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PULL BACK RELATION FOR NON-SPHERICAL KNOTS

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ABSTRACT

We introduce a new relation for high dimensional non-spherical knots, which is motivated by the codimension two surgery theory: a knot is a pull back of another knot if the former is obtained as the inverse image of the latter by a certain degree one map between the ambient spheres. We show that this relation defines a partial order for $(2n-1)$-dimensional simple fibered knots for $n \geq 3$. We also give some related results concerning cobordisms and isotopies of knots together with several important explicit examples.

Keywords: Pull back; codimension two surgery; high dimensional knot; cobordism; fibered knot.

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1. Introduction

In this paper, by a *knot* (or an *m-knot*) we mean an oriented manifold (of dimension m) embedded in a sphere in codimension two. We do not distinguish a knot from its oriented isotopy class if there is no confusion. Note that the embedded manifold may not necessarily be homeomorphic to a sphere. When it is homeomorphic to the standard sphere as an abstract manifold, we say that the knot is *spherical*.

Recall the following notion of cobordism for knots. Two m-knots K_0 and K_1 are said to be *cobordant* if there exists a proper submanifold X of $S^{m+2} \times [0,1]$ diffeomorphic to $K_0 \times [0,1]$ such that $\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})$, where $-K_0$ denotes the knot K_0 with the orientation reversed. Then, for non-spherical knots, the following question naturally arises. If two non-spherical knots are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots? According to the codimension two surgery theory [8], this is true provided that the relevant knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres (for details see Definition 2.1). When such a degree one map exists, we say that the former knot is a *pull back* of the latter.

In this paper, we focus on the pull back relation and clarify its properties and relationship to isotopies and cobordisms of non-spherical knots. Instead of developing a general theory for the pull back relation, we will rather restrict ourselves to the class of odd dimensional fibered knots which satisfy certain connectivity conditions, i.e. *simple fibered knots* (see Definition 2.5).

The pull back relation restricted to simple fibered knots turns out to be very strong. We will show that if two such knots are related by the pull back relation and if their fibers have the same middle dimensional betti number, then they are isotopic (Theorem 2.6). As a corollary, we show that the pull back relation defines a partial order for simple fibered knots (Corollary 2.7). In fact, we will show that a simple fibered knot K_0 is a pull back of another simple fibered knot K_1 if and only if K_0 is isotopic to the connected sum of K_1 with a spherical simple fibered knot (Theorem 2.8).

In Sec. 4, we introduce a new class of simple fibered knots, called *special* simple fibered knots, and show that two such knots are related by a chain of pull back relations if and only if they are cobordant after taking connected sums with spherical knots (Theorem 4.4).

Finally in Sec. 5, we give several explicit examples of knots which enjoy significant properties with respect to the pull back relation and cobordism.

Throughout the paper, we work in the smooth category. All homology and cohomology groups are with integer coefficients. The symbol "$\cong$" denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

2. Results

Let us begin by defining the relation among knots that we are going to consider in this paper.

Definition 2.1. Let K_0 and K_1 be m-knots in S^{m+2}. We say that K_0 is a *pull back* of K_1 if there exists a degree one smooth map $g : S^{m+2} \to S^{m+2}$ with the following properties:

(1) g is transverse to K_1,

(2) $g^{-1}(K_1) = K_0$,
(3) $g|_{K_0} : K_0 \to K_1$ is an orientation preserving simple homotopy equivalence.

In this case, we write $K_0 \succ K_1$.

Remark 2.2. Here are some direct consequences of the definition.

(a) $K \succ K$ for any m-knot K.
(b) $K_0 \succ K_1$ and $K_1 \succ K_2$ imply $K_0 \succ K_2$ for any m-knots K_0, K_1 and K_2.
(c) $K_0 \succ K_1$ and $K_0' \succ K_1'$ imply $K_0 \sharp K_0' \succ K_1 \sharp K_1'$ for any m-knots K_0, K_0', K_1 and K_1'.

Furthermore, if we restrict ourselves to spherical m-knots, then it is not difficult to show that the *trivial m-knot* (or the *m-dimensional unknot*) K_U is the minimal element, i.e., $K \succ K_U$ for every spherical m-knot K, where K_U is defined to be the isotopy class of the boundary of an $(m+1)$-dimensional disk embedded in S^{m+2}.

Remark 2.3. In the terminology of [7], the map g in Definition 2.1 is *weakly h-regular along* K_1. In fact, the above definition is motivated by the following consequence of the codimension two surgery theory.

For an m-knot K, let $N(K)$ be a tubular neighborhood of K in S^{m+2} and set $E(K) = S^{m+2} \setminus \operatorname{Int} N(K)$. We say that K is *exterior 2-connected* if

$$\pi_i(E(K), \partial E(K)) = 0, \quad \forall i \leq 2\,.$$

(This implies, in particular, that K is simply connected.) The codimension two surgery theory [8] implies that if two exterior 2-connected m-knots K_0 and K_1 with $m \geq 5$ are related by the pull back relation, then they are cobordant after taking connected sums with some spherical knots.

Remark 2.4. In Definition 2.1, if the knots K_0 and K_1 are simply connected, then it is enough that $g|_{K_0} : K_0 \to K_1$ is just an orientation preserving homotopy equivalence for item (3).

Definition 2.5. An m-knot K is *fibered* if there exist a trivialization $\tau : N(K) \to K \times D^2$ of the tubular neighborhood $N(K)$ of K in S^{m+2} and a smooth fibration $\varphi : S^{m+2} \setminus K \to S^1$ such that the following diagram is commutative:

$$\begin{array}{ccc} N(K) \setminus K & \xrightarrow{\ \tau\ } & K \times (D^2 \setminus \{0\}) \\ {\scriptstyle \varphi|_{(N(K)\setminus K)}} \searrow & & \swarrow {\scriptstyle p} \\ & S^1, & \end{array}$$

where p denotes the obvious projection. In this case, for each $t \in S^1$, the closure F in S^{m+2} of $\varphi^{-1}(t)$ is called a *fiber* of K. Note that $F = \varphi^{-1}(t) \cup K$ is a compact $(m+1)$-dimensional manifold with boundary $\partial F = K$.

We say that a fibered $(2n-1)$-knot K in S^{2n+1} is *simple* if K is $(n-2)$-connected and its fiber is $(n-1)$-connected (see [3]).

In this paper, we show the following.

Theorem 2.6. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with fibers F_0 and F_1, respectively, where $n \geq 3$. Suppose* $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$. *If $K_0 \succ K_1$, then K_0 and K_1 are orientation preservingly isotopic.*

Corollary 2.7. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. If $K_0 \succ K_1$ and $K_1 \succ K_0$, then K_0 is orientation preservingly isotopic to K_1. In other words, the relation "$\succ$" defines a partial order for simple fibered $(2n-1)$-knots in S^{2n+1} for $n \geq 3$.*

Theorem 2.8. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then $K_0 \succ K_1$ if and only if there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is orientation preservingly isotopic to the connected sum $K_1 \sharp \Sigma$.*

Remark 2.9. We do not know if the above results are valid also for $n = 1$ or $n = 2$.

3. Proofs

In order to prove the theorems, let us first assume that $K_0 \succ K_1$, where K_0 and K_1 are simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then, there exists a degree one smooth map $g : S^{2n+1} \to S^{2n+1}$ as in Definition 2.1. By items (1) and (2), we see that there exist trivializations $N(K_i) = K_i \times D^2$ of sufficiently small tubular neighborhoods $N(K_i)$ of K_i in S^{2n+1}, $i = 0, 1$, such that $g^{-1}(N(K_1)) = N(K_0)$ and $g|_{N(K_0)} : K_0 \times D^2 = N(K_0) \to N(K_1) = K_1 \times D^2$ is identified with $(g|_{K_0}) \times \mathrm{id}_{D^2}$. Note that $g|_{K_0} : K_0 \to K_1$ is an orientation preserving homotopy equivalence.

We see that the trivializations $N(K_0) = K_0 \times D^2$ and $N(K_1) = K_1 \times D^2$ are essentially unique, since both $H^1(K_0)$ and $H^1(K_1)$ vanish. Therefore, we may further assume that $g|_{(N(K_0) \setminus K_0)} : N(K_0) \setminus K_0 \to N(K_1) \setminus K_1$ is compatible with the fibrations $S^{2n+1} \setminus K_0 \to S^1$ and $S^{2n+1} \setminus K_1 \to S^1$.

Set $E(K_i) = S^{2n+1} \setminus \operatorname{Int} N(K_i)$, $i = 0, 1$. Note that g induces a smooth map

$$g_E = g|_{E(K_0)} : E(K_0) \to E(K_1) \tag{3.1}$$

whose restriction to $\partial E(K_0)$ is a homotopy equivalence onto $\partial E(K_1)$.

Let $\widetilde{E}(K_i)$ be the universal cover of $E(K_i)$, $i = 0, 1$. Note that $\widetilde{E}(K_i) \cong F_i \times \mathbf{R}$. Since the smooth map g_E in (3.1) induces an isomorphism between the fundamental groups, it lifts to a smooth map $\widetilde{g}_E : \widetilde{E}(K_0) \to \widetilde{E}(K_1)$, whose restriction to the boundary is a homotopy equivalence and respects the product structures $\partial \widetilde{E}(K_i) \cong \partial F_i \times \mathbf{R}$, $i = 0, 1$. Hence, there exists a continuous map $\psi : (F_0, \partial F_0) \to (F_1, \partial F_1)$ such that $\psi|_{\partial F_0} : \partial F_0 \to \partial F_1$ is an orientation preserving homotopy equivalence. (For example, ψ is the composition

$$F_0 = F_0 \times \{0\} \subset F_0 \times \mathbf{R} \cong \widetilde{E}(K_0) \xrightarrow{\widetilde{g}_E} \widetilde{E}(K_1) \cong F_1 \times \mathbf{R} \to F_1,$$

where the last map is the projection to the first factor.)

Note that ψ induces an isomorphism between $H_{2n}(F_0, \partial F_0)$ and $H_{2n}(F_1, \partial F_1)$, since the boundary homomorphism induces an isomorphism

$$H_{2n}(F_i, \partial F_i) \to H_{2n-1}(K_i), \quad i = 0, 1.$$

By the universal coefficient theorem, it also induces an isomorphism between the cohomology groups $H^{2n}(F_1, \partial F_1)$ and $H^{2n}(F_0, \partial F_0)$.

Let $\tau_i : F_i \to F_i$ be the monodromy diffeomorphism of the fibered knot K_i, $i = 0, 1$. Note that $\tau_i|_{\partial F_i}$ is the identity. Since $\widetilde{g}_E$ is compatible with the covering translations, we see that $\psi \circ \tau_0$ and $\tau_1 \circ \psi$ are homotopic relative to boundary.

Lemma 3.1. *The homomorphisms*

$$\psi^* : H^n(F_1) \to H^n(F_0) \quad \textit{and} \quad \psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$$

are injective and their images are direct summands of $H^n(F_0)$ *and* $H^n(F_0, \partial F_0)$ *respectively.*

Proof. Let us consider the following commutative diagram:

$$\begin{array}{ccc} H^n(F_1) \otimes H^n(F_1, \partial F_1) & \xrightarrow{\smile} & H^{2n}(F_1, \partial F_1) \\ \downarrow{\scriptstyle \psi^* \otimes \psi^*} & & \downarrow{\scriptstyle \psi^*} \\ H^n(F_0) \otimes H^n(F_0, \partial F_0) & \xrightarrow{\smile} & H^{2n}(F_0, \partial F_0), \end{array}$$

where "$\smile$" denotes the cup product. Let $\xi \in H^n(F_1)$ be an arbitrary primitive element. Then, there exists an element $\zeta \in H^n(F_1, \partial F_1)$ such that $\xi \smile \zeta$ is a generator of $H^{2n}(F_1, \partial F_1) \cong \mathbf{Z}$. Since $\psi^* : H^{2n}(F_1, \partial F_1) \to H^{2n}(F_0, \partial F_0)$ is an isomorphism, we see that $(\psi^* \xi) \smile (\psi^* \zeta)$ is also a generator of $H^{2n}(F_0, \partial F_0)$. This means that $\psi^* \xi$ is a primitive element of $H^n(F_0)$. This shows that $\psi^* : H^n(F_1) \to H^n(F_0)$ is an injection and that its image is a direct summand of $H^n(F_0)$. A similar argument shows the corresponding assertion for $\psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$. This completes the proof of Lemma 3.1. □

The above lemma together with the universal coefficient theorem implies that the homomorphisms

$$\psi_* : H_n(F_0, \partial F_0) \to H_n(F_1, \partial F_1) \quad \text{and} \quad \psi_* : H_n(F_0) \to H_n(F_1) \tag{3.2}$$

are surjections.

Let $\Delta_i : H_n(F_i, \partial F_i) \to H_n(F_i)$ be the variation map of the fibered knot K_i, $i = 0, 1$. Recall that for an n-cycle c of $(F_i, \partial F_i)$, $\Delta_i([c])$ is defined to be the homology class represented by $c - \tau_i(c)$, where $[c] \in H_n(F_i, \partial F_i)$ is the homology class represented by c. Note that this is a well-defined homomorphism, since $\tau_i|_{\partial F_i}$ is the identity and the isotopy class of τ_i relative to boundary is uniquely determined.

Note also that the variation maps are isomorphisms (see [5]). Then, we see easily that the following diagram is commutative:

$$\begin{array}{ccc} H_n(F_0, \partial F_0) & \xrightarrow{\Delta_0} & H_n(F_0) \\ {\scriptstyle \psi_*}\downarrow & & \downarrow{\scriptstyle \psi_*} \\ H_n(F_1, \partial F_1) & \xrightarrow{\Delta_1} & H_1(F_1), \end{array} \tag{3.3}$$

since $\psi \circ \tau_0$ and $\tau_1 \circ \psi$ are homotopic relative to boundary.

Proof of Theorem 2.6. If $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$, then Lemma 3.1 implies that the homomorphisms (3.2) are isomorphisms. Then the commutative diagram (3.3) implies that K_0 and K_1 are orientation preservingly isotopic, since the variation map determines and is determined by the Seifert form, which in turn determines the oriented isotopy class of a simple fibered knot (for details see [5, 3, 4]). □

Proof of Corollary 2.7. By Lemma 3.1, we see that $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$. Then the result follows from Theorem 2.6. □

Proof of Theorem 2.8. First, suppose that there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is isotopic to the connected sum $K_1 \sharp \Sigma$. Then by Remark 2.2, we have $\Sigma \succ K_U$ and $K_1 \sharp \Sigma \succ K_1 \sharp K_U$, and hence $K_0 \succ K_1$.

For the converse, let G and G' be the kernels of the homomorphisms ψ_*: $H_n(F_0, \partial F_0) \to H_n(F_1, \partial F_1)$ and $\psi_* : H_n(F_0) \to H_n(F_1)$ respectively. Then we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \to & G & \to & H_n(F_0, \partial F_0) & \xrightarrow{\psi_*} & H_n(F_1, \partial F_1) & \to & 0 \\ & & \downarrow{\scriptstyle \Delta_0|_G} & & \downarrow{\scriptstyle \Delta_0} & & \downarrow{\scriptstyle \Delta_1} & & \\ 0 & \to & G' & \to & H_n(F_0) & \xrightarrow{\psi_*} & H_n(F_1) & \to & 0. \end{array}$$

Since $H_n(F_1, \partial F_1)$ and $H_n(F_1)$ are free, the exact sequences split. This means that the variation map Δ_0 of K_0 is isomorphic to the direct sum of the variation map Δ_1 of K_1 and the isomorphism $\Delta_0|_G : G \to G'$.

Recall that with respect to certain bases, the matrix associated with the variation map is the inverse of the Seifert matrix (for details see [5]). Since $n \geq 3$, every unimodular matrix is realized as the Seifert matrix of a simple fibered $(2n-1)$-knot (see [3, 4]). So we see that there exists a simple fibered $(2n-1)$-knot Σ which realizes $\Delta_0|_G : G \to G'$ as its variation map.

Then, we see that the Seifert matrices for K_0 and $K_1 \sharp \Sigma$ are congruent. Consequently, they are orientation preservingly isotopic to each other by [3, 4].

Furthermore, since K_0 is homotopy equivalent to both K_1 and $K_1 \sharp \Sigma$, we see that Σ should be homeomorphic to a sphere. This completes the proof. □

Remark 3.2. Theorem 2.8 implies in particular that the fiber of K_0 is diffeomorphic to the boundary connected sum of the fiber of K_1 and a certain $(n-1)$-connected $2n$-dimensional manifold with spherical boundary. When K_0 and K_1 are spherical, this is also a consequence of [2, Theorem B].

Definition 3.3. Let us consider the equivalence relation generated by the pull back relation defined in Definition 2.1. When two m-knots K_0 and K_1 in S^{m+2} are equivalent with respect to this equivalence relation, we say that K_0 and K_1 are *pull back equivalent*.

The above definition together with Theorem 2.8 implies the following, whose proof is easy and is left to the reader.

Corollary 3.4. *Two simple fibered* $(2n-1)$*-knots* K_0 *and* K_1 *in* S^{2n+1} *with* $n \geq 3$ *are pull back equivalent if and only if there exist spherical simple fibered* $(2n-1)$*-knots* Σ_0 *and* Σ_1 *in* S^{2n+1} *such that* $K_0 \sharp \Sigma_0$ *is orientation preservingly isotopic to* $K_1 \sharp \Sigma_1$.

4. Special Knots

In this section, we show that for a certain class of simple fibered knots, the pull back equivalence relation is equivalent to the relation generated by connected sums with spherical fibered knots together with the cobordism. For a theory of cobordism of simple fibered knots, refer to [1, 10, 11].

Definition 4.1. Let K be a simple fibered $(2n-1)$-knot with fiber F. Let us denote by $I(K)$ the image of the homomorphism $H_n(K) \to H_n(F)$ induced by the inclusion (or equivalently, the kernel of the homomorphism $H_n(F) \to H_n(F, \partial F)$). The fibered knot K is said to be *special* if its Seifert form restricted to $I(K)$ is unimodular (for a definition of a Seifert form, see [3]).

Lemma 4.2. *A simple fibered* $(2n-1)$*-knot* K *is special if and only if there exist two simple fibered* $(2n-1)$*-knots* K_F *and* K_T *with the following properties*:

(1) K *is orientation preservingly isotopic to* $K_F \sharp K_T$,
(2) *the intersection form of the fiber of* K_F *is the zero form*,
(3) $H_{n-1}(K_T)$ *is a torsion group* (*or equivalently*, $H_n(K_T) = 0$).

Proof. If there exist simple fibered $(2n-1)$-knots K_F and K_T with properties (1)–(3), then the Seifert form of K restricted to $I(K)$ coincides with the Seifert form of K_F. Since K_F is fibered, its Seifert form must be unimodular. Hence, K is special.

Conversely, suppose that the simple fibered knot K is special. Let us consider a basis $e_1, \ldots, e_u, e_{u+1}, \ldots, e_{u+v}$ of $H_n(F)$, where $e_1, \ldots, e_u$ constitute a basis of $I(K)$. This is possible, since $I(K)$ is a direct summand of $H_n(F)$. Then, the Seifert matrix L of K with respect to this basis is of the form

$$L = \begin{pmatrix} L_F & A \\ B & C \end{pmatrix}$$

for some $u \times u$ matrix L_F, $u \times v$ matrix A, $v \times u$ matrix B, and $v \times v$ matrix C. Note that $L_F + (-1)^n({}^tL_F) = 0$ and $A + (-1)^n({}^tB) = 0$, since the homomorphism $H_n(F) \to H_n(F, \partial F) = \mathrm{Hom}(H_n(F), \mathbf{Z})$ can be identified with the intersection form of F and the intersection matrix of F is given by $L + (-1)^n({}^tL)$ (for example, see [3]). Since L_F is unimodular by our hypothesis and $L_F = (-1)^{n+1}({}^tL_F)$, we see that L is congruent to a matrix of the form

$$L' = \begin{pmatrix} L_F & 0 \\ 0 & L_T \end{pmatrix}$$

for some $v \times v$ matrix L_T. Since L' is unimodular, so is L_T. Furthermore, $L_T + (-1)^n({}^tL_T)$ is a nonsingular matrix, since the kernel of the intersection form is generated by $e_1, \ldots, e_u$. Let K_F and K_T be the simple fibered $(2n-1)$-knots realizing L_F and L_T as their Seifert matrices respectively. Then, we can check that conditions (1)–(3) are satisfied. This completes the proof. □

Remark 4.3. In the above lemma, if $H_{n-1}(K)$ is torsion free, then the knot K_T is spherical.

Let us prove the following.

Theorem 4.4. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. Suppose that K_0 is special and that $H_{n-1}(K_0)$ is torsion free. Then the following conditions are all equivalent to each other.*

(1) *$K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$ for some spherical knots Σ_0 and Σ_1.*
(2) *$K_0 \sharp \Sigma'_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma'_1$ for some spherical simple fibered knots Σ'_0 and Σ'_1.*
(3) *K_0 is pull back equivalent to K_1.*

For the proof, we need the following lemma, which is a direct consequence of [1, Theorem 4] (see also [10, 11]). Recall that a $(2n-1)$-knot is *simple* if it is $(n-2)$-connected and it bounds an $(n-1)$-connected $2n$-dimensional compact manifold in S^{2n+1}.

Lemma 4.5. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical simple knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

Proof of Theorem 4.4. The equivalence of (2) and (3) follows from Corollary 3.4. Condition (2) clearly implies condition (1). Thus, we have only to show that (1) implies (2).

Suppose that (1) holds. Since every spherical $(2n-1)$-knot is cobordant to a spherical simple $(2n-1)$-knot by [6], we may assume that Σ_0 and Σ_1 are simple. Then by Lemma 4.5, the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other. By our assumption, these forms are unimodular, and hence K_1 is also special. Therefore, by Lemma 4.2, there exist simple fibered $(2n-1)$-knots $K_F^{(i)}$, $K_T^{(i)}$, $i = 0, 1$, such that

(a) K_i is orientation preservingly isotopic to $K_F^{(i)} \sharp K_T^{(i)}$,
(b) the intersection form of the fiber of $K_F^{(i)}$ is the zero form,
(c) $H_{n-1}(K_T^{(i)})$ is a torsion group,

for $i = 0, 1$. Note that $K_F^{(0)}$ is orientation preservingly isotopic to $K_F^{(1)}$, since their Seifert forms are isomorphic.

Recall that $H_{n-1}(K_0)$ is torsion free by our assumption. Therefore, $K_T^{(i)}$ are spherical knots for $i = 0, 1$. Since $K_0 \sharp K_T^{(1)}$ is orientation preservingly isotopic to $K_F^{(1)} \sharp K_T^{(0)} \sharp K_T^{(1)}$, it is also orientation preservingly isotopic to $K_1 \sharp K_T^{(0)}$. Hence condition (2) holds. This completes the proof. □

Remark 4.6. Let K_F be the simple fibered $(2n-1)$-knot as in Lemma 4.2. Then its Seifert form is skew-symmetric for n even, and is symmetric for n odd. Note that unimodular skew-symmetric matrices have even ranks and the congruence class of such a matrix is uniquely determined by its rank. Therefore, when n is even, the oriented isotopy class of K_F is determined by its rank, which is even. On the other hand, when n is odd, unimodular symmetric matrices are not determined by its rank. For details, refer to [9], for example.

5. Examples

In the previous sections, we have seen that for two simple fibered $(2n-1)$-knots with $n \geq 3$, the following implications hold:

pull back equivalent
$\Longrightarrow$ cobordant after taking connected sums with some spherical knots
$\Longrightarrow$ (simple) homotopy equivalent.

In fact, these implications hold for arbitrary m-knots with $m \geq 5$ which satisfy the connectivity conditions mentioned in Remark 2.3. In this section, we show that the converses of the above two implications do not hold in general by giving several important examples.

Proposition 5.1. *For every odd integer $n \geq 3$, there exists a pair (K_0, K_1) of simple fibered $(2n-1)$-knots with the following properties.*

(1) *The knots K_0 and K_1 are cobordant.*
(2) *The knots K_0 and K_1 are not pull back equivalent.*

Proof. Let us consider the following two matrices:

$$L_0 = \begin{pmatrix} 9 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 25 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that they are both unimodular and that

$$S_0 = L_0 - {}^tL_0 = S_1 = L_1 - {}^tL_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Let us show that L_0 and L_1 are algebraically cobordant in the sense of [1, (1.2), Definition] for $\varepsilon = (-1)^n = -1$.

Set $m = {}^t(5,0,3,0)$ and $m' = {}^t(0,3,0,5)$. Then it is easy to see that the submodule M of $\mathbf{Z}^4$ generated by m and m' constitutes a metabolizer for $L = L_0 \oplus (-L_1)$. Furthermore, M is pure in $\mathbf{Z}^4$: in other words, M is a direct summand of $\mathbf{Z}^4$. Since $S_0 = S_1$ are non-degenerate, we have only to verify the condition c.2 of [1, (1.2), Definition].

Set $S = S_0 \oplus (-S_1) = L - {}^tL$. Let $S^* : \mathbf{Z}^4 \to \mathbf{Z}^4$, $S_0^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ and $S_1^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the adjoints of S, S_0 and S_1 respectively. It is easy to see that $\operatorname{Coker} S_0^* = \operatorname{Coker} S_1^*$ is naturally identified with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Furthermore, we have

$$S^*(m) = {}^tmS = (0,10,0,-6) \quad \text{and} \quad S^*(m') = {}^tm'S = (-6,0,10,0).$$

Therefore, $S^*(M)^\wedge$, the smallest direct summand of $\mathbf{Z}^4$ containing $S^*(M)$, is the submodule of $\mathbf{Z}^4$ generated by $(0,5,0,-3)$ and $(-3,0,5,0)$. Hence, for the natural quotient map $d : \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we have

$$d(S^*(M)^\wedge) = \{(x,x) \,:\, x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\},$$

since $\operatorname{Im} S_i^*$ is generated by $(2,0)$ and $(0,2)$, $i = 0,1$, and $\operatorname{Im} S^*$ is generated by $(2,0,0,0)$, $(0,2,0,0)$, $(0,0,2,0)$ and $(0,0,0,2)$. Therefore, we conclude that the unimodular matrices L_0 and L_1 are algebraically cobordant.

Now, there exists a simple fibered $(2n-1)$-knot K_i which realizes L_i as its Seifert form with respect to the fiber, $i = 0,1$ (see [3, 4]). By [1, Theorem 3], K_0 and K_1 are cobordant.

Let us now show that K_0 and K_1 are not pull back equivalent. By Corollary 3.4, we have only to show that for any spherical simple fibered $(2n-1)$-knots Σ_0 and Σ_1 in S^{2n+1}, $K_0 \sharp \Sigma_0$ is never orientation preservingly isotopic to $K_1 \sharp \Sigma_1$.

Since $K_i \sharp \Sigma_i$ is a fibered knot, we can consider the monodromy on the nth homology group of the fiber, $i = 0,1$. Let us denote by H_i the monodromy matrix of $K_i \sharp \Sigma_i$ and by $\widetilde{L}_i$ its Seifert matrix with respect to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for K_i and that for Σ_i. It is known that $H_i = (-1)^{n+1} \widetilde{L}_i^{-1} ({}^t\widetilde{L}_i)$ (for example, see [3]). Therefore, we have

$$H_0 = \begin{pmatrix} -1 & 0 \\ 18 & -1 \end{pmatrix} \oplus H_0' \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 \\ 50 & -1 \end{pmatrix} \oplus H_1',$$

where H_i' is the monodromy matrix of Σ_i, $i = 0,1$.

Let us consider $\mathrm{Ker}((I + H_i)^2)$, where I is the unit matrix, $i = 0, 1$. Since Σ_i are spherical knots, the monodromy matrices H_i' cannot have the eigenvalue -1. Therefore, $\mathrm{Ker}((I + H_i)^2)$ corresponds exactly to the homology of the fiber of K_i.

Suppose that $K_0 \sharp \Sigma_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma_1$. Then the Seifert form of $K_0 \sharp \Sigma_0$ restricted to $\mathrm{Ker}((I + H_0)^2)$ should be isomorphic to that of $K_1 \sharp \Sigma_1$ restricted to $\mathrm{Ker}((I + H_1)^2)$. This means that L_0 should be congruent to L_1. However, this is a contradiction, since there exists an element $x \in \mathbf{Z}^2$ such that ${}^t x L_0 x = 9$, while such an element does not exist for L_1.

Thus, we conclude that K_0 and K_1 are not pull back equivalent. □

Note that the simple fibered knots K_0 and K_1 constructed above are special; however, $H_{n-1}(K_i)$, $i = 0, 1$, are not torsion free.

Remark 5.2. In fact, we can find infinitely many examples as in the above proposition. For example, we could use the matrices

$$\begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix}$$

for arbitrary relatively prime odd integers p and q. Or we could also use $K_0 \sharp K'$ and $K_1 \sharp K'$, instead of K_0 and K_1, for any simple fibered $(2n-1)$-knot K' whose monodromy does not have the eigenvalue -1.

As has been remarked in Remark 2.3, under a certain connectivity condition, if two m-knots K_0 and K_1 with $m \geq 5$ are pull back equivalent, then they are cobordant after taking connected sums with some spherical knots. The above example shows that the converse is not true in general.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition, which is a slight modification of Lemma 4.5 and is implicitly proved in the proof of Theorem 4.4.

Proposition 5.3. *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

Remark 5.4. In fact, the above proposition is implicitly proved also in [11]. Based on this, Vogt proves the following. The usual $(2n-1)$-dimensional spherical knot cobordism group C_{2n-1} acts on the cobordism semi-group of simple $(2n-1)$-knots with torsion free homologies by connected sum. The orbit space of the action inherits a natural semi-group structure. Then this orbit space contains infinitely many free generators as a commutative semi-group for $n \geq 3$.

Vogt [11] also proves that the action of C_{2n-1} on the cobordism semi-group of simple $(2n-1)$-knots is fixed point free for $n \geq 3$. This can also be proved by using

[1, (5.1), Proposition]. In fact, for an arbitrary spherical simple $(2n-1)$-knot Σ whose Alexander polynomial is nontrivial and irreducible, $K\sharp\Sigma$ is never cobordant to K for any simple $(2n-1)$-knot K, since the Alexander polynomials of $K\sharp\Sigma$ and K do not satisfy a Fox–Milnor type relation necessary to be cobordant (see [1, (5.1), Proposition]).

The following example answers the question raised at the beginning of Sec. 1 negatively.

Example 5.5. Let us consider the following unimodular matrices:

$$L_0 = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^{n+1} & 0 & 0 & 1 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}.$$

Then, for every integer $n \geq 3$, there exist simple fibered $(2n-1)$-knots K_i in S^{2n+1} whose Seifert matrices are given by L_i, $i = 0, 1$ (see [3, 4]). Note that if we denote their fibers by F_i, $i = 0, 1$, then F_1 is orientation preservingly diffeomorphic to $F_0\sharp(S^n \times S^n)$. In particular, K_0 and K_1 are orientation preservingly diffeomorphic to each other.

It is easy to verify that the Seifert form restricted to $I(K_1)$ is the zero form, while it is not zero for K_0. Hence, by Proposition 5.3, $K_0\sharp\Sigma_0$ is never cobordant to $K_1\sharp\Sigma_1$ for any spherical (but not necessarily simple or fibered) knots Σ_0, Σ_1.

Note that for this example, we have $H_{n-1}(K_i) \cong \mathbf{Z} \oplus \mathbf{Z}$, $i = 0, 1$.

Let us give another kind of an example together with another argument.

Example 5.6. Let us consider the unimodular matrices

$$L_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and their associated simple fibered $(2n-1)$-knots K_i, $i = 0, 1$, as in the previous example, for every even integer $n \geq 4$. By a similar argument, we see that K_0 and K_1 are orientation preservingly diffeomorphic to each other.

Suppose that for some spherical $(2n-1)$-knots Σ_i, $i = 0, 1$, $K_0\sharp\Sigma_0$ is cobordant to $K_1\sharp\Sigma_1$. We may assume that Σ_0 and Σ_1 are simple. We see easily that the Alexander polynomials of K_0 and K_1 are given by

$$\Delta_{K_0}(t) = \det(tL_0 + {}^tL_0) = t^2 + t + 1$$

and

$$\Delta_{K_1}(t) = \det(tL_1 + {}^tL_1) = -(t^4 + t^3 - t^2 + t + 1)$$

respectively. Note that the both polynomials are irreducible over $\mathbf{Z}$. If $K_0\sharp\Sigma_0$ is cobordant to $K_1\sharp\Sigma_1$, then by [1, (5.1), Proposition], we see that

$$\Delta_{K_0}(t)\Delta_{\Sigma_0}(t)\Delta_{K_1}(t^{-1})\Delta_{\Sigma_1}(t^{-1}) = t^\lambda f(t)f(t^{-1})$$

for some $\lambda \in \mathbf{Z}$ and $f(t) \in \mathbf{Z}[t, t^{-1}]$ (Fox–Milnor type relation), where $\Delta_{\Sigma_i}(t)$ is the Alexander polynomial of Σ_i, $i = 0, 1$.

Note that we have $|\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3$ and $|\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1$. Since $\Delta_{K_0}(t)$ is irreducible of degree 2, and $\Delta_{K_1}(t)$ is irreducible of degree 4, the above relation leads to a contradiction.

Hence, $K_0\sharp\Sigma_0$ is not cobordant to $K_1\sharp\Sigma_1$ for any spherical (not necessarily simple or fibered) $(2n-1)$-knots Σ_0, Σ_1. Note that we have $H_{n-1}(K_i) \cong \mathbf{Z}_3$, $i = 0, 1$, for this example.

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