PHYSICAL SOLUTIONS OF THE HAMILTON-JACOBI EQUATION

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Abstract. We consider a Lagrangian system on the d-dimensional torus, and the associated Hamilton-Jacobi equation. Assuming that the Aubry set of the system consists in a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow, we study the vanishing-viscosity limit, from the viscous equation to the inviscid problem. Under suitable assumptions, we show that solutions of the viscous Hamilton-Jacobi equation converge to a unique solution of the inviscid problem.

1. Introduction

Let L be a strictly convex and superlinear Lagrangian of class C^3 on the d-dimensional torus $\mathbb{T}^d$, and let H be the associated Hamiltonian via the Legendre transformation:

$$L : \mathbb{T}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$$

and

$$H : \mathbb{T}^d \times \mathbb{R}^d \longrightarrow \mathbb{R},$$

$$H(x, p) = p.v - L(x, v)$$

with $p = \frac{\partial L}{\partial v}(x, v)$ or equivalently $v = \frac{\partial H}{\partial p}(x, p)$.

There is only one constant c such that the Hamilton-Jacobi equation

$$H(x, D\phi(x)) = c. \tag{1}$$

has solutions $\phi : \mathbb{T}^d \longrightarrow \mathbb{R}$. The constant is called "Mañé's critical value" ([21], [22], [9]) or "the effective Hamiltonian" in [8] (see also [5]). To get the existence of solutions, one should not consider classical (that is, C^1) solutions but "viscosity solutions", a notion defined by Crandall, Evans and Lions (see [6], and Section 2). However there might, in general, be several such solutions: obviously, they are defined up to additive constants, but there may also exist different solutions which do not differ by a constant. Roughly speaking, the family of solutions is parametrized by the values taken at the static classes (see [3], and Section 2).

Let us now consider the viscous Hamilton-Jacobi equation:

$$H(x, D\phi(x)) + \varepsilon \triangle \phi(x) = c(\varepsilon) \tag{2}$$

where the torus is equipped with a flat metric and $\triangle$ is the Laplacian. The parameter ε is called the viscosity coefficient. As in the inviscid case, there is only one constant, $c(\varepsilon)$, such that the viscous Hamilton-Jacobi equation admits solutions. However, in the viscous case, the solution is unique, up to an additive constant; we denote it ϕ_ε (see [17]).

Remark 1. In the "mechanical case", namely $L(x, v) = \frac{|v|^2}{2} - F(x)$ or equivalently $H(x, p) = \frac{|p|^2}{2} + F(x)$, this can be seen by using the relation with the Schrödinger equation. In fact, if ϕ_ε is a solution of

$$\frac{1}{2}|D\phi(x)|^2 + F(x) + \varepsilon \triangle \phi(x) = c(\varepsilon), \tag{3}$$

then $\exp(\phi_\varepsilon/2\varepsilon)$ is an eigenfunction of the Schrödinger operator $2\varepsilon^2 \triangle + F(x)$ with eigenvalue $c(\varepsilon)$ (note the unusual sign in front of the Laplacian). Since the eigenfunction is positive, it has to be the ground state, and the associated eigenvalue $c(\varepsilon)$ is simple (see for example [7], Chapter 6.5).

We study the behaviour of ϕ_ε as ε tends to zero. It is a classical fact that the family $(\phi_\varepsilon)_{\varepsilon>0}$ is equilipschitz, so that we can extract subsequences which converge uniformly (see Lemma 2). By the stability theorem for viscosity solutions ([6]), limits as $\varepsilon \to 0$ of such subsequences have to be solutions of the non-viscous equation (1). We call such solutions of (1) "physical solutions", because they are obtained from the viscous equation by passing to the limit. It is then natural to ask: in the case of non-uniqueness of solutions of (1), is there a unique "physical solution" ?

We prove that this is the case, assuming that the "Aubry set" of the Lagrangian system (defined in Section 2) consists in a finite number of hyperbolic periodic orbits of the Euler-Lagrange flow, and a condition on the second derivative to be precisely stated below. Under the name "periodic orbits" we also allow fixed points of the flow. As a particular example, the result applies to a mechanical Lagrangian $L = \frac{1}{2}v^2 - F(x)$, under the assumptions that:

– F reaches its global maximum at a finite number of points $(x_i)_{1\le i\le m}$, and the maxima are non-degenerate;

– if we call $(-k_j(x_i))_{1\le j\le d}$ the eigenvalues of the Hessian of F at x_i, there is only one x_I that minimizes $\sum_{j=1}^{d} \sqrt{k_j(x_i)}$.

In fact, we find an expression of the limit of (ϕ_ε) in terms of x_I and a function called the "Peierls barrier".

Our result is a generalization from the one dimensional case to the d-dimensional case of the results of Jauslin, Kreiss and Moser [20], as well as of Bessi [2]. Our method is very close to the one in [2]: we use a variational representation formula for ϕ_ε, which is a stochastic version of the "Lax-Oleinik formula" used to represent solutions of the non-viscous equation. Then, using standard techniques in stochastic calculus, we estimate the limiting behavior, which allows us to obtain the same conclusion in higher dimensions. See Section 2 for a more detailed comparison between our assumptions and those of [2].

In [1] a closely related problem is addressed, in the case of a mechanical Lagrangian (possibly with a magnetic term): namely, the convergence, as $\varepsilon \to 0$, of the probability measure $\exp(\phi_\varepsilon/\varepsilon)dx$ defined by the eigenfunction of the Schrödinger operator. This certainly implies some restrictions on the behaviour of ϕ_ε. Yet, these restrictions do not allow to deduce the convergence of ϕ_ε towards a single solution of the inviscid equation.

In Section 2 we give the main definitions and assumptions, and a more detailed statement of the result. The theorem is then proved in Section 3.

2. Preliminaries and Statement

2.1. **Viscosity Solutions.** It is well known that there do not necessarily exist global C^1 solutions of equation (1). The appropriate sense of a weak solution is the notion of viscosity solution, introduced in [6]:

Definition 1. A continuous function $\phi : \mathbb{T}^d \to \mathbb{R}$ is called a *viscosity solution* of equation (1) if it satisfies the two properties:

(1) If v is a C^1 function and $\phi - v$ has a local maximum at x then
$$H(x, Dv(x)) \geq c,$$
(2) If v is a C^1 function and $\phi - v$ has a local minimum at x then
$$H(x, Dv(x)) \leq c.$$

Under our assumptions on H, viscosity solutions of (1) are lipschitz (Lemma 2), and exist for a unique value of c ([9]). See also [5]. Besides, a function v is a viscosity solution of (1) if and only if it solves the following fixed point problem: for all $x \in \mathbb{T}^d$ and for all $t \geq 0$,

$$v(x) = \sup_{\gamma:[0,t]\to\mathbb{T}^d,\gamma(0)=x} \{v(\gamma(t)) - \int_0^t L(\gamma(s), \dot\gamma(s))ds\} - ct, \qquad (4)$$

where the sup is taken over all piecewise C^1 curves $\gamma : [0,t] \longrightarrow \mathbb{T}^d$ such that $\gamma(0) = x$. This is called the Hopf-Lax or Lax-Oleinik formula ([9], [7]).

Remark 2. One could also define the notion of viscosity solution for the viscous equation (2), however, in this case, one would obtain nothing more than the classical solution, which is unique up to an additive constant. It is a classical fact that the family $(\phi_\varepsilon)_{\varepsilon>0}$ is equilipschitz, we will provide a proof in Section 2.3. (Lemma 2). A nice feature of the notion of viscosity solution is the stability theorem ([6]), which says that limits of $(\phi_\varepsilon)_{\varepsilon\longrightarrow 0}$ in the uniform topology have to be solutions of the non-viscous equation (1).

Remark 3. There are actually two types of viscosity solutions. In this paper we consider *forward* viscosity solutions, defined above. These solutions are semiconvex, can have non-differentiable minima but not maxima. In the non-autonomous case they are solutions of the "final value problem" (4). By regularizing the Hamilton-Jacobi equation as in (2) and taking the vanishing-viscosity limit, one obtains *forward* viscosity solutions of (1). More often, people consider *backward* viscosity solutions. The latter are defined by reversing both inequalities in Definition 1. This time the solutions are semiconcave, can have non-differentiable maxima but not minima; they are solutions of the "initial value problem", that is, a variational characterization similar to (4), but with $\gamma(t) = x$ instead of $\gamma(0)$. If we regularize the Hamilton-Jacobi equation by subtracting (instead of adding) an elliptic operator term, we obtain in the vanishing-viscosity limit *backwards* viscosity solutions. We will state Theorem 1 in terms of forward solutions, but obviously an analogous result holds with backwards solutions.

2.2. **Aubry set and static classes.** The constant c in equation (1) can be characterized as $\alpha(0)$ where α is Mather's function (see [23], [4], [22]):

$$c = \alpha(0) = -\inf_{\nu}\{\int_{\mathbb{T}^d\times\mathbb{R}^d} L(x,v)d\nu(x,v)\},$$

where the inf is taken over the set of probability measures ν on $\mathbb{T}^d \times \mathbb{R}^d$ which are invariant under the Euler-Lagrange flow of L.

We recall the definition of the Peierls Barrier ([10]) $h : \mathbb{T}^d \times \mathbb{T}^d \to \mathbb{R}$. Define the action of a piecewise C^1 curve $\gamma : [0,T] \to M$ as

$$A(\gamma) = \int_0^T L(\gamma(s), \dot\gamma(s))ds.$$

Given a constant $k \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{T}^d$ let

$$h_T^k(x_1, x_2) \quad = \quad \inf\{A(\gamma) + kT | \gamma : [0,T] \to \mathbb{T}^d \text{ joins } x_1 \text{ and } x_2\},$$

and

$$h^k(x_1, x_2) \quad = \quad \liminf_{T\to\infty} h_T^k(x_1, x_2).$$

Since time T is not bounded, there is only one possible value of k that will make the function h^k different from being identically $-\infty$ or ∞, this is again Mañé's critical value c. We define $h_T = h_T^c$ and $h = h^c$ (it is shown in [12] that, in the case of an autonomous system, h_T actually converges uniformly to h). Note that, given a fixed $y \in \mathbb{T}^d$, the function $x \mapsto -h(x, y)$ is a forward viscosity solution of (1), whereas $x \mapsto h(y, x)$ is a backward solution.

We now define as in [11] the Aubry set $\mathcal{A} \subset \mathbb{T}^d$:

$$\mathcal{A} = \{x \in \mathbb{T}^d, h(x, x) = 0\}.$$

(in the reference [11] it was called the Peierls set.)

Very related to Mather's graph theorem ([23]), it is shown in [11], that the set $\mathcal{A}$ can be lifted, in a unique way, to a set $\tilde{\mathcal{A}} \subset \mathbb{T}^d \times \mathbb{R}^d$. This set projects homeomorphically to $\mathcal{A}$ through the usual projection from $\mathbb{T}^d \times \mathbb{R}^d$ to $\mathbb{T}^d$, and is invariant under the Euler-Lagrange flow. We also call the set $\tilde{\mathcal{A}}$ "Aubry set". See also [4] for some other graph properties.

An crucial property of viscosity solutions (both forward and backward) with respect to the Aubry set is the following: if (x_0, v_0) is an element of $\tilde{\mathcal{A}}$, and $(x_t, v_t)_{t\in\mathbb{R}}$ is its orbit under the Euler-Lagrange flow, then

$$u(x_T) - u(x_0) = \int_0^T L(x_t, v_t)dt + cT \tag{5}$$

for all T, if u is a viscosity solution of (1) (see [9], [13]). Note also that, because of (4), one always have the inequality $u(x_T) - u(x_0) \leq \int_0^T L(x_t, \dot{x}_t)dt + cT$ for an arbitrary curve (x_t).

The "static classes" form a partition of $\mathcal{A}$, defined by the equivalence relation on $\mathcal{A}$: $x \sim y$ if and only if

$$h(x, y) + h(y, x) = 0.$$

In the paper we assume that the Aubry set $\tilde{\mathcal{A}}$ is made up of a finite union of hyperbolic periodic orbits of the Euler-Lagrange flow. This implies in particular that each static class is a periodic orbit.

Viscosity solutions are completely determined by one value taken in each static class, as shown in [3]:

Denote the static classes $S_i, 1 \leq i \leq m$ and choose one point x_i in each static class. For each $i \in [1, m]$, assign a value $\phi_i \in \mathbb{R}$. Because of the general properties recalled above, if there exists a viscosity solution $\phi : \mathbb{T}^d \longrightarrow \mathbb{R}$ of (1) such that $\phi(x_i) = \phi_i$ for all $i \in [1, m]$, we must have $\phi_j - \phi_i \leq h(x_i, x_j)$ for all i, j. Conversely, if this necessary condition is satisfied, then there is a unique viscosity ϕ solution of the Hamilton-Jacobi equation having these prescribed values. In fact it is given by

$$\phi(x) = \max_i \phi_i - h(x, x_i). \tag{6}$$

2.3. **Main result.** In the paper we assume that the Aubry set $\tilde{\mathcal{A}}$ is made up of a finite union of hyperbolic periodic orbits of the Euler-Lagrange flow. This implies in particular that each static class is the projection of a periodic orbit, say, $\gamma_i : [0, T_i] \to \mathbb{T}^d$, $i = 1, \ldots, m$. In all the paper we use a slightly abusive notation: we use the same symbol γ_i to denote the parametrized curve in $\mathbb{T}^d$, its image, which is a subset of the torus, and its lift to the tangent bundle, which is a periodic orbit of the Euler-Lagrange flow.

Fix a point x_i in each static class, for instance $x_i = \gamma_i(0)$, and let $h_i(x) = h(x, x_i)$ (by the properties of viscosity solutions stated in paragraph 2.2, replacing x_i by another point in the same static class would only modify h_i by an additive constant). Because the periodic orbits composing the Aubry set are hyperbolic, we prove below that h_i is C^2 in a neighbourhood of γ_i, and so we may define

$$\lambda_i := \frac{1}{T_i} \int_0^{T_i} \triangle h_i(\gamma_i(t)) dt$$

(In the case of a fixed point of the flow, that is, $T_i = 0$, we let $\lambda_i := \triangle h_i(\gamma_i(x_i))$.) We assume that there is exactly one static class γ_I such that

$$\lambda_I = \min_{1 \leq i \leq m} \lambda_i \tag{7}$$

Finally, we assume that $\partial_x H / |p| + 1$ or equivalently $\partial_x L / |\partial_v L| + 1$ is uniformly bounded. This will be needed only in Lemma 2, and may in fact not be necessary.

Theorem 1. *Under the previous assumptions, the solution ϕ_ε of* (2)*, normalized by $\phi_\varepsilon(x_I) = 0$, converge uniformly to $-h_I$ as $\varepsilon \to 0$.*

Remark 4. As already mentioned, this result is a generalization to higher dimension (but in the autonomous case) of Theorem 1 in [2]. In that paper, the assumption that the Aubry set consists in a finite union of hyperbolic periodic orbits is expressed in a slightly different form, namely:

– the rotation number is rational, and the periodic orbits forming the Mather set are hyperbolic;

– there is no heteroclinic cycle "of zero action" between these periodic orbits (note that, in the low-dimensional case treated by Bessi, the existence of such a cycle implies, anyway, the uniqueness of the solution of (1), so that the problem is particularly easy if the assumption is not satisfied. Such a phenomenon does no longer occur in higher dimension).

To conclude, Bessi needs some combinatorial considerations only valid in the one dimensional case. Instead, we use a characterization of the solutions given in [3] in the inviscid case, and a result of Fathi and Siconolfi [14].

Remark 5. The result in the non autonomous case should be very similar.

Remark 6. The results of this paper can be proved, with minor changes, in any riemannian compact manifold, or replacing the laplacian $\triangle$ by any second-order elliptic operator (of course, the definition of the λ_is has to be modified accordingly).

Remark 7. We have no idea on how to adapt the result to less restrictive assumptions on the nature of the Aubry set. All our arguments break down when the Aubry set has some (transversally) non-isolated points.

Example. Mechanical Lagrangian.

Let $L = \frac{1}{2}v^2 - F(x)$, where F has a finite number of maxima $(x_i)_{1\leq i\leq m}$, which are all non-degenerate. In this case the static classes are the points x_i, and $c = \max F$. We will assume that there is one $I \in [1, m]$ such that

$$\sum_j \sqrt{k_j(x_i)} > \sum_j \sqrt{k_j(x_I)}, \quad i \neq I, \tag{8}$$

where $-k_j(x_i), j = 1, \dots, d$ are the eigenvalues of the Hessian of F at x_i.

Lemma 1. *If ϕ is a viscosity solution of the Hamilton Jacobi equation*

$$\frac{1}{2}|D\phi|^2 + F(x) = c \tag{9}$$

that has a local maximum at x_i, then ϕ is C^3 in a neighbourhood of x_i and the eigenvalues of $\operatorname{Hess}\phi(x_i)$ *are $-\sqrt{k_j(x_i)}, j = 1, \dots d$.*

Proof. Since ϕ has a maximum, using the constant function $v = \phi(x_i)$ we obtain from the viscosity inequality, $\frac{1}{2}|Dv|^2 + F(x_i) \geq c$. Thus x_i is a maximum of F. As in Lemma 5 below, there exists a neighbourhood V of x_i such that for any $x \in V$, the point $(x, D\phi(x))$ belongs to the stable manifold of $(x_i, 0)$ and so $\phi|V$ is as differentiable as F; and ϕ coincides with $-h(., x_i)$ on this neighbourhood. Differentiating (9) twice and evaluating at x_i

$$\begin{aligned} \operatorname{Hess}\phi \cdot D\phi &= -DF \\ \operatorname{Hess}\phi(x_i)\operatorname{Hess}\phi(x_i) &= -\operatorname{Hess}F(x_i). \end{aligned}$$

If the orthonormal basis $\{Y_j\}$ diagonalizes $\operatorname{Hess}\phi(x_i)$, also diagonalizes $\operatorname{Hess}F(x_i)$ and the eigenvalues of $\operatorname{Hess}\phi(x_i)$ are $-\sqrt{k_j(x_i)}$. ☐

Corollary 1. *Let ϕ_ε be the solution of* (3) *such that $\phi_\varepsilon(x_I) = 0$. Then ϕ_ε converges to $-h(., x_I)$, where x_I is the point satisfying* (8).

Remark 8. In the mechanical case, the problem has already been widely studied, in relation with the tunneling effect, and usually with WKB techniques ([18]). What is proved in [18], Chapter 4, is the following: the normalized eigenfunction of the Schrödinger operator $\frac{\exp(\phi_\varepsilon/2\varepsilon)}{||\exp(\phi_\varepsilon/2\varepsilon)||_{L^2}}$ is exponentially close, in L^2-norm, to $\frac{\chi\exp(-h(.,x_I)/2\varepsilon)}{||\chi\exp(-h(.,x_I)/2\varepsilon)||_{L^2}}$ where χ is a smooth function localized in a small neighbourhood of x_I. Our result is different in two aspects: the topology is not the same, and we do not localize things in a neighbourhood of x_I.

2.4. **More Preliminaries: Stochastic Lax Formula and estimates.** The solution to the viscous equation (2) can be characterized by a variational formula analogous to (4). In the viscous case, we need to introduce a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ endowed with a brownian motion $W(t) : \Omega \longrightarrow \mathbb{T}^d$ on the flat d-torus. We denote $\mathbb{E}$ the expectation with respect to the probability measure $\mathbb{P}$.

The solution to equation (2) satisfies Lax's formula

$$\phi_\varepsilon(x) = \sup_v \mathbb{E}\Big(\phi_\varepsilon(X_\varepsilon(\tau)) - \int_0^\tau L(X_\varepsilon(s), v(s))ds - c(\varepsilon)\tau\Big), \tag{10}$$

where v is an admissible progressively measurable control process and X_ε is the solution to the stochastic differential equation

$$\begin{cases} dX_\varepsilon(t) &= v(t)dt + \sqrt{2\varepsilon}\, dW(t) \\ X_\varepsilon(0) &= x. \end{cases} \tag{11}$$

and where τ is a bounded stopping time ([15], Lemma IV 3.1).

Moreover, the sup is achieved in (10), and we have a description of the optimal v: if we introduce the vector field $u_\varepsilon(x) = \frac{\partial H}{\partial p}(x, D\phi_\varepsilon(x))$, and if we consider the solution of the following stochastic differential equation:

$$\begin{cases} dX_\varepsilon(t) & = u_\varepsilon(X_\varepsilon(t))dt + \sqrt{2\varepsilon}\, dW(t) \\ X_\varepsilon(0) & = x, \end{cases} \tag{12}$$

then the optimal control is given by the formula $v(t) = u_\varepsilon(X_\varepsilon(t))$. See for example [15], Theorem IV 11.1.

Although the following lemma is a well known property of the solutions, we give an argument for convenience of the reader.

Lemma 2. *The solutions ϕ_ε of* (2) *are Lipschitz and semiconvex uniformly in ε. Therefore there are always subsequences converging in C^0 norm.*

Proof: We first need to know that $|c(\varepsilon)|$ is bounded independentely of ε, say, by a constant $r > 0$. In fact,

$$\inf H(x,0) \le c(\varepsilon) \le \sup H(x,0),$$

as may be checked by applying Definition 1 to $v = 0$ and x, successively, a local maximum or minimum of a solution ϕ_ε.

By hypothesis, $|\partial_x H(x,p)| \le K(|p|+1)$ for $K > 0$ constant. Since H is superlinear, there is $R > 0$ such that

$$|p| \ge R \Rightarrow H(x,p) \ge r + \sqrt{dK}(|p|+1) \tag{13}$$

where d is the dimension.

We now use an argument we found in [19] originally due to Bernstein. Let $w = |D\phi_\varepsilon|^2$ then

$$Dw = 2\,\mathrm{Hess}\,\phi_\varepsilon D\phi_\varepsilon, \tag{14}$$

$$\triangle w = 2\,\mathrm{Tr}(\mathrm{Hess}\,\phi_\varepsilon)^2 + 2D(\triangle\phi_\varepsilon)\cdot D\phi_\varepsilon \tag{15}$$

Differentating (2), multiplying by $D\phi_\varepsilon$ and using (14)(15)

$$\begin{aligned} \partial_x H \cdot D\phi_\varepsilon + \partial_p H \cdot \mathrm{Hess}\,\phi_\varepsilon D\phi_\varepsilon + \varepsilon D(\triangle\phi_\varepsilon)\cdot D\phi_\varepsilon &= 0 \\ \partial_x H \cdot D\phi_\varepsilon + \frac{1}{2}\partial_p H \cdot Dw + \frac{\varepsilon}{2}\triangle w - \varepsilon\,\mathrm{Tr}(\mathrm{Hess}\,\phi_\varepsilon)^2 &= 0 \end{aligned}$$

Let $x_0 \in \mathbb{T}^d$ be a point where w attains its maximum, then $Dw(x_0) = 0$, $\triangle w(x_0) \le 0$. At the point x_0, we have

$$\begin{aligned} \varepsilon(\triangle\phi_\varepsilon)^2 &\le d\varepsilon\,\mathrm{Tr}(\mathrm{Hess}\,\phi_\varepsilon)^2 \le d\partial_x H\cdot D\phi_\varepsilon \\ (H(x_0, D\phi_\varepsilon) - c(\varepsilon))^2 &\le \varepsilon dK(|D\phi_\varepsilon|+1)|D\phi_\varepsilon|. \end{aligned}$$

Using (13), we then have $\sup_{x\in\mathbb{T}^d} |D\phi_\varepsilon(x)| \le R$ for $0 < \varepsilon \le 1$.

Recall that the function ϕ is semiconvex with constant C means that

$$\phi(x+y) - 2\phi(x) + \phi(x-y) \ge -C|y|^2$$

which implies that $\phi(x) + \dfrac{C}{2}|x|^2$ is convex.

Consider the optimal control $v(t) = u_\varepsilon(X_\varepsilon(t))$ described above; this means that, for all $T > 0$,

$$\phi_\varepsilon(x) = \mathbb{E}\big(\phi_\varepsilon(X_\varepsilon(T)) + \int_0^T L(X_\varepsilon(s), u_\varepsilon(X_\varepsilon(s)))ds\big) - c(\varepsilon)T,$$

where $X_\varepsilon(t)$ is the solution to (12). Take $T = 1$, let $|y| \leq 1$ be an increment and define

$$\chi(s) = (1-s)y$$

and

$$p(s) = X_\varepsilon(s) + \chi(s).$$

The control $u_\varepsilon(X_\varepsilon(s)) + \dot\chi$ is an admissible control, then

$$\phi_\varepsilon(x+y) \geq \mathbb{E}\big(\phi_\varepsilon(p(1)) + \int_0^1 L(X_\varepsilon(s) + \chi(s), u_\varepsilon(X_\varepsilon(s)) + \dot\chi)ds\big) - c(\varepsilon)$$

where

$$dp = dX_\varepsilon + \dot\chi dt = (u_\varepsilon(X_\varepsilon(s)) + \dot\chi)dt + \sqrt{2\varepsilon}dW$$

clearly $p(0) = x + y$ and $p(1) = X_\varepsilon(1)$.

Similarly

$$\phi_\varepsilon(x-y) \geq \mathbb{E}\big(\phi_\varepsilon(X_\varepsilon(1)) + \int_0^1 L(X_\varepsilon(s) - \chi(s), u_\varepsilon(X_\varepsilon(s)) - \dot\chi)ds\big) - c(\varepsilon).$$

Let

$$M = 1 + \sup_{\varepsilon\in(0,1]} |u_\varepsilon(x)|,$$

This is finite since we have that $|u_\varepsilon(x)| = \frac{\partial H}{\partial p}(x, D\phi_\varepsilon(x))$ and by the prevoius part of the lemma $D\phi_\varepsilon(x)$ is uniformly bounded. Define now

$$A = \sup_{|v|\leq M} \|\partial_{xx} L(x,v)\|, B = \sup_{|v|\leq M} \|\partial_{xv} L(x,v)\|.$$

Since $\partial_{vv}L$ is positive definite, an application of Taylor's Theorem gives

$$L(x+\chi, v+\dot\chi) - 2L(x,v) + L(x-\chi, v-\dot\chi) \geq A|\chi|^2 + 2B|\chi||\dot\chi|$$

for $|v| \leq M - 1$. Therefore

$$\begin{aligned}\phi_\varepsilon(x+y) - 2\phi_\varepsilon(x) + \phi_\varepsilon(x-y) &\geq -\int_0^1 (A|\chi(s)|^2 + B|\chi(s)||\dot\chi(s)|)ds \\ &\geq -\big(\frac{A}{3} + B\big)|y|^2.\end{aligned}$$

□

We will need the following

Lemma 3. *Suppose the sequence ϕ_{ε_n} of solutions of* (2) *converges C^0 to ϕ_0. Assume that ϕ_0 is differentiable in an open set V. Then $d\phi_{\varepsilon_n}$ converges to $d\phi_0$ uniformly in every compact subset of V.*

This is an immediate consequence of Lemma 2 and

Theorem 2. [24], *Theorem 25.7.*

Let f_n be a sequence of differentiable convex functions converging pointwise to a differentiable function f. Then Df_n converges pointwise to Df, and in fact uniformly on compact subsets.

3. Proof of the main result

We assume that the static classes consist in a finite number of hyperbolic periodic orbits, $\gamma_i : [0, T_i] \longrightarrow \mathbb{T}^d$, for $1 \leq i \leq m$.

We recall the notation:

$$\lambda_i := \frac{1}{T_i} \int_0^{T_i} \triangle h_i(\gamma_i(t))dt \quad 1 \leq i \leq m,$$

(which is well defined, see Lemma 5), and we assume that there is only one $i \in [1, m]$ for which λ_i is minimal: we denote it I.

A recent result of Fathi and Siconolfi [14] claims the existence of a C^1 critical subsolution f of the Hamilton-Jacobi equation, that is,

$$H(x, Df(x)) \leq c.$$

Moreover, f can be constructed so that the inequality is strict outside the Aubry set $\mathcal{A}$ (it has to be an equality on the Aubry set). The main consequence for our purposes is that

$$L(x, v) + c - Df(x).v \geq L(x, v) + H(x, Df(x)) - Df(x).v \geq 0. \quad (16)$$

for all $(x, v) \in \mathbb{T}^d \times \mathbb{R}^d$, and it is zero if and only if (x, v) belongs to the Aubry set, in other words, here, $(x, v) = (\gamma_i(t), \dot{\gamma}_i(t))$ for some i and t.

Lemma 4. *Let ϕ be a viscosity solution of the Hamilton-Jacobi equation* (1)*, and let $\varphi = \phi - f$; let also $\mathbf{h}(x, y) = h(x, y) + f(x) - f(y)$. Then*

1. $\mathbf{h}(x, y) \leq \mathbf{h}(x, z) + \mathbf{h}(z, y)$.
2. $\mathbf{h}(x, y) \geq 0$, *with equality if and only if $x, y \in \gamma_i$ for some i.*
3. *φ is constant on each γ_i.*
4. *If $x \in \mathbb{T}^d$ is a local maximum of φ, then there exists i such that $x \in \gamma_i$.*

Proof.

Part (1) is straightforward, using the triangle inequality for h. For (2) we have, from the definition of h given above,

$$\begin{aligned}\mathbf{h}(x,y) &= h(x,y)+f(x)-f(y)) \\ &= \liminf_{T\to\infty}\inf_{\gamma;x\longrightarrow y}\int_0^T (L(\gamma(s),\dot\gamma(s))+c)\,ds+f(x)-f(y) \\ &= \liminf_{T\to\infty}\inf_{\gamma;x\longrightarrow y}\int_0^T (L(\gamma(s),\dot\gamma(s))+c-Df(\gamma(s)).\dot\gamma(s))\,ds\end{aligned}$$

It follows from inequality (16) that $\mathbf{h}(x,y)\geq 0$. If x and y are in one γ_i, inequality (16) becomes an equality, so $\mathbf{h}(x,y)=0$. Finally if x and y are not in the same static class, then all curves γ joining x and y in time T will pass through a region where the inequality (16) is strict. Besides, the curves $\gamma_T:[0,T]\longrightarrow\mathbb{T}^d, \gamma_T(0)=x, \gamma_T(T)=y$, achieving the inf of the action, have uniformly bounded velocity, for $T\geq 1$ ([13]). Thus, the time spent by γ_T in a region where inequality (16) is strict is bounded below, independently of $T\geq 1$. This implies that $\mathbf{h}(x,y)>0$.

Part (3): from the fact that

$$\phi(\gamma_i(T))-\phi(\gamma_i(0))=\int_0^T L(\gamma_i(t),\dot\gamma_i(t))dt+cT \tag{17}$$

(which is Equation (5)) it follows that

$$\varphi(\gamma_i(T))-\varphi(\gamma_i(0))=\int_0^T (L(\gamma_i(t),\dot\gamma_i(t))+c-Df(\gamma_i(t)).\dot\gamma_i(t))dt \tag{18}$$

Then, since $L+c-Df$ is zero along the Aubry set, the conclusion follows.

Part (4): It follows from the fact that ϕ is a viscosity solution of (1) that, if x is a local maximum of φ, then

$$H(x,Df(x))\geq c,$$

but we know that this can happen only if x belongs to the Aubry set. □

Lemma 5. *If ϕ is a viscosity solution to the Hamilton Jacobi equation (1) and $\varphi=\phi-f$ has a local maximum at γ_i, then there is a neighborhood V of γ_i such that*

$$\phi(x)=\phi(x_i)-h(x,x_i)$$

for $x\in V$. This implies that $(x,D\phi(x))$ belongs to the stable manifold of $(\gamma_i(0),\frac{\partial L}{\partial v}(\dot\gamma_i(0)))$ under the Hamiltonian flow, and that ϕ is C^3 on V.

Proof. Let us denote V a neighbourhood of x_i on which x_i is a local maximum of φ.

For any j,

$$\begin{aligned}\varphi(x_i) &= \varphi(x_i) - \mathbf{h}(x_i, x_i) && (19)\\ &\geq \varphi(x_j) - \mathbf{h}(x_i, x_j). && (20)\end{aligned}$$

If the strict inequality holds for all $j \neq i$ it follows from the continuity of φ and $\mathbf{h}$ that there is a neighbourhood of γ_i where

$$\begin{aligned}\varphi(x) &= \max_j \varphi(x_j) - \mathbf{h}(x, x_j)\\ &= \varphi(x_i) - \mathbf{h}(x, x_i)\end{aligned}$$

and hence $\phi(x) = \phi(x_i) - h(x, x_i)$.

Let us now deal with the case when the equality occurs in (20) for some $j \neq i$. We first construct some $y \in \partial V$ such that $h(x_i, y) + h(y, x_j) = h(x_i, x_j)$ (and thus $\mathbf{h}(x_i, y) + \mathbf{h}(y, x_j) = \mathbf{h}(x_i, x_j)$). To that end, let $\gamma_T : [0, T] \longrightarrow \mathbb{T}^d$ be a curve joining x_i to x_j and achieving

$$h_T(x_i, x_j) = \inf\{A(\gamma) + cT | \gamma : [0, T] \to \mathbb{T}^d \text{ joins } x_i \text{ and } x_j\}.$$

Let $T_V \in [0, T]$ be the first exit time of γ_T out of V, and $y_T = \gamma_T(T_V) \in \partial V$ be the first point of intersection with ∂V.

We claim that, as T tends to infinity, T_V as well as $T - T_V$ tend to infinity; this follows directly from the fact that $\dot{\gamma}_T(0)$ has to tend to $\dot{\gamma}_i(0)$, and $\dot{\gamma}_T(T)$ has to tend to $\dot{\gamma}_j(0)$. To justify this last point, consider v a limit point of $\dot{\gamma}_T(0)$, and $(\gamma(t), \dot{\gamma}(t))_{t\geq 0}$ the trajectory of (x_i, v) under the Euler-Lagrange flow. From the fact that

$$h_T(x_i, x_j) - h_{T-1}(\gamma_T(1), x_j) = A(\gamma_{T|[0,1]})$$

and taking the limit $T \longrightarrow +\infty$, it follows that

$$h(x_i, x_j) - h(\gamma(1), x_j) = A(\gamma_{|[0,1]}).$$

We have also, by (5),

$$h(\gamma_i(-1), x_j) - h(x_i, x_j) = A(\gamma_{i|[-1,0]})$$

so that

$$h(\gamma_i(-1), x_j) - h(\gamma(1), x_j) = A(\gamma_{|[0,1]}) + A(\gamma_{i|[-1,0]})$$

which means that the curve obtained by gluing $\gamma_{i|[-1,0]}$ with $\gamma_{|[0,1]}$ minimizes the action between its endpoints. In particular, it has to be differentiable, thus $v = \dot{\gamma}(0) = \dot{\gamma}_i(0)$.

We know now that T_V and $T - T_V$ tend to infinity. We have obviously

$$h_{T_V}(x_i, y_T) + h_{T-T_V}(y_T, x_j) = h_T(x_i, x_j).$$

Taking for y a cluster point of (y_T), and using the uniform convergence of h_T to h, we obtain

$$h(x_i, y) + h(y, x_j) = h(x_i, x_j).$$

Finally, since x_i is a maximum of φ on V, we have

$$\begin{aligned}\varphi(x_i) &\geq \varphi(y)\\ &\geq \varphi(x_j) - \mathbf{h}(y, x_j)\\ &= \varphi(x_j) + \mathbf{h}(x_i, y) - \mathbf{h}(x_i, x_j)\\ &= \varphi(x_i) + \mathbf{h}(x_i, y).\end{aligned}$$

But $\mathbf{h}(x_i, y) > 0$, and this contradiction proves that the first alternative holds.

We now come to the last part of the lemma. What we need to check is that, in a neighbourhood of x_i, if x is a point of differentiability of $h(x, x_i)$, then $(x, -Dh(x, x_i))$ is in the (weak) stable manifold of $\frac{\partial L}{\partial v}(\gamma_i(0), \dot\gamma_i(0))$ under the Hamiltonian flow.

Let $\gamma_T : [0, T] \longrightarrow \mathbb{T}^d$ be a curve joining x to x_i and achieving

$$h_T(x, x_i) = \inf\{A(\gamma) + cT | \gamma : [0, T] \to \mathbb{T}^d \text{ joins } x \text{ and } x_i\}.$$

Consider a neighbourhood of the curve γ_i in $\mathbb{T}^d \times \mathbb{R}^d$:

$$V_\kappa = \{(x, v), d((x, v), (\gamma_i(t), \dot\gamma_i(t)) \leq \kappa \text{ for some } t\}.$$

If $d(x, \gamma_i) \leq \delta$ then $\mathbf{h}(x, x_i) \leq C\delta$, where C is the lipschitz constant of $\mathbf{h}$. Since the modified Lagrangian $L+c-Df$ is strictly positive outside the Aubry set, and since the velocity of the action-minimizers γ_T is uniformly bounded for $T \geq 1$, there exists an $M(\kappa) > 0$, independent of $T \geq 1$, such that, if γ_T goes out of V_κ then $\mathbf{h}_T(x, x_i) > M(\kappa)$. If δ has been chosen small enough so that $C\delta < M(\kappa)$, this implies that γ_T has to stay inside V_κ as soon as $d(x, \gamma_i) \leq \delta$. Finally, the hyperbolicity of γ_i implies that, if κ has been suitably chosen, $(x, \dot\gamma_T(0))$ converges to a point (x, v) in the stable manifold of $(\gamma_i(0), \dot\gamma_i(0))$ as T tends to infinity.

We call $(x_t, v_t)_{t\geq 0}$ the trajectory of (x, v) under the Euler-Lagrange flow: we have, for all $t, s \geq 0$,

$$h_T(\gamma_T(t), x_i) - h_{T-s}(\gamma_T(t+s), x_i) = A(\gamma_{T|[t,t+s]})$$

and, passing to the limit $T \longrightarrow +\infty$,

$$h(x_t, x_i) - h(x_{t+s}, x_i) = A(x_{|[t,t+s]}).$$

It is now a standard fact ([13]) that $h_i = h(., x_i)$ is differentiable at each x_t for $t > 0$, and that $-Dh_i(x_t) = \frac{\partial L}{\partial v}(x_t, v_t)$. $\square$

Lemma 6. *If ϕ is a viscosity solution of the Hamilton-Jacobi equation* (1) *such that $\varphi = \phi - f$ has only one local maximum at the orbit γ_I, then*

$$\phi(x) = \phi(x_I) - h(x, x_I). \tag{21}$$

Proof. Reordering the periodic orbits, let us assume without loss of generality that $I = 1$, $\varphi(x_1) = 0 = -\mathbf{h}(x_1, x_1)$ and $\varphi(x_1) \geq \varphi(x_2) \geq \ldots \geq \varphi(x_m)$.

As hypothesis of induction we assume that $\varphi(x_l) = -\mathbf{h}(x_l, x_1)$ for $l \leq i$, and we prove that $\varphi(x_{i+1}) = -\mathbf{h}(x_{i+1}, x_1)$. Suppose on the contrary that $\varphi(x_{i+1}) > -\mathbf{h}(x_{i+1}, x_1)$. From the domination condition we have that $\varphi(x_l) \leq \varphi(x_{i+1}) + \mathbf{h}(x_{i+1}, x_l)$ for any l, if we had an equality for some $l \leq i$,

$$\begin{aligned} \varphi(x_{i+1}) &= \varphi(x_l) - \mathbf{h}(x_{i+1}, x_l) \\ &= -\mathbf{h}(x_l, x_1) - \mathbf{h}(x_{i+1}, x_l) \\ &\leq -\mathbf{h}(x_{i+1}, x_1) \end{aligned}$$

in contradiction with our assumption. So we have the strict inequality $\varphi(x_l) \leq \varphi(x_{i+1}) + \mathbf{h}(x_{i+1}, x_l)$ for all $l \leq i$. On the other hand, it is obvious that $\varphi(x_l) < \varphi(x_{i+1}) + \mathbf{h}(x_{i+1}, x_l)$ for $l \geq i+2$. It follows then that there is a small neighbourhood V_{i+1} of x_{i+1} such that that if x is in V_{i+1} then

$$\varphi(x) = \varphi(x_{i+1}) - \mathbf{h}(x, x_{i+1}),$$

so that φ has a local maximum at x_{i+1}, contradicting our assumption.

Therefore

$$\varphi(x) = \varphi(x_I) - \mathbf{h}(x, x_I)$$

which is equivalent to (21). □

Lemma 7.

$$c'_+(0) = \liminf_{\varepsilon \to 0^+} \frac{c(\varepsilon) - c(0)}{\varepsilon} \geq -\lambda_I$$

Proof. We will prove that

$$\liminf_{\varepsilon \to 0^+} \frac{c(\varepsilon) - c(0)}{\varepsilon} \geq -\lambda_I - r$$

for an arbitrary $r > 0$.

Let Φ be a C^3 function that coincides with $-h_I = -h(., x_I)$ in a neighbourhood V of γ_I.

Consider the vector field $U(x) = \frac{\partial H}{\partial p}(x, D\Phi(x))$, which has γ_I as an attractive periodic orbit. Let X_ε be the solution to

$$\begin{cases} dX_\varepsilon(t) & = U(X_\varepsilon(t))dt + \sqrt{2\varepsilon}\, dW(t) \\ X_\varepsilon(0) & = x_I. \end{cases} \tag{22}$$

Let $\delta > 0$ be sufficiently small to have $\delta\|\Phi\|_{C^3} \leq r$ and $B_\delta(\gamma_I) := \{x, d(x,\gamma_I) \leq \delta\} \subset V$, and define the stopping time

$$\tau(\omega) = \min\{s > 0 : d(X_\varepsilon(s,\omega), \gamma_I(s)) \geq \delta\}. \tag{23}$$

Since

$$\begin{aligned} L(x, U(x)) + H(x, D\Phi(x)) &= D\Phi(x).U(x) \\ H(x, D\Phi(x)) = c(0) & \qquad \text{for all } x \in V, \end{aligned}$$

we have from (10),

$$(c(\varepsilon) - c(0))\mathbb{E}(\tau\wedge\kappa) \geq \mathbb{E}\Big(\phi_\varepsilon(X_\varepsilon(\tau\wedge\kappa)) - \phi_\varepsilon(x_I)) - \int_0^{\tau\wedge\kappa} D\Phi(X_\varepsilon(s)).U(X_\varepsilon(s))ds\Big),$$

for all $\kappa > 0$ (we denote $\tau \wedge \kappa$ the bounded stopping time $\min(\tau, \kappa)$).

An application of Ito's formula gives

$$\begin{aligned} &\mathbb{E}(\Phi(X_\varepsilon(\tau\wedge\kappa)) - \Phi(x_I)) \\ &\qquad = \mathbb{E}\Big(\int_0^{\tau\wedge\kappa} D\Phi(X_\varepsilon(s))U(X_\varepsilon(s))ds + \varepsilon \triangle \Phi(X_\varepsilon(s))ds\Big). \end{aligned}$$

Defining $\psi_\varepsilon = \phi_\varepsilon - \Phi$ we get

$$(c(\varepsilon) - c(0))\mathbb{E}(\tau\wedge\kappa) \geq \mathbb{E}\Big(\psi_\varepsilon(X_\varepsilon(\tau\wedge\kappa)) - \psi_\varepsilon(x_I)) + \varepsilon \int_0^{\tau\wedge\kappa} \triangle\Phi(X_\varepsilon(s))ds\Big).$$

For $s \in [0, \tau(\omega)]$,

$$|\triangle \Phi(X_\varepsilon(s,\omega)) + \triangle h_I(\gamma_I(s))| \leq \|\Phi\|_{C^3}\delta \leq r$$

so that

$$\left|\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle\Phi(X_\varepsilon(s))ds\Big) + \mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_I(\gamma_I(s), x_i)ds\Big)\right| \leq \mathbb{E}(\tau\wedge\kappa)r.$$

Let $M = \sup_{x,\varepsilon} |\psi_\varepsilon(x)|$ (which is finite by Lemma 2), then

$$\frac{c(\varepsilon) - c(0)}{\varepsilon} \geq -\frac{2M}{\varepsilon\mathbb{E}(\tau\wedge\kappa)} - \frac{1}{E(\tau\wedge\kappa)}\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_I(\gamma_I(s))ds\Big) - r.$$

In the case when $T_I > 0$, we can write

$$\begin{aligned}
\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_I(\gamma_I(s))ds\Big) &= \int_0^{+\infty} \triangle h_I(\gamma_I(s))\mathbb{P}(\tau\wedge\kappa > s)ds \\
&= \int_0^{T_I} \triangle h_I(\gamma_I(s))\Big(\sum_{k=0}^{+\infty} \mathbb{P}(\tau\wedge\kappa > s + kT_I)\Big)ds \\
&\leq \int_0^{T_I} \triangle h_I(\gamma_I(s))\Big(1 + \int_0^{+\infty} \mathbb{P}(\tau\wedge\kappa > s + uT_I)du\Big)ds \\
&= \int_0^{T_I} \triangle h_I(\gamma_I(s))\Big(1 + \mathbb{E}(\frac{\tau\wedge\kappa - s}{T_I})\Big)ds
\end{aligned}$$

so that

$$\frac{1}{\mathbb{E}(\tau\wedge\kappa)}\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_I(\gamma_I(s))ds\Big) \leq \frac{1}{T_I}\int_0^{T_I} \triangle h_I(\gamma_I(s))ds + \frac{T_I||h_I||_{C^2(V)}}{\mathbb{E}(\tau\wedge\kappa)}$$

and we can now let κ tend to infinity to obtain:

$$\frac{c(\varepsilon) - c(0)}{\varepsilon} \geq -\frac{2M}{\varepsilon\mathbb{E}(\tau)} - \frac{T_I||h_I||_{C^2(V)}}{\mathbb{E}(\tau)} - \frac{1}{T_I}\int_0^{T_I} \triangle h_I(\gamma_I(s))ds - r.$$

For $T_I = 0$, the same argument would hold, with obvious modifications.

Freidlin and Wentzel ([16], Chapter 4.4) gave an estimate for $E(\tau)$, for a stochastic perturbation of a vector field having a sink. Here the vector field has a sink or an attractive periodic orbit γ_I, but, clearly, the estimates of [16] apply also in the second situation:

$$m = \liminf_{\varepsilon\to 0} \varepsilon \log E(\tau) > 0.$$

Letting now $\varepsilon > 0$ go to zero we obtain

$$c'_+(0) \geq -\lambda_I - r.$$

□

Suppose that a sequence (ϕ_{ε_n}) of solutions of (2) converges to ϕ_0. Let ψ be a C^3 function that coincides with ϕ_0 on a neigbourhood V_i of each γ_i that is a local maximun of $\phi_0 - f$ (such a function ψ exists by Lemma 5). Then $\psi_\varepsilon = \phi_\varepsilon - \psi$ is a solution to the equation

$$H(x, D\psi_\varepsilon + D\psi) + \varepsilon \triangle \psi_\varepsilon + \varepsilon \triangle \psi = c(\varepsilon)$$

which can be written as

$$\tilde{H}_\varepsilon(x, D\psi_\varepsilon) + \varepsilon \triangle \psi_\varepsilon = c(\varepsilon), \tag{24}$$

where the hamiltonian $\tilde{H}_\varepsilon$ has corresponding lagrangian

$$\tilde{L}_\varepsilon(x, v) = L(x, v) - D\psi(x).v - \varepsilon \triangle \psi(x).$$

As in (10), ψ_ε satisfies the variational formulation of Equation (24):

$$\psi_\varepsilon(x) = \sup_u \mathbb{E}\Big(\psi_\varepsilon(X_\varepsilon(\tau)) - \int_0^\tau \tilde{L}(X_\varepsilon(s), u(s))ds - c(\varepsilon)\tau\Big), \tag{25}$$

Lemma 8. *Suppose that the function $\phi_0 - f$ has a local maximum at γ_i; then $i = I$ and*

$$\lim_{n\to\infty} \frac{c(\varepsilon_n) - c(0)}{\varepsilon_n} = -\lambda_I$$

Proof. Let $2r = \min_{j\neq I} \lambda_j - \lambda_I$, and consider the vector field

$$u_\varepsilon(x) = \frac{\partial \tilde{H}_\varepsilon}{\partial p}(x, D\psi_\varepsilon(x)) = \frac{\partial H}{\partial p}(x, D\phi_\varepsilon(x)).$$

Given $1 \leq i \leq m$, let X_ε be the solution to

$$\begin{cases} dX_\varepsilon(t) & = u_\varepsilon(X_\varepsilon(t))dt + \sqrt{2\varepsilon}\, dW(t) \\ X_\varepsilon(0) & = x_i. \end{cases} \tag{26}$$

We know then that $(u_\varepsilon(X_\varepsilon(t)))$ is the optimal control associated to the variational formulation (25), which means that, for all bounded stopping time τ,

$$\begin{aligned} \psi_\varepsilon(x) &= E\Big(\psi_\varepsilon(X_\varepsilon(\tau)) - \int_0^\tau \big(L(X_\varepsilon(s), u_\varepsilon(X_\varepsilon(s)) \\ &- D\psi(X_\varepsilon(s))u_\varepsilon(X_\varepsilon(s))\big) - \varepsilon \int_0^\tau \triangle\psi(X_\varepsilon(s)))ds - c(\varepsilon)\tau\Big). \end{aligned}$$

Let $\delta > 0$ be sufficiently small to have $\delta\|\psi\|_{C^3} \leq r$ and $B_\delta(\gamma_i) \subset V_i$ and define

$$\tau(\omega) = \min\{s > 0 : d(X_\varepsilon(s, \omega), \gamma_i(s)) \geq \delta\}. \tag{27}$$

Since $D\psi(x).u_\varepsilon(x) \leq L(x, u_\varepsilon(x)) + H(x, D\psi(x))$ and $H(x, D\psi(x)) = c(0)$ for $x \in V_i$.

$$((c(\varepsilon) - c(0))\mathbb{E}(\tau\wedge\kappa) \leq \mathbb{E}\Big(\psi_\varepsilon(X_\varepsilon(\tau\wedge\kappa)) - \psi_\varepsilon(x_i) + \varepsilon \int_0^{\tau\wedge\kappa} \triangle\psi(X_\varepsilon(s))ds\Big)$$

for all $\kappa > 0$.

For $s \in [0, \tau(\omega)]$,

$$| \triangle\, \psi(X_\varepsilon(s, \omega)) + \triangle h_i(\gamma_i(s), x_i)| \leq \|\psi\|_{C^3}\delta \leq r$$

so that

$$\left|\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle\psi(X_\varepsilon(s))ds\Big) + \mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_i(\gamma_i(s))ds\Big)\right| \leq E(\tau\wedge\kappa)r.$$

Let $M = \sup_{x,\varepsilon} |\psi_\varepsilon(x)|$, then

$$\frac{c(\varepsilon)-c(0)}{\varepsilon} \leq \frac{2M}{\varepsilon\mathbb{E}(\tau\wedge\kappa)} - \frac{1}{\mathbb{E}(\tau\wedge\kappa)}\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_i(\gamma_i(s))ds\Big)+r.$$

Reasoning as in the proof of Lemma 7, we get

$$\frac{1}{\mathbb{E}(\tau\wedge\kappa)}\mathbb{E}\Big(\int_0^{\tau\wedge\kappa} \triangle h_i(\gamma_i(s))ds\Big)\geq \frac{1}{T_i}\int_0^{T_i} \triangle h_i(\gamma_i(s))ds - \frac{2T_i||h_i||_{C^2(V_i)}}{\mathbb{E}(\tau\wedge\kappa)}$$

and we can now pass to the limit $\kappa \longrightarrow +\infty$ to get

$$\frac{c(\varepsilon)-c(0)}{\varepsilon} \leq \frac{2M}{\varepsilon\mathbb{E}(\tau)} - \frac{1}{T_i}\int_0^{T_i} \triangle h_i(\gamma_i(s))ds - \frac{2T_i||h_i||_{C^2(V_i)}}{\mathbb{E}(\tau)} + r.$$

By Lemma 3, (u_{ε_n}) converges uniformly to $\frac{\partial H}{\partial p}(x, D\phi_0(x))$ in the neighborhood V_i; the estimate of Freidlin and Wentzell for $E(\tau)$ also applies ([16], Chapter 5.3):

$$m = \liminf_{n\to\infty} \varepsilon_n \log E(\tau) > 0,$$

and so, letting n grow we obtain

$$\limsup_{n\to\infty} \frac{c(\varepsilon_n)-c(0)}{\varepsilon_n} \leq -\lambda_i + r,$$

which, by our choice or r, is possible only if $i = I$. □

Proof of Theorem 1 . Let f be a C^1 critical subsolution, strict outside the static classes. Let ϕ_{ε_n} be any sequence of solutions of (1). By Lemma 2 we know that there is a convergent subsequence $\phi_{\varepsilon_{n_k}}$. Let ϕ_0 be the limit. From Lemma 8, we know the only place where $\phi_0 - f$ can have a local maximum is at γ_I. Finally from Lemma 6, we know that $\phi_0(x) = \phi_0(x_I) - h(x, x_I)$. Besides, $\phi_0(x_I) = \lim \phi_{\varepsilon_{n_k}}(x_I) = 0$. □

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