Construction and combinatorics of perfect bases
1) Definition

$$A = \{a_{ij}\; j \in J$$, symmetricizable Cartan matrix

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$, Kac-Moody algebra / \mathbb{C}

$$U = U(m_r)$$, presented by generators $$e_i \; (i \in I)$$

relations $$\sum_{p+q = 1-a_{ij}} (-1)^p e_i^{(p)} e_j e_i^{(q)} = 0 \quad (i \neq j)$$

$$e_i^{(n)} = \frac{e_i^n}{n!}$$, divided power

graded by $$\mathbb{Q}^+ = \{ \sum n_i \epsilon_i \mid n_i \in \mathbb{N} \} \subset \mathfrak{h}^*$$ with deg $$e_i = \alpha_i$$

(with $$U = \bigoplus_{v \in \mathbb{Q}^+} U_v$$)

endowed with an involutive anti-automorphism $$x \mapsto x^\dagger$$ which fixes the $$e_i$$.

Main problem: construct bases of $$U$$ related to this presentation. Already a lot of work on this problem: Gelfand, Zelevinsky, Retakh, Bernstein, Hochster, Lusztig, Kashiwara, Kajihara,...

$$N = \exp \mathfrak{n}_+$$, unipotent group with Lie algebra $$\mathfrak{n}_+$$.

$$R = C[N]$$, algebra of regular functions on $$N$$.

$$\cong U^*$$ graded dual

$$R = \bigoplus_{v \in \mathbb{Q}^+} R_v$$, $$R_v = (U_v)^*$$

$$\mathfrak{n}_+$$ acts by derivations on $$R$$ (left-invariant vector fields on $$N$$)

$$\eta \mapsto \eta^\dagger$$ the transpose of $$x \mapsto x^\dagger$$.

For $$i \in I$$ and $$\eta \in R^{-\{0\}}$$, set $$e_i(\eta) = \max \{ n \in \mathbb{N} \mid e_i^n \eta \neq 0 \}$$

$$\tilde{e}_i \eta = \left( e_i(\eta) \right) \eta$. 

Definition: A linear basis $B$ of $R$ is perfect if:

(P0) $a \in B$

(P1) $B$ is graded with the $Q_+$-gradation

(P2) $\forall b \in B, \forall i \in I, \ \bar{e}^\text{max}_i b \in B$.

Further, $\forall n \in \mathbb{N}$, $\bar{e}^\text{max}_i$ is injective on $\{b \in B | \bar{e}^\text{I}_i(b) = n\}$.

(P3) $B$ is stable under $\dagger$.

Observation: For $i \in I$ and $n \in \mathbb{N}$, let $K_{i,n} = \bar{e}^\text{max}_i \cdot b \in R$.

- $B$ perfect basis $\Rightarrow \{b \in B | \bar{e}^\text{I}_i(b) \leq n\}$ basis of $K_{i,n}$.

(proof: let $\eta \in K_{i,n}$. Write $\eta = \sum_{b \in B} a_b \cdot b$, set $m = \max \{\bar{e}^\text{I}_i(b) | a_b \neq 0\}$.

If we had $m > n$, then we would have $0 = \bar{e}^\text{I}_i(\eta) = \sum_{b \in B} a_b (\bar{e}^\text{max}_i \cdot b)_{\bar{e}^\text{I}_i(b) = m}$, which is impossible.)

- $\bar{e}^\text{I}_i$ induces a linear bijection $K_{i,m}/K_{i,m-1} \longrightarrow K_i$.

(proof of surjectivity: $\bar{e}^\text{I}_i$ is not a zero divisor in $U$)

and $\bar{e}^\text{max}_i$ induces a bijection $\{b \in B | \bar{e}^\text{I}_i(b) = n\} \longrightarrow \{b \in B | \bar{e}^\text{I}_i(b) = 0\}$.

Conclusion: Each perfect basis carries a combinatorial structure:

- A map $\chi : B \rightarrow \mathbb{Q}_+$

- Maps $\bar{e}^\text{I}_i : B \rightarrow \mathbb{N}$

- Bijections $\{b \in B | \bar{e}^\text{I}_i(b) = n\} \overset{\chi}{\longrightarrow} \{b \in B | \bar{e}^\text{I}_i(b) = n+1\}$

- $\bar{e}^\text{max}_i$:

- The involution $b \mapsto b^\dagger$ (Kashiwara's involution $\ast$ / Lusztig's $\sigma$)

- For convenience, define $\bar{e}^\text{I}_i(b) = \bar{e}^\text{I}_i(b) + \langle \bar{e}^\text{I}_i, \chi \rangle b$

and set $\bar{e}^\text{I}_i b = 0$ if $\bar{e}^\text{I}_i(b) = 0$.
2) Usefulness

\[ P_+ = \{ \text{dominant integral weights} \} \]

\[ \lambda \in P_+ \quad \mapsto \quad L(\lambda) \quad \text{irreducible integrable module with hw } \lambda \]

\[ \psi^\lambda : L(\lambda) \rightarrow \mathbb{R} \quad \text{the unique morphism of } \mathfrak{h}_+ \text{-modules that maps } v_{\lambda} \mapsto 1 \]

\[ W = \langle \mathbf{w}_i \mid i \in I \rangle \quad \text{Weyl group} \]

For \( w \in W, \lambda \in P_+ \) \( \mathbf{w}_\lambda \in L(\lambda) \) extremal weight vector (suitably normalized)

Flag minors: any element in \( R \) of the form \( \psi^\lambda (w_{\lambda \lambda}) \).

Proposition: Let \( B \) be a perfect basis of \( R \). Then

1) \( B \) is compatible with all subspaces in \( \psi^\lambda \) and contains all flag minors.

2) for each \( \lambda \in P_+ \), the basis \( \psi^\lambda(B) \) of \( L(\lambda) \) consists of weight vectors and is compatible with all subspaces \( \text{ker } \mathbf{e}_i^{\lambda} \) and \( \text{ker } \mathbf{p}_i^{\lambda} \).

Proof: Let \( \lambda \in P_+ \), set \( \mathbf{v}_\lambda = (\mathbf{v}_\lambda^i, \lambda) \). Then \( \im \psi^\lambda = \{ \eta \in R \mid \eta^+ \in \bigcap_{i \in I} \text{Ker } K_i \mathbf{v}_\lambda \} \).

Application: Let \( B \) be a perfect basis of \( R \), let \( \lambda, \mu, \nu \in P_+ \). Then

\[ \dim \text{Hom}_g (L(\lambda), L(\mu) \otimes L(\nu)) = \# \{ b \in B \mid wt \mathbf{b} = \mu + \nu - \lambda, \mathbf{e}_i(b) \in \langle \omega_i^\mu, \mu \rangle, \mathbf{e}_i(b^+) \in \langle \omega_i^\nu, \nu \rangle \} \]

Proof: \( f \mapsto \psi_f (f(v_\lambda \otimes v_\mu \otimes v_\nu)) \) is an \( \text{bijection} \)

\[ \text{Hom}_g (L(\lambda) \otimes L(\mu) \otimes L(\nu)) \]

\[ \{ \eta \in R \mid wt(\eta) = \mu + \nu - \lambda, \eta^+ \in \bigcap_{i \in I} \text{Ker } K_i (\omega_i^\mu), \eta^+ \in \bigcap_{i \in I} \text{Ker } K_i (\omega_i^\nu) \} \]

Theorem (Kashiwara): \( B \) perfect basis, \( \lambda \in P_+ \). The basis of \( L(\lambda) \) dual to \( \psi^\lambda(B) \) (w.r.t. a contravariant form) is compatible with the Demazure submodules.

(Reference: M. Kashiwara, The crystal base and Littelmann's refined Demazure character formula, §§ 3.1-3.2. Kashiwara proves this for the upper crystal basis, but the argument is general.)
3) Examples

Type A₁: \[ N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \middle| x \in \mathbb{C} \right\} \quad C[N] = C[x] \]

\[ e = \frac{d}{dx} \quad B = \left\{ x^n \middle| n \in \mathbb{N} \right\} \quad \text{(exists and is unique)} \]

Type A₂: \[ N = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{C} \right\} \quad C[N] = C[x, y, z] \]

\[ e_1 = \frac{2}{3x} \quad e_2 = \frac{2}{3y} + x \frac{2}{3y} \]

Flag minors of \( L(\omega_1) \): \( x, z \)

Flag minors of \( L(\omega_2) \): \( y, xy-z \)

\[ B = \left\{ x^a y^b (xy-z)^c \middle| a, b, c \in \mathbb{N} \right\} \cup \left\{ y^a z^b (xy-z)^c \middle| a, b, c \in \mathbb{N} \right\} \]

(Exists and is unique)

(One can check here that in this basis, \( e_1 \) and \( e_2 \) act with coefficients in \( \mathbb{N} \), and that the structure constants of the multiplication belong to \( \mathbb{N} \).)

Type A₃: Still have existence, uniqueness, and explicit formulas

(Reference: A. Berenstein, A. Zelevinsky, String bases for quantum groups of type \( A_n \). This paper is the starting point of the theory of cluster algebras.)

In general: no uniqueness; existence ensured by several constructions:

- Lusztig's dual canonical basis = Kashiwara's upper crystal basis (specialized at \( q=1 \)).

- Basis arising from KLR algebras: \( R_v = G_v(R(V)) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{C} \) (at \( q=1 \))

  (Simple graded \( R(V) \)-modules up to isomorphism and up to a shift in the graduation give a perfect basis)

- Lusztig's dual semicanonical basis (\( A \) symmetric)

- the MV basis, arising from geometric Satake equivalence (\( A \) of finite type)

  (proof: see II)
4) Uniqueness of crystal

Theorem (Berenstein-Kashiwara): Let $B', B''$ be two perfect bases of $R$.

Then $\exists !$ bijection $B' \rightarrow B''$ that preserves the combinatorial data
(wt, $\xi_i, \eta_i, \tilde{c}_i, \tilde{f}_i$). In addition, it commutes with $\dagger$.

Notation: $(B(\infty), \text{wt}, \xi_i, \eta_i, \tilde{c}_i, \tilde{f}_i, \dagger)$ the abstract set with combinatorial data
common to all perfect bases of $R$ (Kashiwara's crystal).

5) Saito's reflections

For $i \in I$ and $b \in B_i$, define $\xi_i^+ b = (\xi_i b^\dagger)^\dagger$ (right action of $U$ on $R$).

Theorem-Definition (Saito): Let $i \in I$.

There are inverse bijections $\{ b \in B(\infty) | \xi_i(b) = 0 \}$ given by

$\sigma_i(b) = \xi_i (\xi_i^+ b) = \xi_i (\xi_i^+ \xi_i^{-\min} b)$

$\xi_i^+ (\xi_i^+) = (\xi_i(b))^\dagger$

if $\xi_i(b) = 0$.

Note that $\text{wt } \sigma_i(b) = \text{wt } b - (\xi_i^+ b - \xi_i (b^\dagger)) \omega_i = \text{wt } b - (\omega_i^\dagger \text{wt } b^\dagger) \omega_i = \omega_i (\text{wt } b)$.

For convenience, set $\xi_i^+ (b) = \sigma_i (\xi_i^+ \omega_i b)$ for all $b \in B(\infty)$, $i \in I$.

Remark (Tingley): If $n \geq \xi_i(b) + \xi_i(b^+) + (\omega_i^\dagger \text{wt } b)$, then $\sigma_i(b) = (\xi_i^+)^{\text{max} \frac{n}{\omega_i}} b$.

(always a $\geq 0$ number)

Proposition: The $\xi_i: B(\infty) \rightarrow B(\infty)$ satisfy the braid relations.

Proof: see III

Notation: For $w = \omega_i \cdots \omega_k$ reduced, set $\xi_w = \xi_i \cdots \xi_k$. 
6) Minkovici-Vilenken polytopes

A of finite type

Notations: \( \Phi \) root system of \( \mathfrak{g} \)

- Weyl fan in \( \mathfrak{g}_R^* \) (described by the root hyperplanes)
- \( \overline{Q}^+_+ = \{ \sum a_i \alpha_i | a_i \in \mathbb{R}_+ \} \subset \mathfrak{g}_R^* \)

Definition (Kannitger): A Gelfand-Graevski-Molevan-Serganov polytope is a woven polytope

\( p \subset \mathfrak{g}_R^* \) when dual fan is a coarsening of \( W \).

Picture in type \( A_2 \)

To a chamber \( wC_i \), corresponds a vertex \( \mu_w \) of \( p \). (Vertices are allowed to be non-distinct)

Lemma (Kannitger): \( G \rightarrow \{ \text{GCMV polytopes} \} \rightarrow \{ \text{collection } (\mu_w) \in (\mathfrak{g}_R^*)^W | \forall w, \mu_w \in \mu_w + w \overline{Q}^+_+ \} \)

\( A \rightarrow \{ \text{collection } (\mu_w) \mid \forall w, \forall j, \mu_{w^j - w} \in \mathbb{R}_+ w \delta_i \} \}

Back to \( B(\varnothing) \).

Definition: For \( b \in B(\varnothing) \) and \( w \in W \), set \( \mu_w(b) = w \cdot \text{wt}(\hat{\sigma}_w^\vee b) \).

Observation: \( wC_i \succ W \Rightarrow (\mu_{w^j}(b) - \mu_j(b)) = w \left[ \text{wt}(\hat{\sigma}_i^\vee b) - \text{wt}(b) \right] = \hat{\epsilon}_i(b) \cdot w \delta_i \geq 0 \)

So \( \text{Conv} \{ \mu_w(b) | w \in W \} \subset \text{GCMV} P(b), \text{the MV polytope of } b \).
7) Lusztig data

A still of finite type.

\[ U_q(g) \] the quantum group \( \mathcal{L}(q) \); generators \( E_i, F_i, k_i^{\pm 1} \)

\[ T_i : U_q(g) \rightarrow U_q(g) \] Lusztig's automorphism

a quantum analogue of \( \text{Ad}(\tilde{a}_i) \).

\[ \tilde{\tau} : U_q(g) \rightarrow U_q(g) \] the bar involution; \( C \)-algebra automorphism, \( \tilde{q} = q^{-1}, \tilde{E}_i = E_i \)

Given \( \tilde{\tau} = (\tilde{a}_1, \ldots, \tilde{a}_n) \) such that \( a_1 \cdots a_n \) reduced decomposition of \( \tilde{a}_{\tilde{\tau}} \):

- enumeration of the positive roots \( \beta_1, \ldots, \beta_n \)

- \( \beta_k = a_1 \cdots a_k \)

- PBW basis of \( U_q(\mathfrak{n}_+) \)

\[ \left\{ E_{\beta_1}^{(n_1)} \cdots E_{\beta_n}^{(n_n)} \right\} \quad \text{such that} \quad E_{\beta_1}^{(n_1)} \cdots E_{\beta_n}^{(n_n)} = E_{\sigma(\beta_1)}^{(n_1)} \cdots E_{\sigma(\beta_n)}^{(n_n)} \quad \text{for all} \quad \sigma \in S_n \]

- \( \tilde{\tau} = (\tilde{a}_1, \ldots, \tilde{a}_n) \) basis of \( U_q(\mathfrak{m}_+) \); independent of \( \beta \) : canonical basis.

Specialization at \( q = 1 \) gives a basis of \( U \), whose dual is perfect.

Theorem (Lusztig): \( \forall \ n \in \mathbb{N}^\ast, \exists ! \) bar-invariant element in \( U_q(\mathfrak{n}_+) \)

\[ \tilde{E}_{\beta_1}^{(n)} (n) = \sum_{m \in \mathbb{N}^\ast} \tilde{E}_{\beta_1}^{(n)} E_{\beta_1}^{(m)} \]

such that \( \tilde{E}_{\beta_1}^{(n)} = 1 \) and \( \tilde{E}_{\beta_1}^{(n)} \in q \mathbb{Z}[q] \) for all \( m \neq n \).

\[ \left\{ \tilde{E}_{\beta_1}^{(n)} \right\} \quad \forall \ n \in \mathbb{N}^\ast \] basis of \( U_q(\mathfrak{m}_+) \), independent of \( \beta \) : canonical basis.

Notation: \( B(\infty) \) denotes the dual canonical basis, whereas a bijection \( B(\infty) \rightarrow \mathbb{N}^\ast \)

\( b \mapsto n = N(\beta, b) \)

\( N(\beta, b) \): Lusztig data of \( b \) in direction \( \beta \).

Theorem (Saito): \( n_1 = \tilde{s}_1(b) ; n_2 = \tilde{s}_1(\tilde{s}_2(b)) \) \( \ldots \) \( n_k = \tilde{s}_1(\tilde{s}_2(\cdots \tilde{s}_k(b))) \) \( \ldots \)

\( ( \tilde{s}_1 \) mimics on \( B(\infty) \) the action of \( T_i^{-1} \) on PBW monomials. \)

Corollaries:
1) \( \tilde{m}_{i_1 \cdots i_k}(b) - \tilde{m}_{i_1 \cdots i_k}(b) = n_k \beta_k \)

( The length of the edges of \( \text{Pol}(b) \) are Lusztig data of \( b \). )

2) \( b \mapsto \text{Pol}(b) \) is injective.
A of finite type

1) Background on geometric Satake equivalence

\( G \) connected alg. gp. s.t. Lie \( G = \mathfrak{g} \)

\( U, B, U_T \)

\( \mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \)

\( X = \text{Hom}(T, \mathbb{C}^*) \)  \( (X = P \text{ if } G \text{ simply connected}) \)

\( T^\vee = X \otimes \mathbb{Z} \mathbb{C}^* \) dual torus

\( \text{Hom}(T^\vee, \mathbb{C}^*) = X^\vee \) dual lattice

\( \tilde{X}^\vee \)

\( G^\vee \) Langlands dual

\( \mathcal{O} = \mathbb{C}[[t]], \ K = \mathbb{C}((t)) \)

\( G^\vee = G^\vee(X)/G^\vee(\mathcal{O}) \) affine Grassmannian of \( G^\vee \)

(like a \( G/P \) with \( P \) parabolic maximal, but for a Kac-Moody group, so infinite dimensional. However, this is the limit of a direct system of projective varieties and closed embeddings, namely the Schubert varieties)

\( \text{Perv} = \{ G^\vee(\mathcal{O})\text{-equiv. perverse sheaves on } G^\vee \text{ with coeff in } \mathcal{O} \text{ and fin. dim. supp.} \} \)

abelian rigid monoidal category

\( H: \text{Perv} \rightarrow \text{Vect} \) exact, faithful, monoidal

\( \Rightarrow \text{Perv} \cong \text{Rep} \ G \) pro-algebraic gp (Saavedra Rivano's theorem)

Beilinson-Drinfeld, Ginzburg, Mirković-Vilonen (+ Lusztig): \( \overline{G} \cong G \)
\[ X = \text{Hom}(C^*, T^V) \]

\[ \lambda \mapsto k^\lambda = \text{image of } t \in k^* \text{ in } T^V(\lambda) \text{ or } G_\lambda. \]

\[ G_\lambda \subset G \text{ the } G^v(0) \text{- orbit of } k^\lambda \]

\[ G_\lambda = \bigsqcup_{\lambda \in X_+} k^\lambda \]

Simple objects in \( \text{Pev} \): \[ J_\lambda = I_C(\overline{G_\lambda}, \lambda) \]

\[ \bigoplus_{H} L(\lambda) \]

2) The MV basis

\[ G^V \]

\[ U \]

\[ B_- \subset N_- \]

\[ U \]

\[ T^V \]

For \( \nu \in X \), let \( T^\nu \subset G_\nu \) the \( N^\nu(\lambda) \)-orbit of \( k^\nu \).

Define \( p : \bR^*_R \to \bR \), \( \omega \mapsto 1 \).

Minkošić–Vilomen:

\[ \forall \lambda \in \text{Pev}, \forall k \in \bZ, \quad \bigoplus_{\nu \in X_+} H^{k}_{T^\nu}(G_\nu, A) \to H^{k}(G_\nu, A) \text{ isomorphism} \]  \((?)\)

\[ \forall \lambda \in X_+, \forall \nu \in X, \quad H^{2p(\nu)}_{T^\nu}(G_\nu, j_\lambda) \cong H^{2p(\nu)}(\overline{G_\nu} \cap T^\nu) \text{ with roots in } C \text{ of pure spin } p(\lambda-\nu) \]

\[ z(\lambda)_\nu = \text{Im}(\overline{G_\nu} \cap T^\nu) \in \bZ \quad \mapsto \quad [z] \in H^{2p(\nu)}_{T^\nu}(G_\nu, j_\lambda) \text{ fundamental class} \]

\[ \bigoplus_{\nu} \{ [z] | z \in z(\lambda)_\nu \} \text{ basis of } H(G_\nu, j_\lambda) = L(\lambda) \]

Theorem (B – Kammüller): Via \( \Psi_\lambda : L(\lambda) \to \bR \), these bases glue together and give a perfect basis of \( \bR \). In this basis, the structure constants of the multiplication \( \in \bN \).
3) Action of $G$ (Gingburg, Vasserot)

Fix the isomorphism $G \cong \mathfrak{g}$.

Action of $T$ defined by $(\phi)$:

$$
\begin{align*}
\text{maximal torus of } \mathfrak{g} \\
\text{basic representation of } \phi' \\
\text{affine Kac-Moody algebra corresponding to } \phi'.
\end{align*}
$$

Consider the Plücker embedding:

$$
\begin{align*}
\mathfrak{g} & \hookrightarrow \mathbb{P}(L(\Lambda_0)) \\
\text{basic representation of } \phi' \\
\end{align*}
$$

(to simplify, assume here $G$ simple of adjoint type)

$L: \mathfrak{g} \otimes \mathbb{C}$ ample line bundle. Set $\mathfrak{e} = (e_i(x)) \cup ?$.

Since $[\mathfrak{h}, \mathfrak{e}] = 2 \mathfrak{e}$, $\mathfrak{h}$ acts on $H^k$ by multiplication by $k$.

we can write $\mathfrak{e} = \sum Q(\alpha_i) \mathfrak{e}_i$ with $\mathfrak{e}_i \in \phi^{\alpha_i}$ ($Q(\alpha_i)$ = square of length of $\alpha_i$, 1 if $\alpha_i$ short root).

Hard Lefschetz $\Rightarrow \exists e_{\mathfrak{h}}$-triple $(\mathfrak{h}, \mathfrak{f}, \mathfrak{e})$ for each $\mathfrak{e} \neq 0$.

Define $\mathfrak{g} = \mathfrak{e}$ by integrating the isomorphism $\phi = \phi'$, i.e., $\mathfrak{e} \rightarrow \mathfrak{e}$.

Geometric translation:

Choose v $\in X$. Minkovíč-Vilonen $\Rightarrow \mathfrak{g}_v = \bigcup_{x \in Q^+_\mathfrak{g}} \mathfrak{g}_x$ and $\bigcap_{x \in Q^+_\mathfrak{g}} \mathfrak{g}_x = \bigcup_{i \in I \subset I} \mathfrak{g}_i$.

For a well-chosen hyperplane $D = \mathbb{P}(L(\Lambda_0))$.

For $k = 2 p(\lambda)$ and $\lambda \in X^+_{\mathfrak{q}}$, $d = 2 p(\lambda)$:

$$
\begin{align*}
H^k(G, \mathfrak{g}) & \leftarrow H^k(\mathfrak{g}, \mathfrak{g}) \xrightarrow{\text{MV}} H^{\mathfrak{g}^\mathfrak{m}}(\mathfrak{g}, \mathfrak{g}) \rightarrow H^{\mathfrak{g}^\mathfrak{m}}(\mathfrak{g}, \mathfrak{g}) \\
\bigcup_{\lambda \in X^+_{\mathfrak{g}}} & \bigcup_{i \in I} H^{k+1}(\mathfrak{g}, \mathfrak{g}) \xrightarrow{\text{MV}} H^{\mathfrak{g}^\mathfrak{m}}(\mathfrak{g}, \mathfrak{g}) \\
\end{align*}
$$
4) Polytopes

Let \( Y \subset G \) be closed, \( T \)-invariant, finite dim.

For \( v \in X \), \( T^v \cdot Y = Y \) meets \( T^v \), and this holds for finitely many \( v \).

If \( Y \) irreducible, then \( \exists v \in X \) s.t. \( Y T^v \) open dense in \( Y \);

concretely, \( t^v \in Y \) and any \( p \in X \) s.t. \( t^v Y \) belongs to \( v \), \( v + Q^+ \).

Denote this \( v \) by \( \mu_i(Y) \).

For \( w \in W \), let \( \mu_{w,v}(Y) = w \cdot \mu_i \left( \tilde{w}^v \cdot Y \right) \), where \( \tilde{w} \in G^v \) lift of \( w \).

Fact: \( \forall x, w, \quad \mu_{x,v}(Y) \in \mu_{w,v}(Y) + w \cdot Q^+ \)

\( \rightarrow \) GMS polytope \( \text{Pol}(Y) \).

Remark: \( Y \subset Z \Rightarrow \mu_{x,v}(Y) \in Z \Rightarrow \mu_i(Y) \in \mu_i(Z) + Q^+ \)

\( \Rightarrow \) plus généralement, \( \forall w, \mu_{w,v}(Y) \in \mu_{w,v}(Z) + w \cdot Q^+ \)

\( \Rightarrow \) \( \text{Pol}(Y) \subset \text{Pol}(Z) \)

Notation: \( \mu_{w,v}(Y) - \mu_i(Y) = \varepsilon_i(Y) \varepsilon_w \)

\( \in \mathbb{N} \)

Fact 1: \( z \in \mathcal{L}(\lambda) \) \( \Rightarrow \mu_i(z) = v \) and \( \mu_{w,v}(z) = \lambda \)

\( \Rightarrow \) Let \( z \in \mathcal{L}(\lambda) \), let \( i \in I \).

Any \( Y \in \mathcal{L}(\lambda), Y \subset Z \subset Z \) satisfies \( \varepsilon_i(Y) \subset \varepsilon_i(Z) \)

Assume \( \varepsilon_i(Z) > 1 \). Then \( \exists! Y \in \mathcal{L}(\lambda) \) such that \( Y \subset Z \)
and \( \varepsilon_i(Y) = \varepsilon_i(Z) - 1 \) (Braverman-Gaitsgory), and with \( D \) as in \( \S 3 \)
the multiplicity of \( [Y] \) in \( [Z], [D] \subset D(X)_X \varepsilon_i(Z) \).
5) End of the proof

a) Look first at $L(\lambda)$ for $\lambda \in X_+$

Take $Z \in \mathcal{Z}(\lambda)_0$, choose $D$ as in §3.

Then $\mathcal{E}_z [Z] = [Z] \cdot D$ so $\mathcal{E}_z [Z] = \sum_{Y \in \mathcal{Z}(\lambda)_{\mu \nu, Y \subset Z}} \frac{\text{multiplicity}}{Q(\lambda)} [Y]

b) For all $Y$ here, $\mathcal{E}_z (Y) \leq \mathcal{E}_z (Z) - 1$; exactly one $Y$ has $\mathcal{E}_z (Y) = \mathcal{E}_z (Z) - 1$, and it appears with coefficient $\mathcal{E}_z (Z)$.

$$\Rightarrow \mathcal{P}_{z, (k)} [Z] = \begin{cases} 0 & \forall \; k > \mathcal{E}_z (Z) \\ [V] & \text{for a } V \in \mathcal{Z}(\lambda)_{\nu + k, x_i} \land k = \mathcal{E}_z (Z) \end{cases}$$

c) For any $\lambda, \rho \in X_+$ and $V \in X$, there exists a linear injection $\mathcal{L}(\lambda)_0 \hookrightarrow \mathcal{L}(\lambda + \rho)_0 \rightarrow (J. \text{Anderson})$

$$Z \mapsto Z^{\rho}$$

whence a linear injection $i : L(\lambda) \hookrightarrow L(\lambda + \rho)$.

Step a) $\Rightarrow i$ is a map of $n_+$-module

$$\begin{array}{c}
\xymatrix{ & L(\lambda) \ar[r]^-{\Psi_\lambda} & R \\
L(\lambda + \rho) \ar[r]_-{\Psi_{\lambda + \rho}} & } \\
\end{array}$$

so the bases of the $L(\lambda)$ glue correctly and give a basis of $R$.

d) Step b) $\Rightarrow$ this basis satisfies (P2)

(I omit the proof of (P3) and of the last assertion of the theorem.)

6) Relation to MV polytopes

Theorem (Kamnitzer's thesis): Let $\lambda \in X_+$, let $Z \in \mathcal{Z}(\lambda)_0$. Suppose $\Psi_\lambda [Z]$ is indexed by $b \in B(\infty, x \lambda)$. Then $\text{Pol} (Z) = \lambda + \text{Pol} (b)$.

Proof: requires finer description of MV cycles + Berenstein, Fomin and Zelevinsky's chamber Anomaly, whose tropicalization describe Lusztig data.
Proposition: The transition matrix between the MV basis and the dual canonical basis is upper unitriangular and the order on $B(\infty)$ given by inclusion of MV polytopes.
III  Preprojective algebras and the semicanonical basis

A symmetric

1) Preprojective algebras

Oriented graph without loops: 
\[ \{ \text{vertices} \} = \mathbb{I} \]
\[ \{ \text{edges} \} = \mathcal{H} : \text{between } i \text{ and } j, \text{ -aj edges in each direction} \]
\[ \cup \]
\[ h \to \mathbb{H} \text{ orientation reversing involution} \]
\[ \varepsilon : \mathcal{H} \to \{ \pm 1 \} \text{ such that } \varepsilon(h) + \varepsilon(h^{-1}) = 0 \]

K field

\[ \Lambda = K \text{-path algebra of the quiver } (\mathbb{I}, \mathcal{H})/ \left\langle \sum_{h \in \mathcal{H}} \varepsilon(h) \mathbb{H}^2 \right\rangle \text{ completed with the ideal generated by the arrows} \]

Example: type \( A_3 \)
\[ (\mathbb{I}, \mathcal{H}) = \begin{array}{c}
\mathbb{I} \\
\mathbb{H} \\
\mathbb{H}
\end{array} \]
\[ \varepsilon(h) = \varepsilon(h)^{-1} \cdot 1 \]

\[ \Lambda \text{-module } M_1 \xrightarrow{M_2} M_2 \xrightarrow{M_3} M_3 \]
\[ \Delta \mathbb{F} = M_{\mathbb{F}} M_{\mathbb{F}} = M_{\mathbb{F}} M_{\mathbb{F}} - M_{\mathbb{F}} M_{\mathbb{F}} = 0 \]

\[ M \text{ a } \Lambda \text{-module } \rightsquigarrow \dim M = \sum_{i \in \mathbb{I}} (\dim M_i) \alpha_i \in \mathbb{Q}_+ \text{ dimension vector} \]

Simple \( \Lambda \)-modules: \( S_i \) \((i \in \mathbb{I})\), \( \Lambda \)-dimensional, concentrated on vertex \( i \);
\[ \dim S_i = \alpha_i \]

\[ M \mapsto \dim M \text{ induces } K(\Lambda \text{-mod}) = \bigoplus_{i \in \mathbb{I}} \mathbb{Z} \alpha_i \]

Duality operation on \( \Lambda \text{-mod} \):
\[ M = (\bigoplus_{i \in \mathbb{I}} M_i, (M_h)) \mapsto M^\dagger = (\bigoplus_{i \in \mathbb{I}} M_i^*, (M_h^*)) \]

Representation spaces:
\[ \forall v = \sum_i \nu_i \alpha_i \in \mathbb{Q}_+, \text{ let } \Lambda(v) \subset \prod_{h \in \mathcal{H}} \text{Hom}_K(K^{\nu_h} \mathbb{I}, K^{\nu_h}) \]
\[ G(v) = \prod_{i \in \mathbb{I}} GL(K) \]

affine variety of \( \Lambda \)-module structures on \( \bigoplus_{i \in \mathbb{I}} K^{\nu_i} \)

"Lusztig's nilpotent varieties"
2) Lusztig's semi-canonical basis

Take here $K = C$.

Let $\nu \in \mathbb{Q}_+$. For a $\Lambda$-module $M$ of dim. vector $\nu$, define $\delta_M : U_\nu \to C$ by:

If $\alpha_1 + \ldots + \alpha_k = \nu$, then (Lusztig; Geiβ- Leclerc - Schröer)

$$\delta_M(e_{i_1} \ldots e_{i_k}) = \chi \left( \left\{ 0 : M_0 < M_1 < \ldots < M_k = M \middle| \dim M_p/M_{p-1} = \alpha_p \right\} \right)$$

A closed subset of the product of flag manifolds

Example: Type $A_2$  

\[
\begin{align*}
C^2 & \overset{0}{\underset{0}{\longrightarrow}} C \\
S_1 \oplus S_2 & \quad \chi(\mathbb{F}) = 2 \\
S_1 \oplus T_1 & \quad \chi(\mathbb{F}) = 2 \\
S_1 \oplus T_2 & \quad \chi(\mathbb{F}) = 0
\end{align*}
\]

in dimension vectors $\nu = 2\alpha_1 + \alpha_2$

\[
\begin{array}{c|c|c|c}
M & \delta_M(e_1 e_2) & \delta_M(e_1 e_2 e_1) & \delta_M(e_2 e_1) \\
\hline
S_1 \oplus S_2 & \chi(\mathbb{F}) = 2 & \chi(\mathbb{F}) = 2 & \chi(\mathbb{F}) = 2 \\
S_1 \oplus T_1 & \chi(\mathbb{F}) = 2 & \chi(\mathbb{F}) = 1 & \chi(\mathbb{F}) = 0 \\
S_1 \oplus T_2 & 0 & 1 & 2
\end{array}
\]

As one can see, the Sene relation $\delta_M(e_1 e_2) - 2 \delta_M(e_1 e_2 e_1) + \delta_M(e_2 e_1) = 0$ is always satisfied, so $\delta_M$ is well defined.

Observation: The $\delta_M$ are not linearly independent. But they span $R_\nu = (U_\nu)^*$. 

Problem: Extract a basis 

$(\Lambda(\nu) \to R_\nu, M \mapsto \delta_M)$ is constructible; for each $Z \in \text{Im} \Lambda(\nu)$, define $\delta_2$ as $\delta_M$ for $M$ general in $Z$.

Theorem (Lusztig): \[
\bigcup_{\nu \in \mathbb{Q}_+} \{ \delta_2 \mid Z \in \text{Im} \Lambda(\nu) \} \text{ is a basis of } R \text{ "dual semi-canonical basis".}
\]
Definition: \( M \) a \( \Lambda \)-module, \( \iota \in \mathbb{I} \).

1. Head of \( M \): \( \text{hd}_i M \): largest quotient of \( M \) isomorphic to \( S_i^{\oplus \infty} \)\
   (\( i \)-th part of the head of \( M \))

Observations: \( \delta_i(M) \neq 0 \Rightarrow \exists N \rightarrow S_i^{\oplus n} \Rightarrow n \leq \dim \text{hd}_i M \)

and for \( n = \dim \text{hd}_i M \), \( e_i^{(n)}M = \delta_i N = 0 \), where \( N = \ker (M \rightarrow \text{hd}_i M) \).

(Not have that the divided power \( n! \) is in the Euler characteristic of the flag variety of \( \text{hd}_i M \).)

Moreover, if \( M \) is generic, then \( N \) is generic.

\[ \delta_i(M)^+ = \delta_i N^+ \]

Conclusion: 1) The dual semicanonical basis is perfect.

2) \( \bigoplus_{\iota \in \mathbb{I}} \mu_\iota \Lambda_\iota \) is canonically induced by \( S(\infty) \) (Kashiwara–Saito)

an ingredient of the proof of Lusztig's theorem.

3) Reflection functors

Let \( i \in \mathbb{I} \).

Local description around \( i \) of a \( \Lambda \)-module \( M \):

\[ \bigoplus_{k \in \mathbb{K}} \mu_k \Lambda_k \rightarrow M_i \rightarrow \bigoplus_{\alpha \in \mathbb{A}} \mu_{\alpha(i)} \Lambda_{\alpha(i)} \]

for brevity: \( \widetilde{M} \rightarrow \mu_{\text{in}(i)} \rightarrow M_i \rightarrow \mu_{\text{out}(i)} \rightarrow \widetilde{M} \)

Note: \( \text{hd}_i M = \text{co} \mu_{\text{in}(i)} \) set \( \text{soc}_i M = \text{ker} \mu_{\text{out}(i)} \) is socle of \( M \)

Define \( \Sigma_i M \) by replacing in \( M \) the part (\( \ast \)) by \( \widetilde{M} \rightarrow \mu_{\text{in}(i)} \rightarrow \text{ker} \mu_{\text{out}(i)} \rightarrow \widetilde{M} \)

\( \Sigma_i^+ M \)

Still get \( \Lambda \)-modules, because \( \widetilde{M} \rightarrow \widetilde{M} \) hasn't changed, and at vertex \( i \), the composed of the two maps is zero.
Facts: These functors induce equivalences of categories
\[
\left\{ M \in \Lambda\text{-mod} \mid \text{ld}_i M = 0 \right\} \xleftarrow{\Sigma_i} \left\{ M \in \Lambda\text{-mod} \mid \text{soc}_i M = 0 \right\}
\]
Moreover \( \text{ld}_i M = 0 \Rightarrow \dim \Sigma_i M = \delta_i (\dim M) \).

Theorem (B.): Let \( \lambda \in I \), set \( T_\lambda = \text{Ad}(\lambda) \in \text{Aut}(U(g)) \) \( (\lambda = \exp(z) \exp(-z) \exp(z)) \).
Let \( \nu \in Q_+ \), let \( M \in \Lambda\text{-mod} \) s.t. \( \text{dim} M = \nu \) and \( \text{ld}_i M = 0 \). Let \( x \in U_\nu \)
such that \( T_\lambda(x) \in U_\nu \). Then \( \langle \delta_M, \nu \rangle = \langle \delta_{\Sigma_i M}, T_\lambda(x) \rangle \).

Theorem (B.-Kamnitzer): Let \( b \in B(\infty) \), \( M \) general in \( \Lambda_b \Rightarrow \Sigma_i M \) general in \( \Lambda_{\hat{c}b} \).
(Loosely stated).
(Another interpretation of the Saito reflection: they now act on irreducible components of nilpotent varieties.)

4) Tilting theory in \( \Lambda\text{-mod} \)
\[ I_\lambda = \text{ann}_\Lambda \Sigma_i \lambda \text{. Then } \Sigma_i = \text{Hom}_\Lambda (I_\lambda, ?) \text{ and } \Sigma_i^\perp = I_\lambda \otimes_\Lambda ? \text{.} \]

Theorem (Buan-Iyama-Reiten-Scott)
1) The \( \Lambda\)-bimodules \( I_\lambda \) satisfy the braid relations: set \( I_{\lambda \lambda} = I_{\lambda \lambda} \otimes_\Lambda \text{ red.} \)
\( \text{red } \lambda = \lambda_1 \ldots \lambda_k \text{ reduced.} \)
(Consequence: the \( \Sigma_i \) satisfy the braid relations, hence the \( \hat{\Sigma}_i \) also do.)
2) \( I_{\lambda \lambda} \text{ tilting } \Lambda\text{-bimodule, } \text{End}_\Lambda (I_{\lambda \lambda}) = \Lambda \).

Brenner-Butler theory \( \Rightarrow \)
- \( I_{\lambda \lambda} \) defines a (in fact, two) torsion pair in \( \Lambda\text{-mod} \).
- Each \( M \in \Lambda\text{-mod} \) has a largest quotient \( M/N \) such that \( \text{Hom}_\Lambda (I_{\lambda \lambda}, M/N) = 0 \),
  namely \( N = \ker \left( I_{\lambda \lambda} \otimes_\Lambda \text{Hom}_\Lambda (I_{\lambda \lambda}, M) \rightarrow M \right) \).
Write \( M^{\text{tr}} \) for \( N \).
Examples: \( M^{\text{tr}} M, M^{\text{tr}} = \ker (M \rightarrow \text{ld}_i M) \), \( M^{\text{tr}} \subset M^{\text{tr}} \) if \( \lambda(\lambda
u) = \lambda(\lambda) + \lambda(\nu) \).
Prop (B.-Kamnitzer-Tracy): Let \( b \in B(\infty) \), \( M \) general in \( \Lambda_b \), \( w \in W \). Then \( \mu_w(b) = -\dim M^w \).

Proof: \( w = s_{i_1} \ldots s_{i_k} \) reduced.

\[ \text{Hom}_\Lambda(I_n, M) = \Sigma_{i_k} \cdots \Sigma_{i_1} M \text{ in general in } \Lambda_{n}, \text{ where } b' = \widehat{s}_{i_k} \cdots \widehat{s}_{i_1} b = \widehat{s}_{i_k} \cdots \widehat{s}_{i_1} M. \]

\[ \dim M^w = \dim \text{Hom}_\Lambda(I_n, M) = -w \dim b' = -\mu_w(b). \]

5) Harder-Narasimhan polytopes

A finite length category

\( T \subseteq A \rightarrow [T] \in K(A) \)

Definition: \( P(T) = \text{convex hull in } K(A)_R = K(A) \otimes \mathbb{R} \text{ of } [X] \text{ for } X \in T. \)

HN polytope of \( T \)

(Convex hull of a finite number of points).

Faces of \( P(T) \): each \( \Theta \in K(A)_R^* \) defines \( P_\Theta(T) = \{ x \in P(T) \mid \langle x, \Theta \rangle = \text{sup } P(T) \} \)

Fact: \( \{ X \in T \mid [X] \in P_\Theta(T) \} \) has a smallest element, \( T_\Theta^{\min} \), and a largest one, \( T_\Theta^{\max} \).

Exercise: Let \( \Theta_B = \{ T \in A \mid \langle \Theta, [T] \rangle = 0 \text{ and } \forall X \in T, \langle \Theta, [X] \rangle \leq 0 \} \)

(\( \Theta \)-ministable objects) Then \( \Theta_B \) is a biadjoint subcategory, \( T_\Theta^{\max} / T_\Theta^{\min} \in \Theta_B \),

and for \( i: R_\Theta \rightarrow A \),

\[ P_\Theta(T) = [T_\Theta^{\min}] + K_A \left( P \left( T_\Theta^{\max} / T_\Theta^{\min} \right) \right) \]

HN polytope relative to \( R_\Theta \)

(Hereditary property: a face of a HN polytope is a HN polytope,
flattened and shifted.)
Case $A = \Lambda$-mod

$$K(\Lambda \text{-mod}) = \bigoplus_{i \in I} \mathbb{Z} x_i, \quad K(\Lambda \text{-mod})_\mathbb{R} = f^*_R.$$ Denote $C_0 \subset f^*_R$ dominant chamber.

$$[M] = \dim M$$

Theorem (B.-Kamnitzer-Tingley): Assume $A$ of finite type.

Let $\Theta \in f^*_R$. Then \{ $\omega \in W$ | $\omega \Theta \in C_0$ \} has a shortest element, $w_1$, and longest element, $w_2$.

Then for each $\Lambda$-module $M$, $M^\min_\Theta = M^{w_2}$ and $M^\max_\Theta = M^{w_1}$.

In particular, $P(M)$ is GAGMS and equals \{convex hull ({$\dim M^w$ | $w \in W$})\}.

Corollary: $b \in B_\infty$, $M$ general in $A_b$ $\Rightarrow$ $P(e)(b) = -P(M)$.

Example: Type $A_2$, $M = S^\Theta_a \oplus S^\Theta_b \oplus T^\Theta_c \oplus T^\Theta_d$.

![Diagram](image)

$M$ general $\Rightarrow$ $a \cap b = 0$ $\Rightarrow$ one of the two diagonals is $\parallel$ to the opposite side.

All 2-faces of type $A_2$ of an MV polytope have this property ("Tropical Bleicher Relations"). This characterizes MV polytopes among all lattice GAGMS polytopes (see also TPR for 2-faces of type $B_2$ and $G_2$).

In view of the last Corollary in Part I, this condition translates to relations between Lustzig data $N_i(2,b)$ for $b$ fixed, $i$ variable. These relations are equivalent to Lustzig's piecewise linear bijections.