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Cobordism of Nonspherical Links

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0 Introduction and definitions

0.1. After the knot theory developed by Kervaire [K1], [K2] and Levine [L1], [L2], in which they gave a classification of spherical knots up to cobordism, Lê [Lê] showed that the algebraic one-dimensional spherical knots have a particular behavior. More precisely, Lê proved that cobordant algebraic knots of dimension one are isotopic. Some 20 years later, du Bois and Michel proved that in high dimensions (i.e., $2n-1$ with $n \geq 3$), things are completely different: du Bois and Michel found for, any $n \geq 3$, examples of cobordant algebraic spherical knots, of dimension $2n-1$, which are not isotopic.

Using the spherical knots given by du Bois and Michel, we construct the first examples of cobordant nonspherical links which are nonisotopic. The links we construct here are fibered and nonalgebraic.

Let us be more precise.

0.2. A *link* is an $(n-2)$-connected, oriented, smooth, closed, $(2n-1)$-dimensional submanifold of S^{2n+1}. A *knot* is a spherical link (i.e., a link abstractly homeomorphic to S^{2n-1}). For any link K, there exists a smooth, compact, oriented $2n$-submanifold F of S^{2n+1}, having K as boundary; such a manifold F is called a *Seifert surface* for K.

0.3. Following Kervaire [K1], we say that two links K_0 and K_1, abstractly diffeomorphic to the same manifold $\mathcal{K}$, are *cobordant* if there exists an embedding Φ, $\Phi\colon\ \mathcal{K} \times [0,1] \to S^{2n+1} \times [0,1]$, such that

$$\Phi(\mathcal{K} \times \{0\}) = K_0 \text{ and } \Phi(\mathcal{K} \times \{1\}) = -K_1,$$

where $-K_1$ is the link K_1 with the orientation reversed.

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0.4. For any $2n$-dimensional oriented, smooth, submanifold F of S^{2n+1}, we denote by G the quotient of $H_n(F)$[1] by its $\mathbb{Z}$-torsion.

The *Seifert form* associated to F is the bilinear form $A\colon\ G \times G \to \mathbb{Z}$ defined as follows (see also [K2, p. 88] or [L2, p. 185]): let (x, y) be in $G \times G$; then $A(x, y)$ is the linking number in S^{2n+1} of x and $i_+(y)$, where $i_+(y)$ is the cycle y "pushed" in $(S^{2n+1} \setminus F)$ by the positively oriented vector field normal to F in S^{2n+1}.

By definition, a *Seifert form for a link* K is the Seifert form associated to a Seifert surface for K.

0.5. A *simple link* is a link which has an $(n-1)$-connected Seifert surface. A link K is a *simple fibered link* if there exists a differentiable fibration $\varphi\colon\ S^{2n+1} \setminus K \to S^1$, being trivial on $U \setminus K$, where U is a "small" open tubular neighbourhood of K, and having $(n-1)$-connected fibers, the adherence of which are Seifert surfaces for K.

An *algebraic link* is a link $K(f)$ associated to a holomorphic germ f with an isolated singularity. Furthermore, Milnor's theory of singular complex hypersurfaces implies that algebraic links are simple fibered links [M].

0.6. Let $\mathcal{A}$ be the set of bilinear forms defined on free $\mathbb{Z}$-modules G of finite rank. Let ε be $+1$ or -1. If A is in $\mathcal{A}$, let us denote by A^T the transpose of A, by S the ε-symmetric form $A + \varepsilon A^T$ associated to A, by $S^*\colon\ G \to G^*$ the adjoint of S (G^* being the dual $\mathrm{Hom}_{\mathbb{Z}}(G; \mathbb{Z})$ of G), and by $\overline{S}\colon\ \overline{G} \times \overline{G} \to \mathbb{Z}$ the ε-symmetric nondegenerated form induced by S on $\overline{G} = G_{/\mathrm{Ker}\, S^*}$. A submodule M of G is pure if $G_{/M}$ is torsion free. If M is any submodule of G, let us denote by $M^\wedge$ the smallest pure submodule of G which contains M. In fact, $M^\wedge$ is equal to $(M \otimes \mathbb{Q}) \cap G$. For a submodule M of G, we denote by $\overline{M}$ the image of M in $\overline{G}$.

Definition. Let $A\colon\ G \times G \to \mathbb{Z}$ be a bilinear form in $\mathcal{A}$. The form A is *Witt associated to zero* if the rank m of G is even and if there exists a pure submodule M of rank $m/2$ in G such that A vanishes on M; such a module M is called a *metabolizer* for A.

0.7. Definition. Let $A_i\colon\ G_i \times G_i \to \mathbb{Z}$, $_{i=0,1}$, be two bilinear forms in $\mathcal{A}$. Let G be $G_0 \oplus G_1$ and let A be $(A_0 \oplus -A_1)$. The form A_0 is *algebraically cobordant* to A_1 if there exist a metabolizer M for A such that $\overline{M}$ is pure in $\overline{G}$, an isomorphism φ from $\mathrm{Ker}\, S_0^*$ to $\mathrm{Ker}\, S_1^*$, and an isomorphism θ from $\mathrm{Tors}\,(\mathrm{Coker}\, S_0^*)$ to $\mathrm{Tors}\,(\mathrm{Coker}\, S_1^*)$ which satisfy the two following conditions:

(c.1) $M \cap \mathrm{Ker}\, S^* = \{(x, \varphi(x)); x \in \mathrm{Ker}\, S_0^*\}$;

(c.2) $d(S^*(M)^\wedge) = \{(x, \theta(x)); x \in \mathrm{Tors}\,(\mathrm{Coker}\, S_0^*)\}$, where d is the quotient map from G^* to $\mathrm{Coker}\, S^*$.

[1] We denote by $H_n(F)$ the nth homology group of F with integer coefficients.

Remark. In the previous definition, $S_i = A_i + \varepsilon A_i^T$ is the intersection form on $H_n(F_i)$, $\operatorname{Ker} S_i^*$ is the image of $H_n(K_i)$ in $H_n(F_i)$, and $\operatorname{Coker} S_i^*$ is isomorphic to $\tilde{H}_{n-1}(K_i)$. So for spherical links, both $\operatorname{Ker} S_i^*$ and $\operatorname{Coker} S_i^*$ are zero, and conditions (c.1) and (c.2) vanish.

In order to prove that the links we will construct are cobordant, we will use the following theorem.

0.8. Theorem ([BM]). If $n \geq 3$, two simple fibered links, of dimension $2n-1$, are cobordant if and only if they have algebraically cobordant Seifert forms. □

1 Statement of result and proof

1.1. We will prove the following proposition.

Proposition. For all $n \geq 3$, there exist cobordant nonspherical fibered links of dimension $2n-1$ which are not isotopic. □

Proof. Let us fix $n \geq 3$. We will use the spherical knots K_0 and K_1 of dimension $2n-1$, constructed by du Bois and Michel in [DM]. These knots are the first example of cobordant and nonisotopic algebraic spherical knots. Now we will use them to construct some nonspherical fibered links.

Let K_i, with $i = 0, 1$, be the algebraic link of dimension $2n-1$ associated to the isolated singularity at zero of the germs of holomorphic functions $h_i\colon (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ defined by

$$h_i(x_0, \dots, x_n) = g_i(x_0, x_1) + x_2^p + x_3^q + \sum_{k=4}^{n} x_k^2,$$

with $g_0(x_0, x_1) = (x_0 - x_1)((x_1^2 - x_0^3)^2 - x_0^{s+6} - 4x_1 x_0^{(s+9)/2})((x_0^2 - x_1^5)^2 - x_1^{r+10} - 4x_0 x_1^{(r+15)/2})$, and $g_1(x_0, x_1) = (x_0 - x_1)((x_1^2 - x_0^3)^2 - x_0^{r+14} - 4x_1 x_0^{(r+17)/2})((x_0^2 - x_1^5)^2 - x_1^{s+2} - 4x_0 x_1^{(s+7)/2})$.

Here $s \geq 11$ and $s \neq r + 8$ are odd, and p and q are distinct prime numbers which do not divide the product $\epsilon = 330(30 + r)(22 + s)$ ([DM, p. 166]). We denote by A_i, $i = 0, 1$ the Seifert form associated to K_i defined on a free $\mathbb{Z}$-module of finite rank H_i.

Let L be the algebraic link of dimension $2n-1$ associated to the isolated singularity at zero defined by the germ

$$\begin{aligned} f\colon (\mathbb{C}^{n+1}, 0) &\to (\mathbb{C}, 0) \\ (x_0, \dots, x_n) &\mapsto \sum_{k=0}^{n} x_k^2. \end{aligned}$$

According to [D, Proposition 2.2], this algebraic link has $A = ((-1)^{n(n+1)/2})$, defined on a free $\mathbb{Z}$-module of rank one G, as a Seifert matrix.

We construct L_i the connected sum of L and K_i for $i = 0, 1$. The Seifert form for L_i is the integral bilinear form $A \oplus A_i$ defined on a free $\mathbb{Z}$-module $G_i = G \oplus H_i$ of finite rank. The links L_i are simple fibered since $A \oplus A_i$ is unimodular (see [KW, Chapter V, §3]) and the links L and K_i are simple.

According to [A, Theorem 4], the links L_0 and L_1, which are the connected sum of two algebraic links, cannot be algebraic.

Using Theorem 0.8, we will prove that L_0 is cobordant to L_1. First let us get some notation: $B = A \oplus A_0 \oplus -(A \oplus A_1)$, $S = B + (-1)^n B^T$, $\Sigma = A + (-1)^n A^T$.

We have $A = (\pm 1)$ so Tors Coker$S_i^* \neq \{0\}$ or Ker$S_i^* \neq \{0\}$; hence L_i, $i = 0, 1$ are not spherical links.

Let M be the metabolizer for $A_0 \oplus -A_1$ given by du Bois and Michel. The module $N = \Delta_G \oplus M$, where $\Delta_G = \{x \oplus x, x \in G\}$, is a metabolizer for B. In order to have $A \oplus A_0$ algebraically cobordant to $A \oplus A_1$, we have to show that N fulfills (c.1) and (c.2) in 0.7.

(1) If $\operatorname{Ker} S^* = \{0\}$, then $\overline{N} = N$; hence condition (c.1) of 0.7 is fulfilled. Furthermore, Tors Coker $S^* =$ CokerΣ^*. This implies condition (c.2) of 0.7, and the two Seifert forms of L_0 and L_1 are algebraically cobordant.

(2) If $\operatorname{Ker} S^* = G$, then $N \cap \operatorname{Ker} S^* = \Delta_G$, and the metabolizer N fulfills (c.1) in 0.7; we also have that $\overline{N} = \overline{M} = M$ is pure. Moreover, Tors Ker$S^* = \{0\}$, so the Seifert forms of L_0 and L_1 are algebraically cobordant.

Now we are going to prove that the links considered are not isotopic. Let τ_i be the monodromy associated to the fibered link L_i; if there exists an integer e such that $(\tau_i^e - 1)\, G_i = 0$, then e is called an *exponent* for L_i.

Recall that the *e-twist* group for K_i is defined as follows: assuming $(t^e - 1)^2\, H_i = 0$, if e is an exponent for K_i, then the e-twist group associated to K_i is the group denoted by $GT^e(h_i)$ (or $GT^e(K_i)$), which is the $\mathbb{Z}$-torsion subgroup of the quotient $\operatorname{Ker}(t_i^e - 1)/(t_i^e - 1)H_i$.

According to the monodromy theorem (Breiskorn-Grothendieck), the e-twist group is well defined for one-dimensional algebraic links, and du Bois and Michel showed that

(1) ϵ is an even exponent for the algebraic knots associated to g_0 and g_1; for all multiples k of ϵ, the finite abelian groups $GT^k(g_0)$ and $GT^k(g_1)$ have distinct orders;

(2) the k-twist group for h_0 and h_1 are well defined, and $GT^k(h_i) = (GT^k(g_i))^{(p-1)(q-1)}$ $i = 0, 1$.

Let k be a multiple of $\epsilon = 330(30 + r)(22 + s)$. For a fibered link L, the matrices A of Seifert form and τ of the monodromy are related together by : $\tau = (-1)^n A^{-1} A^T$. Hence for $i = 0, 1$ we have $\tau_i = (\pm \mathrm{Id}) \oplus t_i$. Thus $GT^k(L_i)$ is well defined, and we have $GT^k(L_i) = GT^k(h_i)$.

Finally, $GT^k(L_0)$ and $GT^k(L_1)$ have distinct order and the $\mathbb{Z}[t, t^{-1}]$-modules $H_n(G_0)$ and $H_n(G_1)$ are not isomorphic. Hence the links L_0 and L_1 are not isotopic. ∎

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