, $(b',b) \in B^{(2)}$. The Wigner transform $\mathcal{W}a(ir,b,ir',b')$ is also perfectly well defined, and $\mathcal{L}$ has been constructed so that $PS\mathcal{L}a(ir,b,ir',b') = \mathcal{W}a(ir,b,ir',b')$. We see that

$$\begin{aligned} PS(G^t\mathcal{L}a)(ir,b,ir',b') &= e^{-i\frac{(r^2-r'^2)t}{2}}PS(\mathcal{L}a)(ir,b,ir',b') \\ &= e^{-i\frac{(r^2-r'^2)t}{2}}\mathcal{W}a(ir,b,ir',b') = \mathcal{W}(V^ta)(ir,b,ir',b') = PS(\mathcal{L}V^ta)(ir,b,ir',b'), \end{aligned}$$

and inverting this formula we get that $G^t\mathcal{L}a = \mathcal{L}V^ta$, for all $a \in \mathcal{S}_0^0$ (and this equality holds pointwise).

We can now easily extend the intertwining formula to $a \in L^2_W(G \times \mathbf{R}, dg \times dp(r))$. Using formula (5.2), we see (in a tautological way) that $\mathcal{L}$ is an isometry from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to $\mathcal{H}_{PS}(\mathbf{D})$, and we have $G^t \circ \mathcal{L} = \mathcal{L} \circ V^t$, where both sides are bounded operators from $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ to $\mathcal{H}_{PS}(\mathbf{D})$.

□

Comparing with Lemma 5.6, we note that the norm on $\mathcal{H}_{PS}(\mathbf{D})$ is not equivalent to the norm on $L^2_W(G \times \mathbf{R}, dg \times dp(r))$ (the ratio of the $drdr'$-densities is unbounded), so that $\mathcal{L}$ is unbounded on $L^2(G \times \mathbf{R}, dg \times dp(r))$.

In the Section 6, we will mimick this construction to build two Hilbert spaces $\mathcal{H}_W(\mathbf{X}_\Gamma)$ and $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ formed of Γ-invariant symbols, such that $\mathcal{L}$ sends $\mathcal{H}_W(\mathbf{X}_\Gamma)$ isometrically to $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$, and such that the intertwining formula $G^t \circ \mathcal{L} = \mathcal{L} \circ V^t$ holds between these two spaces. On the quotient, $\mathcal{H}_W(\mathbf{X}_\Gamma)$ will be naturally identified with the space of Hilbert-Schmidt operators via the quantization procedure Op_Γ, but will *not* be equivalent to $L^2\left((\Gamma\backslash G) \times \mathbf{R}\right)$.

Remark 5.2. Since G^t preserves the variable r, Proposition 5.20 still holds if we modify the definition of $\mathcal{L}a(g,R)$ by a constant depending only on R. Thus, we have the choice of a normalization factor for $\mathcal{L}$. We note that $G^t 1 = 1$ and $V^t 1 = 1$, so that it is quite natural to renormalize $\mathcal{L}$ to have formally $\widehat{\mathcal{L}}1 = 1$. This means dividing $\mathcal{L}a(g,R)$ by

$$\int (1+u^2)^{-(\frac{1}{2}+iR)} e^{-2i(R-r)\tau} dr du d\tau = \pi \int (1+u^2)^{-(\frac{1}{2}+iR)} du.$$

The function $\mu_0(s) = \int_{-\infty}^{+\infty} (1+u^2)^{-s} du$ ($\Re e(s) > \frac{1}{2}$) extends meromorphically to the whole complex plane by $\mu_0(s) = \frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}$ (see p. 65-66 in [He]).

The renormalized $\widehat{\mathcal{L}}$ now satisfies $\widehat{PS}_{(ir,b),(-ir',b')}(\widehat{\mathcal{L}}a) = W_{(ir,b),(-ir',b')}(a)$ if we define the normalized $\widehat{PS}$-distributions by

$$\widehat{PS}_{(ir,b),(-ir',b')} = \pi \mu_0 \left(\frac{1}{2} + i\frac{r+r'}{2}\right) PS_{(ir,b),(-ir',b')}. \tag{5.23}$$

Remark 5.3. When working on the quotient, we will need the following properties of $\mathcal{L}$, obtained from its explicit expression (5.19).

First note that, if $a \in \mathcal{S}^0_0$, then $\mathcal{L}a(g,R)$ has a holomorphic extension to $R \in \mathbf{C}$

Assume that $a(z,b,r) \in \mathcal{S}^0_0$ satisfies in addition, for every $\epsilon > 0$, p, all $q > 0$, every non-negative integer n, and every K-left-invariant differential operator D acting on B, a bound of the form

$$\sup_{(r,b)} e^{q|r|} \left| \frac{\partial^n}{\partial r^n} Da(\bullet, b, r) \right|_{\mathcal{C}^p(G/K)} < +\infty, \tag{5.24}$$

in $\{|\Im m(r)| < \frac{\epsilon}{2}\}$. In other words, we strenghthen the definition of $\mathcal{S}^0_0$ by asking that a decay superexponentially fast in r, instead of superpolynomially fast. We will denote by $\mathcal{S}^0_\alpha$ the space of such symbols.

Then, for any fixed $g \in G$ and $R \in \mathbf{C}$, the map $t \mapsto \mathcal{L}a(ga_t, R)$, originally defined for $t \in \mathbf{R}$, has a holomorphic extension to $t \in \mathbf{C}$. In particular, $(G^t \mathcal{L}a)(g,R)$ is well defined for $R \in \mathbf{C}$.

Remark 5.4. Let us briefly discuss the case of the wave flow, $e^{it\sqrt{-\triangle - 1/4}}$. The corresponding quantum evolution is

$$\beta^t(\mathrm{Op}(a)) = e^{-it\sqrt{-\triangle-1/4}}\,\mathrm{Op}(a) e^{it\sqrt{-\triangle-1/4}} =: \mathrm{Op}(U^t a). \tag{5.25}$$

The explicit expression of U^t is given in [Z1, Z3].

Since $e^{it\sqrt{-\triangle-1/4}}e_{(ir,b)} = e^{itr}e_{(ir,b)}$ and $e^{it\sqrt{-\triangle-1/4}}e_{(-ir,b)} = e^{itr}e_{(-ir,b)}$ for $r > 0$, we see that U^t defines a unitary operator on $L^2_W(G \times \mathbf{R}, dg \times dp(r))$, and that $U^t_\sharp W_{e_{(ir,b)},e_{(-ir',b')}} = e^{it(r-r')}W_{e_{(ir,b)},e_{(-ir',b')}}$ for $r, r' > 0$.

We also have $g^t_\sharp PS_{e_{(ir,b)},e_{(-ir',b')}} = e^{it(r-r')}PS_{e_{(ir,b)},e_{(-ir',b')}}$ (where g^t is the unit-speed geodesic flow).

It follows that $(\mathcal{L} \circ U^t)_\sharp PS_{e_{(ir,b)},e_{(-ir',b')}} = (g^t \circ \mathcal{L})_\sharp PS_{e_{(ir,b)},e_{(-ir',b')}}$ for $r, r' > 0$. So in this sense, $\mathcal{L}$ also intertwines the wave group and the unit-speed geodesic flow. But because we restricted to positive values of r, the result is not apriori as strong as for the Schrödinger flow.

A further defect is that $e^{it\sqrt{-\triangle-1/4}}$ does not preserve the Schwartz spaces $\mathcal{C}^p(\mathbf{D})$, because $\sqrt{\cdot}$ is not a holomorphic function on the complex plane. On a compact quotient, there is also a problem with the definition of $e^{it\sqrt{-\triangle-1/4}}$ for low eigenvalues : in particular, it is not unitary. The latter problem may be circumvented by using $e^{it\sqrt{-\triangle}}$. But these difficulties explain why we prefer in this article to work with the Schrödinger group $e^{it\triangle}$. We discuss the intertwining of the wave group and geodesic flow further in [Z4], where the intertwining involves a modification of $\mathcal{L}$.

5.7. **Remarks and questions about an "exact" Egorov theorem.** In the study of quantum chaos in the semiclassical régime, one works with the flow $(e^{it\hbar\frac{\triangle}{2}})$, in the limit $\hbar \longrightarrow 0$. In this case, one considers "semiclassical symbols" also depending on $\hbar > 0$: usually one considers smooth functions $a_\hbar$ depending on $\hbar > 0$, having an asymptotic expansion

$$a_\hbar \sim \sum_{k=0}^{+\infty} \hbar^k a_k, \tag{5.26}$$

the expansion being valid in all the C^ℓ–seminorms on compact sets. In this setting, one works with the operators $\mathrm{Op}_\hbar(a) \stackrel{\text{def}}{=} \mathrm{Op}(a(z, b, \hbar r))$.

Introducing the operator $M_\hbar a(z, b, r) \stackrel{\text{def}}{=} a(z, b, \hbar r)$, it is natural in this context to introduce the notations $\widehat{\mathcal{L}}_\hbar = M_\hbar^{-1} \circ \widehat{\mathcal{L}} \circ M_\hbar$, and $V^t_\hbar = M_\hbar^{-1} \circ V^{t\hbar} \circ M_\hbar$. Note that $M_\hbar^{-1} \circ G^{\hbar t} \circ M_\hbar = G^t$.

In the semiclassical setting, the intertwining relation reads $\widehat{\mathcal{L}}_\hbar \circ V^t_\hbar a_\hbar = G^t \circ \widehat{\mathcal{L}}_\hbar a_\hbar$ if we assume, say, that $a_\hbar \in \mathcal{S}^0_0$ (4.7) and that the expansion (5.26) holds in all the $\mathcal{S}^0_0$–seminorms. Using the stationary phase method, both sides of the equality $\widehat{\mathcal{L}}_\hbar \circ V^t_\hbar a_\hbar = G^t \circ \widehat{\mathcal{L}}_\hbar a_\hbar$ can be expanded into powers of $\hbar$, and on both sides the coefficient of $\hbar^0$ is $G^t a_0$: this is an expression of the so-called "Egorov theorem", which says that $V^t_\hbar a_\hbar$ has an expansion starting with $G^t a_0 + O(\hbar)$, combined with the fact that $\widehat{\mathcal{L}}_\hbar = I + O(\hbar)$.

The intertwining relation of Theorem 7 at least formally defines a quantization procedure of $a_\hbar \in \mathcal{S}^0_0$ (4.7) on $\mathbf{D}$ for which the Egorov theorem is exact : if we know that $\widehat{\mathcal{L}}_\hbar$ is invertible, and have an explicit expression for its inverse, we can then define

$$\widetilde{\mathrm{Op}_\hbar}(a) = \mathrm{Op}_\hbar(\widehat{\mathcal{L}}_\hbar^{-1} a), \tag{5.27}$$

and this new quantization procedure will have the property that

$$e^{-it\hbar\frac{\triangle}{2}}\widetilde{\mathrm{Op}_\hbar}(a)e^{it\hbar\frac{\triangle}{2}} = \widetilde{\mathrm{Op}_\hbar}(G^t a). \tag{5.28}$$

Such an exact intertwining relation is often called the "exact" Egorov property, and is so far only known in the euclidean case, where the Weyl quantization $\mathrm{Op}_\hbar^W$ has the property that $e^{-it\hbar\frac{\triangle}{2}}\mathrm{Op}_\hbar^W(a)e^{it\hbar\frac{\triangle}{2}} = \mathrm{Op}_\hbar(G^t a)$ (where a is a function on $T^*\mathbf{R}^d = \mathbf{R}^d \times \mathbf{R}^d$ with reasonable smoothness and decay properties, $\triangle$ is the euclidean laplacian on $\mathbf{R}^d$, and G^t is the euclidean geodesic flow).

But in our case the exact Egorov theorem (5.28) is only 'formal' unless we can invert $\widehat{\mathcal{L}}$ to define the quantization (5.27) and determine the properties of the inverse. Being able to compute $\widehat{\mathcal{L}}^{-1}$ amounts to computing $\mathcal{L}^{-1}$, and this can be done as follows : we must have $W_{(ir,b),(-ir',b')}(\mathcal{L}^{-1}a) = PS_{(ir,b),(-ir',b')}(a)$, and we can recover the expression of $\mathcal{L}^{-1}a$ using the inversion formula Proposition 5.2. We find

$$\mathcal{L}^{-1}a(z,b,r) = e^{-(\frac{1}{2}+ir)\langle z,b\rangle}\int_{b'\in B, r'>0, \tau\in\mathbf{R}} \frac{a(b',b,\tau;\frac{r+r'}{2})}{|b-b'|^{1+ir+ir'}} e^{(ir-ir')\tau} e^{(\frac{1}{2}-ir')\langle z,b'\rangle} db'dp(r')d\tau.$$

To express this in group theoretic terms, let us consider the special case $z = 0, b = 1$ (in the disc model), corresponding to $g = e \in G$. Using the calculations of §2.4, we find

$$\mathcal{L}^{-1}a(e,r) = \int a\left(n_u a_{\tau+\frac{\log(1+u^2)}{2}}; \frac{r+r'}{2}\right)\left(1+u^2\right)^{\frac{-1+ir+ir'}{2}} e^{(ir-ir')\tau}\frac{2^{-(1+ir+ir')}}{\pi} dudp(r')d\tau.$$

More generally, using G-equivariance of the formulae,

$$\mathcal{L}^{-1}a(g,r) = \int a\left(gn_u a_{\tau+\frac{\log(1+u^2)}{2}}; \frac{r+r'}{2}\right)\left(1+u^2\right)^{\frac{-1+ir+ir'}{2}} e^{(ir-ir')\tau}\frac{2^{-(1+ir+ir')}}{\pi} dudp(r')d\tau.$$

Then (5.27) defines a new quantization procedure satisfying the exact Egorov property.

Open questions :

(1) The problem in making this more precise is that the regularity properties of $\mathcal{L}^{-1}$ are not well understood due to the fact that $\mathcal{L}$ smoothes along the stable foliation and so its inverse must make certain smooth functions very rough. A further complication is that, contrary to what is usually expected from a quantization procedure, $\widetilde{\mathrm{Op}_\hbar}(a)$ is not the multiplication by a if $a = a(z)$ is a function on G/K.

(2) The operator $\widehat{\mathcal{L}}_\hbar$ has a formal expansion into powers of $\hbar$,

$$\widehat{\mathcal{L}}_\hbar \sim \sum_{k=0}^{+\infty} \hbar^k \mathcal{L}_k, \tag{5.29}$$

obtained by applying *formally* the stationary phase method in the integral defining $\widehat{\mathcal{L}}_\hbar$ (see (5.19)).

Note that we have normalized $\widehat{\mathcal{L}}_\hbar$ so that $\mathcal{L}_0$ is the identity. The higher order term $\mathcal{L}_k$ is a polynomial of degree $2k$ in H, X_+ and $\frac{\partial}{\partial r}$, with coefficients depending on r. In the form (5.29), $\widehat{\mathcal{L}}_\hbar$ acts naturally on the space of semiclassical symbols (5.26) modulo $\hbar^\infty$. It preserves the space of Γ-invariant semiclassical symbols.

The derivative at $t = 0$ of intertwining relation reads

$$rH\widehat{\mathcal{L}}_\hbar = \widehat{\mathcal{L}}_\hbar\left(rH - i\hbar\left(\frac{H^2}{4} + X_+^2\right)\right),$$

in other words

$$[rH, \mathcal{L}_k] = -i\hbar \mathcal{L}_{k-1} \left(\frac{H^2}{4} + X_+^2 \right) \tag{5.30}$$

for $k \geq 1$, and

$$[rH, \mathcal{L}_0] = 0.$$

Our argument proves existence of a solution to this family of equations (with $\mathcal{L}_k$ polynomial of degree $2k$ in H, X_+ and $\frac{\partial}{\partial r}$), and allows in theory to compute explicitly the $\mathcal{L}_k$s. The calculation, however, seems tedious, and leave it as an open question to give the explicit expression of $\mathcal{L}_k$. We also leave it open to prove uniqueness of the solution, assuming the normalization $\mathcal{L}_0 = I$.

6. Intertwining the geodesic flow and the Schrödinger group on a compact quotient

In this section, we define the Hilbert spaces $\mathcal{H}_W = \mathcal{H}_W(\mathbf{X}_\Gamma)$ and $\mathcal{H}_{PS} = \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ and the quotient intertwining operator $\mathcal{L}_\Gamma$ (and its adjoint), and prove the main results of this article: Theorems 3 and 4.

We begin with some orienting remarks that are intended to clarify the introduction of the Hilbert spaces $\mathcal{H}_W(\mathbf{X}_\Gamma)$ and $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ below. In Theorem 7, we proved the intertwining relation

$$\mathcal{L} \circ V^t = G^t \circ \mathcal{L} \tag{6.1}$$

as a pointwise equality for $a \in \mathcal{S}_0^0$. Ideally, one would like to generalize formula (6.1) to Γ-invariant symbols : one would like such a formula to hold as a pointwise equality for $a \in \Pi\mathcal{S}_0^0$, or as an almost-everywhere equality for $a \in L^2(\Gamma\backslash G \times \mathbf{R}, dg \times dp(r))$. But it is difficult even to define $\mathcal{L}$ on such automorphic symbols: if $\tilde{a} \in \mathcal{S}_0^0$, $\mathcal{L}\tilde{a}$ does not seem to decay and the periodization $\Pi\mathcal{L}\tilde{a}$ is problematic. On $L^2(G \times \mathbf{R}, dg \times dp(r))$, we saw that $\mathcal{L}$ was unbounded, and its action on $L^2(\Gamma\backslash G \times \mathbf{R}, dg \times dp(r))$ is likely to be worse. Moreover, as discussed in the introduction, there is a spectral obstruction to interwining V_Γ^t and G^t on the quotient, since the spectrum of g^t on $L^2(\Gamma\backslash G)$ is continuous (a manifestation of the mixing of the geodesic flow), whereas the spectrum of the quantum evolution α^t on the space of Hilbert-Schmidt operators is discrete. We circumvent the spectral obstruction by intertwining the flows on different kinds of Hilbert spaces. As is well known, the spectrum of an operator depends on the space where it acts : our intertwining formula on the quotient will hold on a Hilbert space $\mathcal{H}_{PS}$ where the geodesic flow has discrete spectrum. And $\mathcal{L}$ will not be bounded on $\mathcal{H}_{PS}$, it will send $\mathcal{H}_{PS}$ to another Hilbert space $\mathcal{H}_W$ on which V^t has also discrete spectrum.

Circumventing the spectral obstruction is not new, but our particular way of doing it is new. In the theory of resonances for the geodesic flow, it is now standard to consider spectrum of the geodesic flow on spaces other than L^2. Instead, one finds some Banach spaces of distributions where the geodesic flow has some discrete spectrum (see the work of Ruelle and others, e.g. [BT, BKL, BL, FRS, GL, Liv, Ru87, Rugh92, Rugh96]). This approach was especially developed to describe the correlation spectrum of the geodesic flow for smooth (or Hölder) functions. The Hilbert spaces we construct are not the ones arising in the resonance theory, although they have some common elements : as recalled in [AZ], the distributions ϵ_{ν_j}

(1.10) are the generalized eigenfunctions arising in the resonance expansion of the geodesic flow, and they are also precisely the off-diagonal Patterson-Sullivan distributions $PS_{\nu_j,\frac{i}{2}}$, associated with the pairs of eigenfunctions ϕ_j and the constant function $\equiv 1$, with spectral parameters, respectively, ν_j and $\frac{i}{2}$. However, as far as the other off-diagonal Patterson-Sullivan distributions $PS_{\nu_j,-\nu_k}$ are concerned, we saw they are also generalized eigenfunctions of the geodesic flow, but they do not appear in the usual resonance theory.

In the next section, we shall construct a Hilbert space of distributions $\mathcal{H}_{PS}$, on which the geodesic flow acts, with dual eigenbasis the whole family $PS_{\nu_j,-\nu_k}$. In Definition 1, we introduced the Patterson-Sullivan distributions $PS_{\nu_j,-\nu_k} = PS_{(j,\nu_j),(k,-\nu_k)}$ associated with the pair of eigenfunctions (ϕ_j, ϕ_k) and the choice of spectral parameters $(\nu_j, -\nu_k)$. The space $\mathcal{H}_{PS}$ will be constructed by transposing the analysis done in §5, replacing the distributions $PS_{e_{(\nu,b)},e_{(-\nu',b')}}$ by the family $PS_{\nu_j,-\nu_k}$. A drawback is that elements of $\mathcal{H}_{PS}$ cannot be characterized in a simple way in terms of their regularity/growth properties (for instance, $\mathcal{H}_{PS}$ does not coincide with one of the known Sobolev spaces). However, we will give sufficient conditions for a distribution to belong to $\mathcal{H}_{PS}$ in Section 7.

As mentioned in the introduction, there is some flexibility in the definitions of $\mathcal{H}_W$, resp. $\mathcal{H}_{PS}$. In essence our norms are defined by the countable family of semi-norms $\{|W^{\Gamma}_{j,k}(a)|\}$, resp. $\{|PS_{\nu_j,-\nu_k}(a)|\}$. The interwining operator $\mathcal{L}$ takes one family to the other. Hence it is norm preserving if we define the Hilbert space norms or Banach space norms in terms of these semi-norms.

Remark 6.1. Although we prefer Hilbert spaces here, from the point of view of dynamics it might be of interest to consider the distributions

$$ps_{\nu_j,-\nu_k} := \frac{T_{j,\nu_j}(db)\overline{T_{k,-\nu_k}}(db')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} \quad \text{on} \quad B^{(2)},$$

and to define a Banach space by completing $C_0^\infty(B^{(2)})$ under the norm

$$||a||_{ps} = \sup_{(\nu_j,\nu_k)} |PS_{\nu_j,-\nu_k}(a)| = \sup_{\nu_j,\nu_k} \left| \int_{B\times B} \frac{a(b,b')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} T_{j,\nu_j}(db)\overline{T_{k,-\nu_k}}(db') \right|. \tag{6.2}$$

The semi-norms thus resemble 'Riesz norms' with respect to the distributions $T_{j,\nu_j}(db)\overline{T_{k,-\nu_k}}(db')$.

6.1. **$\mathcal{H}_W$ and Hilbert-Schmidt symbols.** The aim of the section is to present an *ad hoc* construction of two Hilbert spaces $\mathcal{H}_W$ and $\mathcal{H}_{PS}$, having respectively the families $(W_{\nu_j,-\nu_k})$ and $(PS_{\nu_j,-\nu_k})$ as dual orthonormal bases, with the following properties : V^t acts unitarily on $\mathcal{H}_W$, G^t acts unitarily on $\mathcal{H}_{PS}$, $\mathcal{L}$ sends $\mathcal{H}_W$ isometrically to $\mathcal{H}_{PS}$, and the intertwining relation (6.1) holds on these spaces. The two Hilbert space are modelled on the space of Hilbert-Schmidt symbols on the quotient, and the constructions mimicks what was done at the end of §5.6.

We start with the Hilbert space $\mathrm{HS}(\mathbf{X}_\Gamma) \simeq L^2(\mathbf{X}_\Gamma \times \mathbf{X}_\Gamma)$ of Hilbert-Schmidt operators on the compact quotient $\mathbf{X}_\Gamma = \Gamma\backslash G/K$. On this space, the quantum evolution α^t has the orthonormal spectral expansion

$$\alpha^t = \sum_{j,k} e^{it\frac{(\nu_j^2-\overline{\nu_k}^2)}{2}} (\phi_j \otimes \phi_k^*) \otimes (\phi_j \otimes \phi_k^*)^* \tag{6.3}$$

(here, $\nu_k^2 \in \mathbf{R}$, but we wrote $\overline{\nu_k}^2$ to underline the fact that the dependence of the expression is antiholomorphic w.r.t. ϕ_k, ν_k). The Hilbert-Schmidt norm is defined by $\|A\|^2_{\mathrm{HS}(\mathbf{X}_\Gamma)} = \mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}(AA^\dagger)$, associated with the scalar product $\langle A, B\rangle_{\mathrm{HS}(\mathbf{X}_\Gamma)} = \mathrm{Tr}(AB^\dagger)$. Starting with a Γ-invariant symbol a belonging to $\Pi\mathcal{S}^0_0$, we obtain a Hilbert-Schmidt operator $\mathrm{Op}_\Gamma(a) \in \mathrm{HS}(\mathbf{X}_\Gamma)$, with norm

$$\begin{aligned} \|\mathrm{Op}_\Gamma(a)\|^2_{\mathrm{HS}(\mathbf{X}_\Gamma)} &= \mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}\mathrm{Op}_\Gamma(a)\,\mathrm{Op}_\Gamma(a)^\dagger \\ = \sum_{j,k} |\mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}\mathrm{Op}_\Gamma(a)\phi_j\otimes\phi_k^*|^2 &= \sum_{j,k}|W^\Gamma_{j,k}(a)|^2. \end{aligned}$$

Here, $\mathrm{Tr}_{L^2(\mathbf{X}_\Gamma)}$ denotes the trace of a trace-class operator on $L^2(\mathbf{X}_\Gamma)$.

This suggests to define the Hilbert space $\mathcal{H}_W$ as follows:

Definition 6.1. $\mathcal{H}_W(\mathbf{X}_\Gamma)$ is the completion of the symbol space $\Pi\mathcal{S}^0_0$ with respect to the semi-norm

$$\|a\|^2_W = \sum |W^\Gamma_{j,k}(a)|^2.$$

Proposition 7.3 shows that indeed this series converges for $a \in \Pi\mathcal{S}^0_0$. The scalar product on $\mathcal{H}_W$ is defined by

$$\langle a, b\rangle_W = \sum W^\Gamma_{j,k}(a)\overline{W^\Gamma_{j,k}(b)}.$$

More generally, we may introduce a positive weight $\rho(\nu_j,\nu_k) \geq 0$ and define the weighted Hilbert space with inner product

$$\langle a, b\rangle_{W,\rho} = \sum \rho(\nu_j,\nu_k) W^\Gamma_{j,k}(a)\overline{W^\Gamma_{j,k}(b)}. \tag{6.4}$$

Since the proofs of the results are essentially the same in the weighted and unweighted cases, we only write them in the unweighted case for notational simplicity.

Here we have to stress two important facts :

- $\|.\|_W$ is only a semi-norm, because we can have $\mathrm{Op}_\Gamma(a) = 0$ although $a \neq 0$. The reader should not be too surprised by this fact, which already occurs in the euclidean case when one wants to study the Weyl quantization $\mathrm{Op}^W(a)$ of a symbol $a(x,\xi)$ (where $(x,\xi)\in T^*\mathbf{R}^d$), $2\pi\mathbf{Z}^d$-periodic in the x-variable. If $a \neq 0$, then $\mathrm{Op}^W(a)$ defines a non-vanishing operator on $L^2(\mathbf{R}^d)$, but the periodization of this operator may vanish when acting on the torus $L^2(\mathbf{R}^d/2\pi\mathbf{Z}^d)$. Actually, this happens if $a(x,\xi)$ vanishes when ξ is a half-integer. In the hyperbolic setting, a similar phenomenon occurs. Although the operator $\mathrm{Op}(a)$ is non-zero on $L^2(\mathbf{D})$, its periodization $\mathrm{Op}_\Gamma(a)$ can vanish when acting on $L^2(\mathbf{X}_\Gamma)$. The difficulty is that there no easy characterization of the symbols a such that $\mathrm{Op}_\Gamma(a) = 0$.
- Another related issue is the following. On the universal cover $\mathbf{D}$, we have seen that the Hilbert-Schmidt norm of $\mathrm{Op}(a)$ coincides with the L^2-norm of a, seen as a function on $S\mathbf{D}\times\mathbf{R}$ endowed with the measure $e^{\langle z,b\rangle}dzdb \times dp(r)$. This is no longer true on the quotient. In other words, for a Γ-invariant symbol a, $\|a\|_W$ is not the L^2-norm of a on $S\mathbf{X}_\Gamma\times\mathbf{R}$ (see §7.2). Again, the same phenomenon already occurs in the euclidean case.

By definition of $\mathcal{H}_W$, $W^\Gamma_{j,k}$ is a bounded linear functional on $\mathcal{H}_W$ for all j,k, in other words $W^\Gamma_{j,k}\in\mathcal{H}^*_W$, where $\mathcal{H}^*_W$ is the dual Hilbert space to $\mathcal{H}_W$. The Riesz theorem endows $\mathcal{H}^*_W$

with a dual inner product, and the $W^\Gamma_{j,k}$ form an orthonormal basis of $\mathcal{H}^*_W$. For $b \in \mathcal{H}_W$, the series

$$\sum_{j,k} \overline{W^\Gamma_{j,k}(b)} W^\Gamma_{j,k} \tag{6.5}$$

converges in $\mathcal{H}^*_W$. In fact, the operator $\sum_{j,k} \overline{W^\Gamma_{j,k}(\bullet)} W^\Gamma_{j,k} : \mathcal{H}_W \to \mathcal{H}^*_W$ is just the standard (antilinear, unitary) isomorphism $b \to \langle \bullet, b\rangle_W$ from $\mathcal{H}_W \to \mathcal{H}^*_W$.

As described earlier, we can define V^t_Γ acting on $\Pi\mathcal{S}^0_0$ by $V^t_\Gamma a = \Pi V^t \tilde{a}$, if $a = \Pi\tilde{a}$ and $\tilde{a} \in \mathcal{S}^0_0$. We then note that the evolution V^t_Γ can be extended to $\mathcal{H}_W$, and is obviously unitary, since we have $W^\Gamma_{j,k}(V^t_\Gamma a) = e^{it\frac{\nu_j^2 - \overline{\nu_k}^2}{2}} W^\Gamma_{j,k}(a)$ for $\Pi\mathcal{S}^0_0$ (this comes from (4.10) and (4.15) and holds as well for low eigenvalues, i.e. real ν_k or ν_j). The Wigner distributions form an orthonormal basis of eigenfunctions of the adjoint $V^t_{\Gamma\sharp}$ in $\mathcal{H}^*_W$: we have $V^t_{\Gamma\sharp}(W^\Gamma_{j,k}) = e^{it\frac{\nu_j^2 - \overline{\nu_k}^2}{2}} W^\Gamma_{j,k}$.

It is difficult to find a full characterization of elements in $\mathcal{H}_W$ in terms of usual Sobolev spaces. However, in §7, we shall give a sufficient condition for a function to belong to $\mathcal{H}_W$, in terms of its regularity and decay rate at infinity (in the r variable).

6.2. $\mathcal{H}_{PS}$**.** Next we define another Hilbert space $\mathcal{H}_{PS}$ involving Patterson-Sullivan distributions. Before we proceed to the construction, let us check the Γ-invariance of the Patterson-Sullivan distributions :

Proposition 6.1. *The distribution $PS_{\nu_j,-\nu_k}$ is Γ-invariant.*

Proof. Recall that

$$PS_{\nu_j,-\nu_k}(db', db, d\tau) = \frac{T_{\nu_j}(db)\overline{T_{-\nu_k}}(db')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} e^{(\nu_j+\overline{\nu_k})\tau} d\tau.$$

We use the formulae of Section 3, and the fact that $\gamma_\sharp^{-1} T_{\nu_j}(db) = e^{-(\frac{1}{2}+\nu_j)\langle\gamma\cdot o,\gamma\cdot b\rangle} T_{\nu_j}(db)$ for $\gamma \in \Gamma$.

$$\gamma_\sharp^{-1} PS_{\nu_j,-\nu_k}(db', db, d\tau) = \frac{\gamma_\sharp^{-1} T_{\nu_j}(db)\gamma_\sharp^{-1}\overline{T_{-\nu_k}}(b')}{|\gamma.b-\gamma.b'|^{1+\nu_j-\overline{\nu_k}}} e^{(\nu_j+\overline{\nu_k})(\tau+\frac{\langle\gamma o,\gamma b\rangle - \langle\gamma\cdot o,\gamma\cdot b'\rangle}{2})} d\tau$$

$$= e^{-(\frac{1}{2}+\nu_j)\langle\gamma\cdot o,\gamma\cdot b\rangle} e^{-(\frac{1}{2}-\overline{\nu_k})\langle\gamma\cdot o,\gamma\cdot b'\rangle} \frac{T_{\nu_j}(db)\overline{T_{-\nu_k}}(db')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} e^{\frac{1}{2}(1+\nu_j-\overline{\nu_k})(\langle\gamma\cdot o,\gamma\cdot b\rangle+\langle\gamma o\cdot,\gamma\cdot b'\rangle)} e^{(\nu_j+\overline{\nu_k})(\tau+\frac{\langle\gamma\cdot o,\gamma\cdot b\rangle-\langle\gamma\cdot o,\gamma\cdot b'\rangle}{2})} d\tau$$

$$= \frac{T_{\nu_j}(db)\overline{T_{-\nu_k}}(db')}{|b-b'|^{1+\nu_j-\overline{\nu_k}}} e^{(\nu_j+\overline{\nu_k})\tau} d\tau = PS_{\nu_j,-\nu_k}(db', db, d\tau).$$

□

Definition 6.2. We define $\mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ as the closure of $\Pi\mathcal{S}^0_0$ under the seminorm

$$\|f\|^2_{PS} \stackrel{\text{def}}{=} \sum_{j,k} |PS^\Gamma_{\nu_j,-\nu_k}(f)|^2. \tag{6.6}$$

Proposition 7.5 shows that indeed this series converges for $a \in \Pi\mathcal{S}^0_0$. The associated scalar product is

$$\langle f, g\rangle_{PS} \stackrel{\text{def}}{=} \sum_{j,k} PS^\Gamma_{\nu_j,-\nu_k}(f)\overline{PS^\Gamma_{\nu_j,-\nu_k}(g)}. \tag{6.7}$$

More generally, we introduce a weight $\rho(\nu_j, \nu_k)$ as in (6.4) and define

$$\langle f, g\rangle_{PS,\rho} \stackrel{\text{def}}{=} \sum_{j,k} \rho(\nu_j, \nu_k)\ PS^{\Gamma}_{\nu_j,-\nu_k}(f)\overline{PS^{\Gamma}_{\nu_j,-\nu_k}(g)}. \tag{6.8}$$

Here we have to choose a value of the spectral parameter ν_j for each j, and we make the standard choice : $\nu_j \in [0, \frac{1}{2}] \cup i\mathbf{R}_+$. Unlike our construction of $\mathcal{H}_W$, the definition of $\mathcal{H}_{PS}$ depends on the choice of spectral parameters. We then have $PS^{\Gamma}_{\nu_j,-\nu_k} \in \mathcal{H}^*_{PS}$, and $\sum_{j,k} \overline{PS^{\Gamma}_{\nu_j,-\nu_k}(\bullet)} PS^{\Gamma}_{\nu_j,-\nu_k} : \mathcal{H}_{PS} \to \mathcal{H}^*_{PS}$ is the standard (antilinear, unitary) isomorphism from $\mathcal{H}_{PS} \to \mathcal{H}^*_{PS}$.

6.3. **Proof of Theorem 4.** The theorem will be proved from Proposition 5.7 (we already mentioned that it holds as well for $\nu, \nu' \in \mathbf{C}$ and test functions in $\mathcal{S}^0_0$) and from the relations

$$W^{\Gamma}_{j,k}(a) = W_{j,k}(\tilde{a}) = \int W_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\tilde{a})dT_{\nu_j}(db)\overline{dT_{-\nu_k}}(db'),$$

$$PS^{\Gamma}_{\nu_j,-\nu_k}(a) = PS_{\nu_j,-\nu_k}(\tilde{a}) = \int PS_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\tilde{a})dT_{\nu_j}(db)\overline{dT_{-\nu_k}}(db'), \tag{6.9}$$

which are valid for any smooth and Γ-invariant a, and any $\tilde{a}$ such that $a = \Pi\tilde{a}$, provided $\tilde{a}$ is smooth and decays fast enough so that all terms are well-defined. Take $a \in \Pi\mathcal{S}^0_0$, and any $\tilde{a}$ such that $a = \Pi\tilde{a}$. We know from the calculations in §5 that

$$\begin{aligned} PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a}) &= \int PS_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\mathcal{L}\tilde{a})dT_{\nu_j}(db)\overline{dT_{-\nu_k}}(db') \\ &= \int W_{e_{(\nu_j,b)},e_{(-\nu_k,b')}}(\tilde{a})dT_{\nu_j}(db)\overline{dT_{-\nu_k}}(db') = W^{\Gamma}_{j,k}(a). \end{aligned}$$

This proves Theorem 4. Proposition 1.1 is proved the same way, by combining Proposition 5.3 on the universal cover, with formula (6.9).

6.4. **Proof of Theorem 3 .**

Definition 6.3. Define the operator $\mathcal{L}_{\Gamma\sharp} : \mathcal{H}^*_{PS}(\mathbf{X}_\Gamma) \to \mathcal{H}^*_W(\mathbf{X}_\Gamma)$ by stipulating that it takes $PS^{\Gamma}_{\nu_j,-\nu_k}$ to $W^{\Gamma}_{j,k}$. Define the operator $\mathcal{L}_\Gamma : \mathcal{H}_W(\mathbf{X}_\Gamma) \to \mathcal{H}_{PS}(\mathbf{X}_\Gamma)$ to be the adjoint of $\mathcal{L}_{\Gamma\sharp}$. More generally, we define the operator and its adjoint with respect to the weighted Hilbert spaces $\mathcal{H}^*_{PS,\rho}(\mathbf{X}_\Gamma) \to \mathcal{H}^*_{W,\rho}(\mathbf{X}_\Gamma)$.

To be precise, we describe the adjoint of $\mathcal{L}_{\Gamma\sharp}$ explicitly. Let $a \in \Pi\mathcal{S}^0_\alpha$, that is, $a = \Pi\tilde{a}$ with $\tilde{a} \in \mathcal{S}^0_\alpha$ (see Remark 5.3). Then by definition,

$$PS^{\Gamma}_{\nu_j,-\nu_k}(\mathcal{L}_\Gamma a) = PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a}), \tag{6.10}$$

and this definition does depend on the choice of $\tilde{a}$.

Remark 6.2. Since the kernel of $\mathcal{L}$ is not smooth and not rapidly decaying away from the diagonal, it seems difficult to define its action directly on Γ-invariant symbols. That explains why we defined the adjoint $\mathcal{L}_{\Gamma\sharp}$ first. Our definition of $\mathcal{L}_\Gamma(a)$ only used the action of $\mathcal{L}$ on $\tilde{a}$; we did not need to define the integral operator $\mathcal{L}$ directly on Γ-invariant symbols.

To prove Theorem 3, we start from the fact proved in Proposition 7.5 that $\Pi\mathcal{S}^0_\alpha$ is dense in $\Pi\mathcal{S}^0_0$ for the $\mathcal{H}_{PS}$-norm, hence is dense in $\mathcal{H}_{PS}$. From Remark 5.3, for $\tilde{a} \in \mathcal{S}^0_\alpha$, $G^t\mathcal{L}\tilde{a}(g,R)$ is well-defined for all $R \in \mathbf{C}$, and for all j,k,

$$\begin{aligned} PS_{\nu_j,-\nu_k}(G^t\mathcal{L}\tilde{a}) &= e^{it\frac{\nu_j^2-\overline{\nu_k}^2}{2}} PS_{\nu_j,-\nu_k}(\mathcal{L}\tilde{a}) \\ &= e^{it\frac{\nu_j^2-\overline{\nu_k}^2}{2}} W_{j,k}(\tilde{a}) \\ &= W_{j,k}(V^t\tilde{a}) = PS_{\nu_j,-\nu_k}(\mathcal{L}\circ V^t\tilde{a}). \end{aligned}$$

We can deduce the following :

- the geodesic flow G^t_Γ induces an isometry of $\mathcal{H}_{PS}$;
- $\mathcal{L}_{\Gamma\sharp}$ is an isometry from $\mathcal{H}^*_{PS}$ to $\mathcal{H}^*_W$, and $\mathcal{L}_\Gamma$ is an isometry from $\mathcal{H}_W$ to $\mathcal{H}_{PS}$.
- the family $PS^\Gamma_{\nu_j,-\nu_k}$ forms an independent family in $\mathcal{H}^*_{PS}$: if $\sum_{j,k}\alpha_{jk}PS^\Gamma_{\nu_j,-\nu_k} = 0$ with $\sum|\alpha_{jk}|^2 < +\infty$, then $\alpha_{jk} = 0$. This comes from the fact that the $W_{j,k}$ form an independent family in $\mathcal{H}^*_W$, and $\mathcal{L}_{\Gamma\sharp}PS^\Gamma_{\nu_j,-\nu_k} = W^\Gamma_{j,k}$. It follows that the family $PS^\Gamma_{\nu_j,-\nu_k}$ is an orthonormal basis of $\mathcal{H}^*_{PS}$.

By the Definition 6.3 and by (6.10), it follows then that

$$\mathcal{L}_\Gamma \circ V^t_\Gamma = G^t_\Gamma \circ \mathcal{L}_\Gamma,$$

as an equality between operators from $\mathcal{H}_{W,\rho}(\mathbf{X}_\Gamma)$ to $\mathcal{H}_{PS,\rho}(\mathbf{X}_\Gamma)$. This completes the proof of Theorem 3.

7. Further discussion about the Hilbert spaces $\mathcal{H}_{W,\rho}$ and $\mathcal{H}_{PS,\rho}$

We now discuss the elements of the Hilbert spaces $\mathcal{H}_W$ and $\mathcal{H}_{PS}$ in more detail : we describe sufficient conditions for a function to belong to $\mathcal{H}_W$ and $\mathcal{H}_{PS}$, in terms of regularity and decay. We use the regularity properties of the boundary values T_{ν_j} of eigenfunctions, described by Otal [O]. In the automorphic case, the regularity properties can be read off directly from the automorphy equation (see e.g. [MS, MS2]).

7.1. Hölder continuity of T_{ν_j}.

Following Otal [O], we say that a function F defined on $\mathbf{R}$ is 2π-periodic if there is a constant C such that $F(x+2\pi) = F(x) + C$, for all x. If F is locally integrable, its derivative DF yields a well-defined distribution on $S^1 = \mathbf{R}/2\pi\mathbf{Z}$,

$$DF(\varphi) = -\int_0^{2\pi} \frac{\partial\varphi}{\partial\theta} F(\theta)d\theta + \varphi(0)\left[F(2\pi) - F(0)\right],$$

for every smooth function φ on S^1. For $0 \leq \delta \leq 1$ we say that a 2π-periodic function $F : \mathbf{R} \to \mathbf{C}$ is δ-Hölder if $|F(\theta) - F(\theta')| \leq C|\theta-\theta'|^\delta$. The smallest constant is denoted $||F||_\delta$. We denote the Banach space of δ-Hölder functions with norm $||F||_\delta$ by Λ_δ.

We recall:

Theorem 7.1. *([O]) Suppose that ϕ is a laplacian eigenfunction of eigenvalue $-s(1-s) = -\left(\frac{1}{4}+r^2\right)$, with $s = \frac{1}{2}+ir$ and $\Re e(s) \geq 0$. Assume that $||\phi||_\infty < \infty$ and $||\nabla\phi||_\infty < \infty$. Then its Helgason boundary value $T_{s,\phi}$ is the derivative of a $\Re e(s)$-Hölder periodic function F.*

In addition, letting $\delta = \Re e(s)$, we have

$$\|F\|_\delta \leq \frac{C}{|C_s|}|s|\left(||\phi||_\infty + ||\nabla\phi||_\infty\right),$$

where $C > 0$ is an absolute constant, and $C_s = \int_0^{+\infty}\int_0^{2\pi} e^{-(1+s)t}P^s(\tanh\frac{t}{2},\theta)d\theta dt$; where P is the Poisson kernel of the unit disc.

Outside of the finite number of "small eigenvalues" of $\mathbf{X}_\Gamma$, we have $\Re e(s) = \frac{1}{2}$ and hence T_{ν_j} is the derivative of a Hölder 1/2-continuous function. The upper bound on $\|F\|_\delta$ given by Otal's proof is quite crude, but will be sufficient for our purposes.

The behavior of C_s for $s = \frac{1}{2} + ir$ and $r \longrightarrow \pm\infty$ can be evaluated by the stationary phase method. The calculation is routine but not completely straightforward because the domain of integration is non-compact. However, for the sake of brevity we omit the details. One finds that $C_s \sim Cr^{-1/2}$, with $C \neq 0$.

7.2. **$\mathcal{H}_W$ and $L^2_W(G\times\mathbf{R}, dg\times dp(r))$.** In this section, we clarify the relation between Hilbert-Schmidt inner product, which induces the inner product $\langle,\rangle_W$ on symbols, and $L^2_W(G \times \mathbf{R}, dg \times dp(r))$. The second term in the following proposition is the discrepancy between the $||\cdot||_W$ and the L^2 norm on symbols (again, we stress the fact that this discrepancy would also appear in a euclidean situation). We denote $\mathcal{D}$ a fundamental domain for the action of Γ on $\mathbf{D}$.

Proposition 7.2. *Let $a \in \Pi\mathcal{S}^0_0$. Then $||a||^2_W$ (see Definition 6.1) is given by*

$$\begin{aligned} ||a||^2_W \;&=\; \int_{\mathcal{D}}\int_B\int_{\mathbf{R}_+} |a(z,b,r)|^2 e^{\langle z,b\rangle}\,\mathrm{Vol}(dz)dp(r)db \\ &+ \sum_{\gamma\in\Gamma\setminus\{e\}}\int_{z\in\mathcal{D},(b,r)\in B\times\mathbf{R}_+} a(z,b,r)\overline{a(\gamma\cdot z,b,r)}e^{(\frac{1}{2}+ir)\langle z,b\rangle}e^{(\frac{1}{2}-ir)\langle\gamma\cdot z,b\rangle}dp(r)db\,\mathrm{Vol}(dz). \end{aligned}$$

Proof. We recall that $||a||^2_W = \mathrm{Op}_\Gamma(a)||^2_{HS(\mathbf{X}_\Gamma)}$ and that $K^\Gamma_a(z,w) = \sum_\gamma K_a(z,\gamma w)$. The kernel K_a is invariant by the diagonal action of $\Gamma :\ K_a(\gamma z,\gamma w) = K_a(z,w)$.

Unfolding the composition formula for the kernels, we have:

$$K^\Gamma_a \circ K^\Gamma_b(z,w) = \int_{\mathbf{D}} K_a(z,v)K^\Gamma_b(v,w)\,\mathrm{Vol}(dv).$$

Hence $K^\Gamma_a \circ K^{\Gamma\dagger}_b(z,w) = \int_{\mathbf{D}} K_a(z,v)\overline{K^\Gamma_b(w,v)}\,\mathrm{Vol}(dv)$. Taking the trace.

$$Tr(K^\Gamma_a \circ K^{\Gamma\dagger}_b) = \sum_{\gamma\in\Gamma}\int_{z\in\mathcal{D}}\int_{v\in\mathbf{D}} K_a(z,v)\overline{K_b(\gamma\cdot z,v)}\,\mathrm{Vol}(dv)\,\mathrm{Vol}(dz).$$

The rest of the calculation proceeds as in Proposition 4.1, using the Fourier inversion formula. The first term corresponds to $\gamma = e$ and the second term to $\gamma \neq e$.

□

7.3. **$\mathcal{H}_W$.** We now describe a large class of elements of the Hilbert space $\mathcal{H}_W$. That is, we determine sufficient conditions on a so that $\mathrm{Op}_\Gamma(a)$ is Hilbert-Schmidt, or in terms of Wigner distributions, so that

$$||a||^2_W = \sum_{j,k}|W^\Gamma_{j,k}(a)|^2 < \infty. \tag{7.1}$$

For brevity we restrict to the unity weight $\rho \equiv 1$. If we choose decaying weights such as $\rho(\nu_j, \nu_k) = \max\{\nu_j, \nu_k\}^{-r}$ then the Hilbert space becomes larger and simpler to describe.

In the following, $\langle x \rangle = (1 + |x|^2)^{1/2}$. If C is an operator, we define $ad(\triangle)C = [\triangle, C]$. We denote $\lambda_j = -\left(\frac{1}{4} + r_j^2\right)$ the laplacian eigenvalues. The following is of course not optimal, but gives an adequate idea of a large class of elements in $\mathcal{H}_W$. In the following, $||a_r||_{C^k}$ denotes the C^k norm of the function $a_r(z, b) = a(z, b, r)$ in the (z, b) variables.

Proposition 7.3. *There exists $C > 0$ such that, for all $a \in \Pi\mathcal{S}_0^0$,*

$$||a||_W \leq C \sup_j \langle r_j \rangle^6 ||a_{r_j}||_{C^6}.$$

Proof. It suffices to prove the following

Lemma 7.4. *Let Y be as in (2.2) and let*

$$|||a||| \stackrel{\text{def}}{=} \sup_j \sup_{(z,b)\in\mathcal{D}} \langle \lambda_j \rangle^2 \left[|(I - Y^2)\triangle_z^2 a| + r_j |(I - Y^2)\triangle_z \nabla_z a| + r_j^2 |(I - Y^2)\nabla_z^2 a| + r_j^2 |(I - Y^2)\nabla_z a| \right].$$

Then there exists $C > 0$ such that, for all $a \in \Pi\mathcal{S}_0^0$,

$$||a||_W \leq C|||a|||.$$

To prove this, we first note that, by Weyl's law, $\sum_{j,k} \langle \lambda_j \rangle^{-2} \langle \lambda_j - \lambda_k \rangle^{-2} < \infty$ in dimension two.

We will also use the expansion of ϵ_{ν_j} (1.10) into K-Fourier series, which takes the form

$$\epsilon_{\nu_j} = \sum_{m\in\mathbf{Z}} \phi_{j,m},$$

with $Y\phi_{j,m} = 2im\phi_{j,m}$. We use the fact that $\|\phi_{j,m}\|_{L^2(\Gamma\backslash G)} = 1$, proved in [Z1] (the full definition of $\phi_{j,m}$ can be found in Proposition 2.2 of [Z1], in particular, $\phi_{j,0} = \phi_j$).

Let us write $B = \mathrm{Op}(b) \stackrel{\text{def}}{=} (ad(\triangle)^2 \,\mathrm{Op}(a)) \circ \triangle^2$. Then,

$$\begin{aligned}
W_{j,k}^{\Gamma}(b) &= \textstyle\sum_{m\in\mathbf{Z}} \langle b_{r_j}, \phi_j \phi_{k,m} \rangle \\
&= \textstyle\sum_{m\in\mathbf{Z}} \langle 2m \rangle^{-2} \langle b_{r_j}, (I - Y^2)\phi_j \phi_{k,m} \rangle \\
&= \textstyle\sum_{m\in\mathbf{Z}} \langle 2m \rangle^{-2} \langle (I - Y^2) b_{r_j}, \phi_j \phi_{k,m} \rangle \\
&\leq \sup |(I - Y^2) b_{r_j}| \textstyle\sum_{m\in\mathbf{Z}} \langle 2m \rangle^{-2} \langle |\phi_j|, |\phi_{k,m}| \rangle \\
&\leq C \sup |(I - Y^2) b_{r_j}|,
\end{aligned}$$

where C is a uniform constant. Here we use that $\langle |\phi_j|, |\phi_{k,m}| \rangle \leq 1$ by the Schwartz inequality and the fact that $||\phi_j||_{L^2} = ||\phi_{k,m}||_{L^2} = 1$.

It follows that

$$
\begin{aligned}
|W^{\Gamma}_{j,k}(b)| \leq C\ \langle\lambda_j\rangle^{-2}\langle\lambda_j-\lambda_k\rangle^{-2} \sup|(I-Y^2)b_{r_j}| \\
\leq C\ \langle\lambda_j\rangle^{-2}\langle\lambda_j-\lambda_k\rangle^{-2} \\
\sup_{(z,b)\in\mathcal{D}} \langle\lambda_j\rangle^{2}[|(I-Y^2)\triangle_z^2 a_{r_j}| + r_j|(I-Y^2)\triangle_z\nabla_z a_{r_j}| \\
+ r_j^2|(I-Y^2)\nabla_z^2 a_{r_j}| + r_j^2|(I-Y^2)\nabla_z a_{r_j}|]
\end{aligned}
$$

Here, we use that the complete symbol of $\mathrm{Op}(a)\circ\triangle^2$ is $(\frac{1}{4}+r^2)^2 a(z,b,r)$. Further the complete symbol of $ad(\triangle)\,\mathrm{Op}(a)$ is given by $\triangle_z a + 2\left(\frac{1}{2}+ir\right)\nabla_z a.\nabla_z\langle z,b\rangle$. □

7.4. **The Hilbert space $\mathcal{H}_{PS}$.** We now consider the analogous question of conditions on a so that

$$
||a||^2_{PS} \stackrel{\text{def}}{=} \sum_{j,k} |PS^{\Gamma}_{\nu_j,-\nu_k}(a)|^2 < \infty. \tag{7.2}
$$

Again we restrict to weight $\rho\equiv 1$ for brevity. The discussion would be simpler again if we used decaying weight.

Proposition 7.5. *There exists $C>0$ such that, for all $a\in\Pi\mathcal{S}^0_0$,*

$$
||a||_{PS} \leq C \sup_{r,|\Im m(r)|\leq\frac{1}{2}} \langle r\rangle^{12}||a_r||_{C^3}.
$$

This follows from

Lemma 7.6. *We have, for any M,*

$$
\begin{aligned}
||a||^2_{PS} \leq \sum_{j,k} |\nu_j|^{3/2}|\nu_k|^{3/2}(\|\phi_j\|_\infty+\|\nabla\phi_j\|_\infty)(\|\phi_k\|_\infty+\|\nabla\phi_k\|_\infty)\langle\nu_j+\overline{\nu_k}\rangle^{-M} \\
\sup_{(b',b)\in B\times B}\Big[|b-b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b',b,i(\nu_j+\overline{\nu_k})), \\
\frac{\partial}{\partial b'}|b-b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b',b,i(\nu_j+\overline{\nu_k})), \\
\frac{\partial}{\partial b}|b-b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b',b,i(\nu_j+\overline{\nu_k})), \\
\frac{\partial^2}{\partial b\partial b'}|b-b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b',b,i(\nu_j+\overline{\nu_k}))\Big]
\end{aligned}
$$

Proof. We use the relation

$$
PS^{\Gamma}_{\nu_j,-\nu_k}(a) = \int \frac{1}{|b-b'|^{1+\nu_j-\overline{\nu_k}}}\mathcal{R}\chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b',b,i(\nu_j+\overline{\nu_k}))T_{\nu_j}(db)\overline{T_{-\nu_k}}(db'), \tag{7.3}
$$

which is obtained from (5.11). Since $\chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}$ is compactly supported on G, then the Radon-Fourier transform $\mathcal{R}a$ is compactly supported in the variables $(b',b)\in B^{(2)}$, so the singularity of $|b-b'|$ on the diagonal is not a problem. It follows by repeated integration by parts in ∂t that if $a\in C^M_c(G)$, then $\mathcal{R}a(b,b',i(\nu_j+\overline{\nu_k})) = O(\langle\nu_j+\overline{\nu_k}\rangle^{-M})$.

Let us call F_{ν_j} the Hölder function such that $T_{\nu_j} = F'_{\nu_j}$, in the sense of §7.1. We use the formula

$$\begin{aligned}\int \varphi(b', b)T_{\nu_j}(db)\overline{T_{-\nu_k}(db')} &= \varphi(0,0)[F_{\nu_j}(2\pi) - F_{\nu_j}(0)]\overline{[F_{-\nu_k}(2\pi) - F_{-\nu_k}(0)]}\\ &\quad -[F_{\nu_j}(2\pi) - F_{\nu_j}(0)]\int \frac{\partial}{\partial b'}\varphi(b', 0)\overline{F_{-\nu_k}}(b')db'\\ &\quad -\overline{[F_{-\nu_k}(2\pi) - F_{-\nu_k}(0)]}\int \frac{\partial}{\partial b}\varphi(0, b)F_{\nu_j}(b)db\\ &\quad +\int \frac{\partial^2}{\partial b\partial b'}\varphi(b', b)F_{\nu_j}(b)\overline{F_{-\nu_k}}(b')db\, db',\end{aligned}$$

valid for every smooth function φ on $B \times B$.

It follows that

$$\begin{aligned}&\left|\int \varphi(b', b)T_{\nu_j}(db)\overline{T_{-\nu_k}(db')}\right|\\ &\quad \le \|F_{\nu_j}\|_{\delta_j}\|F_{-\nu_k}\|_{\delta_k} \sup_{(b',b)\in B\times B}\left(|\varphi(b', b)|, |\frac{\partial}{\partial b'}\varphi(b', b)|, |\frac{\partial}{\partial b}\varphi(b', b)|, |\frac{\partial^2}{\partial b\partial b'}\varphi(b', b)|\right)\end{aligned}$$

where $\delta_j = \frac{1}{2} + \Re e(\nu_j)$, and the Hölder norm $\|.\|_\delta$ is the one appearing in Theorem 7.1.

We can then write

$$\begin{aligned}|PS_{\nu_j,-\nu_k}(a)| &\le \langle \nu_j + \overline{\nu_k}\rangle^{-M}\|F_{\nu_j}\|_{\delta_j}\|F_{-\nu_k}\|_{\delta_k}\\ &\quad \sup_{(b',b)\in B\times B}\Big[|b - b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M \chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b', b, i(\nu_j + \overline{\nu_k})),\\ &\quad \frac{\partial}{\partial b'}|b - b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M \chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b', b, i(\nu_j + \overline{\nu_k})),\\ &\quad \frac{\partial}{\partial b}|b - b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M \chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b', b, i(\nu_j + \overline{\nu_k})),\\ &\quad \frac{\partial^2}{\partial b\partial b'}|b - b'|^{-(1+\nu_j-\overline{\nu_k})}\mathcal{R}\langle\partial t\rangle^M \chi a_{\frac{\nu_j-\overline{\nu_k}}{2i}}(b', b, i(\nu_j + \overline{\nu_k}))\Big]\end{aligned}$$

By Theorem 7.1,

$$\|F_{\nu_j}\|_{\delta_j} = O(|\nu_j|^{3/2})(\|\phi_j\|_\infty + \|\nabla\phi_j\|_\infty).$$

Moreover, by the well-known local Weyl law estimates, $\|\phi_j\|_\infty = O(|\nu_j|^{\frac{1}{2}})$ and $\|\nabla\phi_j\|_\infty = O(|\nu_j|^{\frac{3}{2}})$. We find

$$|PS_{\nu_j,-\nu_k}(a)| \le \langle \nu_j + \overline{\nu_k}\rangle^{-M}|\nu_j|^3|\nu_k|^3 \max\left(|\nu_j|, |\nu_k|\right)\|a_{\frac{\nu_j-\overline{\nu_k}}{2i}}\|_{C^{M+1}}.$$

Using the Weyl law in dimension 2, $|\lambda_j| \sim Cj$, one sees that the series

$$\sum_{j,k}\langle r_j - r_k\rangle^{-M}|r_j|^3|r_k|^3 \max\left(|r_j|, |r_k|\right)\left\langle \frac{r_j + r_k}{2}\right\rangle^{-N}$$

converges for $M > 1$ and $N > 11$. The result follows.

□

We stress again the fact that there is nothing optimal in this upper bound.

8. Appendix

In this section we sketch the proof of Theorem 5. We closely follow the proof in Section 4 of [AZ].

By the generalized Poisson formula and the definition of $\mathrm{Op}(a)$,

(8.1)
$$\langle \mathrm{Op}_\Gamma(a)\phi_{ir_j}, \phi_{ir_k}\rangle = \int_{B\times B}\left(\int_{\mathbf{D}} \chi a(z,b) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} e^{(\frac{1}{2}+ir_k)\langle z,b'\rangle}\,\mathrm{Vol}(dz)\right) T_{ir_j}(db)\overline{T_{-ir_k}(db')}.$$

Here we are only interested in real values of r_j, r_k, since we consider the asymptotics $r_j \to +\infty$ and $|r_j - r_k|$ bounded. We apply stationary phase to the simplify the inner $\mathbf{D}$ integral. More precisely, in [AZ] and in this article, we rewrite the integral in the form

$$\langle \mathrm{Op}_\Gamma(a)\phi_j, \phi_k\rangle = 2^{(1+ir_j+ir_k)}\int L_{ir_k}\chi a(b', b, \tau) PS_{ir_j,-ir_k}(db', db, d\tau),$$

as was shown in Theorem 4, and we then replace $L_{ir_k}\chi a(b', b, \tau)$ by its expansion into powers of r_k^{-1}, obtained by the method of stationary phase.

There is one detail that we did not discuss in [AZ], and that was mentioned to us by Michael Schröder (see [SchDiss]). The PS-distributions have a singularity of the form $|b - b'|^{-(1+ir_j+ir_k)}$ on the diagonal ($b' = b$), and thus can only be integrated along functions that vanish on a neighbourhood of the diagonal. The function $L_{ir_k}\chi a$ does not satisfy this condition, and it is for a very special reason that its integral along $PS_{ir_j,-ir_k}$ can be defined : its singularity exactly cancels with $|b-b'|^{-(1+ir_j+ir_k)}$. However, when replacing $L_{ir_k}\chi a$ by its stationary phase expansion, one would have to justify the fact that each term, including the remainder term, can be integrated along $PS_{ir_j,-ir_k}$. This is not easy and wasn't discussed in [AZ].

It is actually simpler to carry out the localization step away from the diagonal with the original inner integral (8.1). We see that the function

$$\int_{\mathbf{D}} \chi a(z,b) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} e^{(\frac{1}{2}+ir_k)\langle z,b'\rangle}\,\mathrm{Vol}(dz) \tag{8.2}$$

is integrated against $T_{ir_j}(db)\overline{T_{-ir_k}(db')}$, and with the latter there is no issue on the diagonal.

The critical set in the oscillatory integral (8.2) occurs where $\nabla\langle z,b\rangle = -\nabla\langle z,b'\rangle$. So $z \in \gamma_{b',b}$. There is a neighbourhood V of the diagonal such that $\gamma_{b',b}$ does not intersect the support of χa for $(b', b) \in V$. We take a smooth function f on $B\times B$, supported in V, that is identically 1 on a neighbourhood of the diagonal, and divide the integral (8.1) into

$$\int_{B\times B} f(b',b)\left(\int_{\mathbf{D}} \chi a(z,b) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} e^{(\frac{1}{2}+ir_k)\langle z,b'\rangle}\,\mathrm{Vol}(dz)\right) T_{ir_j}(db)\overline{T_{-ir_k}(db')}$$
$$+ \int_{B\times B}(1-f(b',b))\left(\int_{\mathbf{D}} \chi a(z,b) e^{(\frac{1}{2}+ir_j)\langle z,b\rangle} e^{(\frac{1}{2}+ir_k)\langle z,b'\rangle}\,\mathrm{Vol}(dz)\right) T_{ir_j}(db)\overline{T_{-ir_k}(db')}.$$

For the first term, the phase has no critical point, and we integrate by parts using

$$\frac{1}{|\nabla_z\langle z,b\rangle - \nabla_z\langle z,b'\rangle|^2}\left(\nabla_z\langle z,b\rangle - \nabla_z\langle z,b'\rangle\right)\cdot\nabla.$$

Since T_{ir_j}, T_{-ir_k} have polynomial bounds in r_j, r_k, repeated partial integration shows that this first integral is $O(\langle r_k \rangle^{-\infty})$.

The second term, because of the cut-off $(1 - f(b', b))$, is now supported away from the diagonal, and can be rewritten as $2^{(1+ir_j+ir_k)} \int (1 - f(b', b)) L_{ir_k} \chi a(b', b, \tau) PS_{ir_j,-ir_k}(db', db, d\tau)$. The proof of Section 4 in [AZ] now applies without problem.

References

[AN] N. Anantharaman and S. Nonnenmacher, Half-delocalization of eigenfunctions for the Laplacian on an Anosov manifold, Annales Inst. Fourier 57, number 6 (2007), 2465-2523.

[AZ] N. Anantharaman and S. Zelditch, Quantum ergodicity and Patterson-Sullivan distributions, Annales Henri Poincaré Volume 8, Number 2, 361-426 (2007).

[B] V. Baladi, Periodic orbits and dynamical spectra. Ergodic Theory Dynam. Systems 18 (1998), no. 2, 255–292.

[BT] V. Baladi and M. Tsuji, Anisotropic Hölder and Sobolev spaces for hyperbolic diffeomorphisms, Ann. Inst. Fourier (Grenoble) 57 (2007), no. 1, 127 - 154.

[BR] J. Bernstein and A. Reznikov, Analytic continuation of representations and estimates of automorphic forms, Ann. of Math. 150 (1999), 329–352.

[BR2] J. Bernstein and A. Reznikov, Estimates of automorphic functions. Mosc. Math. J. 4 (2004), no. 1, 19–37, 310.

[Bis] J. M. Bismut, The hypoelliptic laplacian and orbital integrals, preprint 2009.

[BKL] M. Blank, G. Keller, and C. Liverani, Ruelle-Perron-Frobenius spectrum for Anosov maps. Nonlinearity 15 (2002), no. 6, 1905–1973.

[BL] O. Butterley and C. Liverani , Smooth Anosov flows: correlation spectra and stability, J. Modern Dynamics, *1*, 2, (2007) 301?322.

[D] Deitmar, Anton Invariant triple products. Int. J. Math. Math. Sci. 2006, Art. ID 48274, 22 pp

[Eg74] Eguchi, M. *The Fourier transform of the Schwartz space on a semisimple Lie group.* Hiroshima Math. J. 4 (1974), 133–209.

[Eg79] Eguchi, M. *Asymptotic expansions of Eisenstein integrals and Fourier transform on symmetric spaces.* J. Funct. Anal. 34 (1979), no. 2, 167–216.

[FRS] F. Faure, N. Roy, and J. Sjöstrand, Semi-classical approach for Anosov diffeomorphisms and Ruelle resonances. Open Math. J. 1 (2008), 35–81.

[FP] M. Feingold and A. Peres, Distribution of matrix elements of chaotic systems. Phys. Rev. A (3) 34 (1986), no. 1, 591–595.

[F] G. B. Folland, *Harmonic Analysis in Phase Space*, Ann. Math. Studies 122, Princeton U. Press, Princeton, N.J. 1989.

[GGP] I. M. Gelfand, M. I. Graev and I. I. Pyatetskii-Shapiro, *Representation theory and automorphic functions.* W. B. Saunders Co., Philadelphia, Pa.-London-Toronto, Ont. 1969.

[GL] S. Gouëzel and C. Liverani, Banach spaces adapted to Anosov systems. Ergodic Theory Dynam. Systems 26 (2006), no. 1, 189–217.

[G] V. Guillemin, Lectures on spectral theory of elliptic operators. Duke Math. J. 44 (1977), no. 3, 485–517.

[HC66] Harish-Chandra, *Discrete series for semisimple Lie groups.* II. Explicit determination of the characters. Acta Math. 116 1966 1–111.

[He] S. Helgason, *Topics in harmonic analysis on homogeneous spaces.* Progress in Mathematics, 13. Birkhäuser, Boston, Mass., 1981.

[He2] S. Helgason, *Groups and geometric analysis. Integral geometry, invariant differential operators, and spherical functions.* Corrected reprint of the 1984 original. Mathematical Surveys and Monographs, 83. American Mathematical Society, Providence, RI, 2000.

[HilSc] J. Hilgert, M. Schröder, *PattersonSullivan distributions for rank one symmetric spaces of the noncompact type*, http://arxiv.org/abs/0909.2142

[Ho I] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volume I. Distribution theory and Fourier analysis.* Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 256. Springer-Verlag, Berlin, 1990.

[Ho III] L. Hörmander, *The Analysis of Linear Partial Differential Operators, Volume III.* Pseudo-differential operators. Classics in Mathematics. Springer, Berlin, 2007.

[L] S. Lang, $\mathrm{SL}_2(R)$. Reprint of the 1975 edition. Graduate Texts in Mathematics, 105. Springer-Verlag, New York, 1985.

[Liv] Liverani, C. Fredholm determinants, Anosov maps and Ruelle resonances. Preprint 2005.

[LS] W.Luo and P.Sarnak, Quantum variance for Hecke eigenforms, Annales Scient. de l'École Norm. Sup. 37 (2004), p. 769-799.

[M] D. M. Mayer, The thermodynamic formalism approach to Selberg's zeta function for $\mathrm{PSL}(2, Z)$. Bull. Amer. Math. Soc. (N.S.) 25 (1991), no. 1, 55–60.

[MS] S.D. Miller and W. Schmid, Automorphic distributions, L-functions, and Voronoi summation for GL(3). Ann. of Math. (2) 164 (2006), no. 2, 423–488.

[MS2] S. D. Miller and W. Schmid, The Rankin-Selberg method for automorphic distributions. Representation theory and automorphic forms, 111–150, Progr. Math., 255, Birkhuser Boston, Boston, MA, 2008.

[N] P.J. Nicholls, *The Ergodic Theory of Discrete Groups*, London Math. Soc. Lect. Notes Series 143, Cambridge Univ. Press, Cambridge 143.

[O] J.P. Otal, Sur les fonctions propres du laplacien du disque hyperbolique. (French. English, French summary) [About eigenfunctions of the laplacian on the hyperbolic disc] C. R. Acad. Sci. Paris Sér. I Math. 327 (1998), no. 2, 161–166.

[Pol] M. Pollicott, Formulae for residues of Dynamical Zeta functions (notes, posted on http://www.warwick.ac.uk/ masdbl/preprints.html).

[Po2] M. Pollicott, Some applications of thermodynamic formalism to manifolds with constant negative curvature. Adv. Math. 85 (1991), no. 2, 161–192.

[Ru87] D. Ruelle, Resonances for Axiom A flows. J. Differential Geom. 25 (1987), no. 1, 99–116.

[Rugh92] H.H. Rugh, The correlation spectrum for hyperbolic analytic maps. Nonlinearity 5 (1992), no. 6, 1237–1263.

[Rugh96] H. H. Rugh, Generalized Fredholm determinants and Selberg zeta functions for Axiom A dynamical systems. Ergodic Theory Dynam. Systems 16 (1996), no. 4, 805–819.

[SZ] P. Sarnak and P. Zhao, in preparation.

[SchDiss] M. Schröder, Patterson-Sullivan distributions for symmetric spaces of the noncompact type (PhD thesis, http://ubdok.uni-paderborn.de/servlets/DocumentServlet?id=12308).

[SV] L. Silberman and A. Venkatesh, On Quantum unique ergodicity for locally symmetric spaces I, Geom. Funct. Anal. 17 (2007), no. 3, 960–998.(math.RT/0407413).

[SV2] L. Silberman and A. Venkatesh, Entropy bounds for Hecke eigenfunctions on division algebras, to appear in GAFA.

[Sog] C. D. Sogge, Fourier integrals in classical analysis. Cambridge Tracts in Mathematics, 105. Cambridge University Press, Cambridge, 1993.

[St] E. M. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals.* With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[W] S. A. Wolpert, Semiclassical limits for the hyperbolic plane. Duke Math. J. 108 (2001), no. 3, 449–509.

[Z1] S. Zelditch, Uniform distribution of eigenfunctions on compact hyperbolic surfaces, Duke Math. J. 55 (1987), 919–941.

[Z2] S. Zelditch, Quantum transition amplitudes for ergodic and for completely integrable systems. J. Funct. Anal. 94 (1990), no. 2, 415–436.

[Z3] S. Zelditch, Pseudodifferential analysis on hyperbolic surfaces. J. Funct. Anal. 68 (1986), no. 1, 72–105.

[Z4] S. Zelditch, Patterson-Sullivan distributions and invariant trilinear functionals (in preparation).

[Zh] P. Zhao, Quantum variance of Maass-Hecke cusp forms, PhD Dissertation, Ohio State (2009).

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