PULL BACK RELATION FOR NON-SPHERICAL KNOTS

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ABSTRACT

We introduce a new relation for high dimensional non-spherical knots, which is motivated by the codimension two surgery theory: a knot is a pull back of another knot if the former is obtained as the inverse image of the latter by a certain degree one map between the ambient spheres. We show that this relation defines a partial order for \((2n-1)\)-dimensional simple fibered knots for \(n \geq 3\). We also give some related results concerning cobordisms and isotopies of knots together with several important explicit examples.

Keywords: Pull back; codimension two surgery; high dimensional knot; cobordism; fibered knot.

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1. Introduction

In this paper, by a knot (or an \(m\)-knot) we mean an oriented manifold (of dimension \(m\)) embedded in a sphere in codimension two. We do not distinguish a knot from its oriented isotopy class if there is no confusion. Note that the embedded manifold may not necessarily be homeomorphic to a sphere. When it is homeomorphic to the standard sphere as an abstract manifold, we say that the knot is spherical.

Recall the following notion of cobordism for knots. Two \(m\)-knots \(K_0\) and \(K_1\) are said to be cobordant if there exists a proper submanifold \(X\) of \(S^{m+2} \times [0,1]\) diffeomorphic to \(K_0 \times [0,1]\) such that \(\partial X = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})\), where \(-K_0\) denotes
the knot \( K_0 \) with the orientation reversed. Then, for non-spherical knots, the following question naturally arises. If two non-spherical knots are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots? According to the codimension two surgery theory [8], this is true provided that the relevant knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres (for details see Definition 2.1). When such a degree one map exists, we say that the former knot is a pull back of the latter.

In this paper, we focus on the pull back relation and clarify its properties and relationship to isotopies and cobordisms of non-spherical knots. Instead of developing a general theory for the pull back relation, we will rather restrict ourselves to the class of odd dimensional fibered knots which satisfy certain connectivity conditions, i.e. simple fibered knots (see Definition 2.5).

The pull back relation restricted to simple fibered knots turns out to be very strong. We will show that if two such knots are related by the pull back relation and if their fibers have the same middle dimensional betti number, then they are isotopic (Theorem 2.6). As a corollary, we show that the pull back relation defines a partial order for simple fibered knots (Corollary 2.7). In fact, we will show that a simple fibered knot \( K_0 \) is a pull back of another simple fibered knot \( K_1 \) if and only if \( K_0 \) is isotopic to the connected sum of \( K_1 \) with a spherical simple fibered knot (Theorem 2.8).

In Sec. 4, we introduce a new class of simple fibered knots, called special simple fibered knots, and show that two such knots are related by a chain of pull back relations if and only if they are cobordant after taking connected sums with spherical knots (Theorem 4.4).

Finally in Sec. 5, we give several explicit examples of knots which enjoy significant properties with respect to the pull back relation and cobordism.

Throughout the paper, we work in the smooth category. All homology and cohomology groups are with integer coefficients. The symbol \( \cong \) denotes a diffeomorphism between manifolds or an appropriate isomorphism between algebraic objects.

2. Results

Let us begin by defining the relation among knots that we are going to consider in this paper.

**Definition 2.1.** Let \( K_0 \) and \( K_1 \) be \( m \)-knots in \( S^{m+2} \). We say that \( K_0 \) is a pull back of \( K_1 \) if there exists a degree one smooth map \( g : S^{m+2} \rightarrow S^{m+2} \) with the following properties:

1. \( g \) is transverse to \( K_1 \),
(2) \( g^{-1}(K_1) = K_0 \).
(3) \( g|_{K_0} : K_0 \to K_1 \) is an orientation preserving simple homotopy equivalence.

In this case, we write \( K_0 \succ K_1 \).

**Remark 2.2.** Here are some direct consequences of the definition.

(a) \( K \succ K \) for any \( m \)-knot \( K \).
(b) \( K_0 \succ K_1 \) and \( K_1 \succ K_2 \) imply \( K_0 \succ K_2 \) for any \( m \)-knots \( K_0, K_1 \) and \( K_2 \).
(c) \( K_0 \succ K_1 \) and \( K_0 \sim K_1 \) imply \( K_0 \sim K_1 \) for any \( m \)-knots \( K_0, K_1 \) and \( K_0 \sim K_1 \).

Furthermore, if we restrict ourselves to spherical \( m \)-knots, then it is not difficult to show that the trivial \( m \)-knot (or the \( m \)-dimensional unknot) \( K_U \) is the minimal element, i.e., \( K \succ K_U \) for every spherical \( m \)-knot \( K \), where \( K_U \) is defined to be the isotopy class of the boundary of an \((m + 1)\)-dimensional disk embedded in \( S^{m+2} \).

**Remark 2.3.** In the terminology of [7], the map \( g \) in Definition 2.1 is weakly \( h \)-regular along \( K_1 \). In fact, the above definition is motivated by the following consequence of the codimension two surgery theory.

For an \( m \)-knot \( K \), let \( N(K) \) be a tubular neighborhood of \( K \) in \( S^{m+2} \) and set \( E(K) = S^{m+2} \setminus \text{Int } N(K) \). We say that \( K \) is exterior 2-connected if

\[
\pi_i(E(K), \partial E(K)) = 0, \quad \forall i \leq 2.
\]

(This implies, in particular, that \( K \) is simply connected.) The codimension two surgery theory [8] implies that if two exterior 2-connected \( m \)-knots \( K_0 \) and \( K_1 \) with \( m \geq 5 \) are related by the pull back relation, then they are cobordant after taking connected sums with some spherical knots.

**Remark 2.4.** In Definition 2.1, if the knots \( K_0 \) and \( K_1 \) are simply connected, then it is enough that \( g|_{K_0} : K_0 \to K_1 \) is just an orientation preserving homotopy equivalence for item (3).

**Definition 2.5.** An \( m \)-knot \( K \) is fibered if there exist a trivialization \( \tau : N(K) \to K \times D^2 \) of the tubular neighborhood \( N(K) \) of \( K \) in \( S^{m+2} \) and a smooth fibration \( \varphi : S^{m+2} \setminus K \to S^1 \) such that the following diagram is commutative:

\[
\begin{array}{ccc}
N(K) \setminus K & \xrightarrow{\tau} & K \times (D^2 \setminus \{0\}) \\
\varphi|_{(N(K) \setminus K)} & \nearrow p & S^1,
\end{array}
\]

where \( p \) denotes the obvious projection. In this case, for each \( t \in S^1 \), the closure \( F \) in \( S^{m+2} \) of \( \varphi^{-1}(t) \) is called a fiber of \( K \). Note that \( F = \varphi^{-1}(t) \cup K \) is a compact \((m + 1)\)-dimensional manifold with boundary \( \partial F = K \).

We say that a fibered \((2n-1)\)-knot \( K \) in \( S^{2n+1} \) is simple if \( K \) is \((n-2)\)-connected and its fiber is \((n-1)\)-connected (see [3]).
In this paper, we show the following.

**Theorem 2.6.** Let $K_0$ and $K_1$ be simple fibered $(2n - 1)$-knots in $S^{2n+1}$ with fibers $F_0$ and $F_1$, respectively, where $n \geq 3$. Suppose $\text{rank } H_n(F_0) = \text{rank } H_n(F_1)$. If $K_0 \succeq K_1$, then $K_0$ and $K_1$ are orientation preservingly isotopic.

**Corollary 2.7.** Let $K_0$ and $K_1$ be simple fibered $(2n - 1)$-knots in $S^{2n+1}$ with $n \geq 3$. If $K_0 \succeq K_1$ and $K_1 \succeq K_0$, then $K_0$ is orientation preservingly isotopic to $K_1$. In other words, the relation $\succeq$ defines a partial order for simple fibered $(2n - 1)$-knots in $S^{2n+1}$ for $n \geq 3$.

**Theorem 2.8.** Let $K_0$ and $K_1$ be simple fibered $(2n - 1)$-knots in $S^{2n+1}$ with $n \geq 3$. Then $K_0 \succeq K_1$ if and only if there exists a spherical simple fibered $(2n - 1)$-knot $\Sigma$ in $S^{2n+1}$ such that $K_0$ is orientation preservingly isotopic to the connected sum $K_1 \# \Sigma$.

**Remark 2.9.** We do not know if the above results are valid also for $n = 1$ or $n = 2$.

### 3. Proofs

In order to prove the theorems, let us first assume that $K_0 \succeq K_1$, where $K_0$ and $K_1$ are simple fibered $(2n - 1)$-knots in $S^{2n+1}$ with $n \geq 3$. Then, there exists a degree one smooth map $g : S^{2n+1} \to S^{2n+1}$ as in Definition 2.1. By items (1) and (2), we see that there exist trivializations $N(K_i) = K_i \times D^2$ of sufficiently small tubular neighborhoods $N(K_i)$ of $K_i$ in $S^{2n+1}$, $i = 0, 1$, such that $g^{-1}(N(K_i)) = N(K_0)$ and $g|_{N(K_0)} : K_0 \times D^2 \cong N(K_0) \to N(K_1) = K_1 \times D^2$ is identified with $(g|_{K_0}) \times \text{id}_{D^2}$. Note that $g|_{K_0} : K_0 \to K_1$ is an orientation preserving homotopy equivalence.

We see that the trivializations $N(K_0) = K_0 \times D^2$ and $N(K_1) = K_1 \times D^2$ are essentially unique, since both $H^1(K_0)$ and $H^1(K_1)$ vanish. Therefore, we may further assume that $g|_{(N(K_0) \setminus K_0)} : N(K_0) \setminus K_0 \to N(K_1) \setminus K_1$ is compatible with the fibrations $S^{2n+1} \setminus K_0 \to S^1$ and $S^{2n+1} \setminus K_1 \to S^1$.

Set $E(K_i) = S^{2n+1} \setminus \text{Int } N(K_i)$, $i = 0, 1$. Note that $g$ induces a smooth map

$$g_{E} = g|_{E(K_0)} : E(K_0) \to E(K_1)$$

(3.1)

whose restriction to $\partial E(K_0)$ is a homotopy equivalence onto $\partial E(K_1)$.

Let $\tilde{E}(K_i)$ be the universal cover of $E(K_i)$, $i = 0, 1$. Note that $\tilde{E}(K_i) \cong F_i \times \mathbf{R}$. Since the smooth map $g_{E}$ in (3.1) induces an isomorphism between the fundamental groups, it lifts to a smooth map $\tilde{g}_{E} : \tilde{E}(K_0) \to \tilde{E}(K_1)$, whose restriction to the boundary is a homotopy equivalence and respects the product structures $\partial \tilde{E}(K_i) \cong \partial F_i \times \mathbf{R}$, $i = 0, 1$. Hence, there exists a continuous map $\psi : (F_0, \partial F_0) \to (F_1, \partial F_1)$ such that $\psi|_{\partial F_0} : \partial F_0 \to \partial F_1$ is an orientation preserving homotopy equivalence.

(For example, $\psi$ is the composition

$$F_0 = F_0 \times \{0\} \subset F_0 \times \mathbf{R} \cong \tilde{E}(K_0) \xrightarrow{\tilde{g}_{E}} \tilde{E}(K_1) \cong F_1 \times \mathbf{R} \to F_1,$$

where the last map is the projection to the first factor.)
Note that $\psi$ induces an isomorphism between $H_{2n}(F_0, \partial F_0)$ and $H_{2n}(F_1, \partial F_1)$, since the boundary homomorphism induces an isomorphism

$$H_{2n}(F_i, \partial F_i) \to H_{2n-1}(K_i), \quad i = 0, 1.$$ 

By the universal coefficient theorem, it also induces an isomorphism between the cohomology groups $H^{2n}(F_1, \partial F_1)$ and $H^{2n}(F_0, \partial F_0)$.

Let $\tau_i : F_i \to F_i$ be the monodromy diffeomorphism of the fibered knot $K_i$, $i = 0, 1$. Note that $\tau_i|_{\partial F_i}$ is the identity. Since $\tilde{g}_E$ is compatible with the covering translations, we see that $\psi \circ \tau_i$ and $\tau_i \circ \psi$ are homotopic relative to boundary.

**Lemma 3.1.** The homomorphisms

$$\psi^* : H^n(F_1) \to H^n(F_0) \quad \text{and} \quad \psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$$

are injective and their images are direct summands of $H^n(F_0)$ and $H^n(F_0, \partial F_0)$ respectively.

**Proof.** Let us consider the following commutative diagram:

$$
\begin{array}{ccc}
H^n(F_1) \otimes H^n(F_1, \partial F_1) & \xrightarrow{\sim} & H^{2n}(F_1, \partial F_1) \\
\downarrow \psi^* \otimes \psi^* & & \downarrow \psi^* \\
H^n(F_0) \otimes H^n(F_0, \partial F_0) & \xrightarrow{\sim} & H^{2n}(F_0, \partial F_0),
\end{array}
$$

where “$\sim$” denotes the cup product. Let $\xi \in H^n(F_1)$ be an arbitrary primitive element. Then, there exists an element $\zeta \in H^n(F_1, \partial F_1)$ such that $\xi \sim \zeta$ is a generator of $H^{2n}(F_1, \partial F_1) \cong \mathbb{Z}$. Since $\psi^* : H^{2n}(F_1, \partial F_1) \to H^{2n}(F_0, \partial F_0)$ is an isomorphism, we see that $(\psi^* \xi) \sim (\psi^* \zeta)$ is also a generator of $H^{2n}(F_0, \partial F_0)$. This means that $\psi^* \xi$ is a primitive element of $H^n(F_0)$. This shows that $\psi^* : H^n(F_1) \to H^n(F_0)$ is an injection and that its image is a direct summand of $H^n(F_0)$. A similar argument shows the corresponding assertion for $\psi^* : H^n(F_1, \partial F_1) \to H^n(F_0, \partial F_0)$. This completes the proof of Lemma 3.1.

The above lemma together with the universal coefficient theorem implies that the homomorphisms

$$\psi_* : H_n(F_0, \partial F_0) \to H_n(F_1, \partial F_1) \quad \text{and} \quad \psi_* : H_n(F_0) \to H_n(F_1) \quad (3.2)$$

are surjections.

Let $\Delta_i : H_n(F_i, \partial F_i) \to H_n(F_i)$ be the variation map of the fibered knot $K_i$, $i = 0, 1$. Recall that for an $n$-cycle $c$ of $(F_i, \partial F_i)$, $\Delta_i([c])$ is defined to be the homology class represented by $c - \tau_i(c)$, where $[c] \in H_n(F_i, \partial F_i)$ is the homology class represented by $c$. Note that this is a well-defined homomorphism, since $\tau_i|_{\partial F_i}$ is the identity and the isotopy class of $\tau_i$ relative to boundary is uniquely determined.
Note also that the variation maps are isomorphisms (see [5]). Then, we see easily that the following diagram is commutative:

\[
\begin{array}{ccc}
H_n(F_0, \partial F_0) & \xrightarrow{\Delta_0} & H_n(F_0) \\
\psi \downarrow & & \downarrow \psi \\
H_n(F_1, \partial F_1) & \xrightarrow{\Delta_1} & H_1(F_1),
\end{array}
\]

(3.3)

since \(\psi \circ \tau_0\) and \(\tau_1 \circ \psi\) are homotopic relative to boundary.

**Proof of Theorem 2.6.** If \(\text{rank } H_n(F_0) = \text{rank } H_n(F_1)\), then Lemma 3.1 implies that the homomorphisms (3.2) are isomorphisms. Then the commutative diagram (3.3) implies that \(K_0\) and \(K_1\) are orientation preservingly isotopic, since the variation map determines and is determined by the Seifert form, which in turn determines the oriented isotopy class of a simple fibered knot (for details see [5, 3, 4]).

**Proof of Corollary 2.7.** By Lemma 3.1, we see that \(\text{rank } H_n(F_0) = \text{rank } H_n(F_1)\). Then the result follows from Theorem 2.6.

**Proof of Theorem 2.8.** First, suppose that there exists a spherical simple fibered \((2n-1)\)-knot \(\Sigma\) in \(S^{2n+1}\) such that \(K_0\) is isotopic to the connected sum \(K_1 \sharp \Sigma\). Then by Remark 2.2, we have \(\Sigma \succeq K_U\) and \(K_1 \sharp \Sigma \succeq K_1 \sharp K_U\), and hence \(K_0 \succeq K_1\).

For the converse, let \(G\) and \(G'\) be the kernels of the homomorphisms \(\psi: H_n(F_0, \partial F_0) \rightarrow H_n(F_1, \partial F_1)\) and \(\psi_*: H_n(F_0) \rightarrow H_n(F_1)\) respectively. Then we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
0 & \rightarrow & G \\
\downarrow{\Delta_0|G} & & \downarrow{\Delta_0} \\
0 & \rightarrow & G' \\
\downarrow{\Delta_0|G} & & \downarrow{\Delta_1} \\
& & \\
0 & \rightarrow & H_n(F_0, \partial F_0) \xrightarrow{\psi} H_n(F_1, \partial F_1) \\
& & \downarrow{\Delta_1} \\
& & \rightarrow 0 \\
& & \\
0 & \rightarrow & H_n(F_0) \xrightarrow{\psi_*} H_n(F_1) \\
& & \downarrow{\Delta_1} \\
& & \rightarrow 0.
\end{array}
\]

Since \(H_n(F_1, \partial F_1)\) and \(H_n(F_1)\) are free, the exact sequences split. This means that the variation map \(\Delta_0\) of \(K_0\) is isomorphic to the direct sum of the variation map \(\Delta_1\) of \(K_1\) and the isomorphism \(\Delta_0|G: G \rightarrow G'\).

Recall that with respect to certain bases, the matrix associated with the variation map is the inverse of the Seifert matrix (for details see [5]). Since \(n \geq 3\), every unimodal matrix is realized as the Seifert matrix of a simple fibered \((2n-1)\)-knot (see [3, 4]). So we see that there exists a simple fibered \((2n-1)\)-knot \(\Sigma\) which realizes \(\Delta_0|G: G \rightarrow G'\) as its variation map.

Then, we see that the Seifert matrices for \(K_0\) and \(K_1 \sharp \Sigma\) are congruent. Consequently, they are orientation preservingly isotopic to each other by [3, 4].

Furthermore, since \(K_0\) is homotopy equivalent to both \(K_1\) and \(K_1 \sharp \Sigma\), we see that \(\Sigma\) should be homeomorphic to a sphere. This completes the proof.

**Remark 3.2.** Theorem 2.8 implies in particular that the fiber of \(K_0\) is diffeomorphic to the boundary connected sum of the fiber of \(K_1\) and a certain \((n-1)\)-connected \(2n\)-dimensional manifold with spherical boundary. When \(K_0\) and \(K_1\) are spherical, this is also a consequence of [2, Theorem B].
Definition 3.3. Let us consider the equivalence relation generated by the pull back relation defined in Definition 2.1. When two $m$-knots $K_0$ and $K_1$ in $S^{m+2}$ are equivalent with respect to this equivalence relation, we say that $K_0$ and $K_1$ are pull back equivalent.

The above definition together with Theorem 2.8 implies the following, whose proof is easy and is left to the reader.

Corollary 3.4. Two simple fibered $(2n-1)$-knots $K_0$ and $K_1$ in $S^{2n+1}$ with $n \geq 3$ are pull back equivalent if and only if there exist spherical simple fibered $(2n-1)$-knots $\Sigma_0$ and $\Sigma_1$ in $S^{2n+1}$ such that $K_0 \sharp \Sigma_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma_1$.

4. Special Knots

In this section, we show that for a certain class of simple fibered knots, the pull back equivalence relation is equivalent to the relation generated by connected sums with spherical fibered knots together with the cobordism. For a theory of cobordism of simple fibered knots, refer to [1, 10, 11].

Definition 4.1. Let $K$ be a simple fibered $(2n-1)$-knot with fiber $F$. Let us denote by $I(K)$ the image of the homomorphism $H_n(K) \to H_n(F)$ induced by the inclusion (or equivalently, the kernel of the homomorphism $H_n(F) \to H_n(F, \partial F)$). The fibered knot $K$ is said to be special if its Seifert form restricted to $I(K)$ is unimodular (for a definition of a Seifert form, see [3]).

Lemma 4.2. A simple fibered $(2n-1)$-knot $K$ is special if and only if there exist two simple fibered $(2n-1)$-knots $K_F$ and $K_T$ with the following properties:

1. $K$ is orientation preservingly isotopic to $K_F \sharp K_T$,
2. the intersection form of the fiber of $K_F$ is the zero form,
3. $H_{n-1}(K_T)$ is a torsion group (or equivalently, $H_n(K_T) = 0$).

Proof. If there exist simple fibered $(2n-1)$-knots $K_F$ and $K_T$ with properties (1)–(3), then the Seifert form of $K$ restricted to $I(K)$ coincides with the Seifert form of $K_F$. Since $K_F$ is fibered, its Seifert form must be unimodular. Hence, $K$ is special.

Conversely, suppose that the simple fibered knot $K$ is special. Let us consider a basis $e_1, \ldots, e_u, e_{u+1}, \ldots, e_{u+v}$ of $H_n(F)$, where $e_1, \ldots, e_u$ constitute a basis of $I(K)$. This is possible, since $I(K)$ is a direct summand of $H_n(F)$. Then, the Seifert matrix $L$ of $K$ with respect to this basis is of the form

$$L = \begin{pmatrix} L_F & A \\ B & C \end{pmatrix}$$
for some $u \times u$ matrix $L_F$, $u \times v$ matrix $A$, $v \times u$ matrix $B$, and $v \times v$ matrix $C$. Note that $L_F + (-1)^n(L_F) = 0$ and $A + (-1)^n(B) = 0$, since the homomorphism $H_n(F) \to H_n(F, \partial F) = \text{Hom}(H_n(F), \mathbb{Z})$ can be identified with the intersection form of $F$ and the intersection matrix of $F$ is given by $L + (-1)^n(L)$ (for example, see [3]). Since $L_F$ is unimodular by our hypothesis and $L_F = (-1)^{n+1}(tL_F)$, we see that $L$ is congruent to a matrix of the form

$$L' = \begin{pmatrix} L_F & 0 \\ 0 & L_T \end{pmatrix}$$

for some $v \times v$ matrix $L_T$. Since $L'$ is unimodular, so is $L_T$. Furthermore, $L_T + (-1)^n(tL_T)$ is a nonsingular matrix, since the kernel of the intersection form is generated by $e_1, \ldots, e_u$. Let $F_K$ and $T_K$ be the simple fibered $(2n-1)$-knots realizing $L_F$ and $L_T$ as their Seifert matrices respectively. Then, we can check that conditions (1)--(3) are satisfied. This completes the proof.

**Remark 4.3.** In the above lemma, if $H_{n-1}(K)$ is torsion free, then the knot $K_T$ is spherical.

Let us prove the following.

**Theorem 4.4.** Let $K_0$ and $K_1$ be simple fibered $(2n-1)$-knots with $n \geq 3$. Suppose that $K_0$ is special and that $H_{n-1}(K_0)$ is torsion free. Then the following conditions are all equivalent to each other.

1. $K_0 \sharp \Sigma_0$ is cobordant to $K_1 \sharp \Sigma_1$ for some spherical knots $\Sigma_0$ and $\Sigma_1$.
2. $K_0 \sharp \Sigma'_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma'_1$ for some spherical simple fibered knots $\Sigma'_0$ and $\Sigma'_1$.
3. $K_0$ is pull back equivalent to $K_1$.

For the proof, we need the following lemma, which is a direct consequence of [1, Theorem 4] (see also [10, 11]). Recall that a $(2n-1)$-knot is simple if it is $(n-2)$-connected and it bounds an $(n-1)$-connected $2n$-dimensional compact manifold in $S^{2n+1}$.

**Lemma 4.5.** Let $K_0$ and $K_1$ be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical simple knots $\Sigma_0$ and $\Sigma_1$, then the Seifert forms of $K_0$ and $K_1$ restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.

**Proof of Theorem 4.4.** The equivalence of (2) and (3) follows from Corollary 3.4. Condition (2) clearly implies condition (1). Thus, we have only to show that (1) implies (2).

Suppose that (1) holds. Since every spherical $(2n-1)$-knot is cobordant to a spherical simple $(2n-1)$-knot by [6], we may assume that $\Sigma_0$ and $\Sigma_1$ are simple. Then by Lemma 4.5, the Seifert forms of $K_0$ and $K_1$ restricted to $I(K_0)$ and $I(K_1)$,
respectively, are isomorphic to each other. By our assumption, these forms are unimodular, and hence \( K_1 \) is also special. Therefore, by Lemma 4.2, there exist simple fibered \((2n - 1)\)-knots \( K_F^{(i)}, K_T^{(i)}, i = 0, 1 \), such that

(a) \( K_1 \) is orientation preservingly isotopic to \( K_F^{(i)} \# K_T^{(i)} \),
(b) the intersection form of the fiber of \( K_F^{(i)} \) is the zero form,
(c) \( H_{n-1}(K_T^{(i)}) \) is a torsion group,

for \( i = 0, 1 \). Note that \( K_F^{(0)} \) is orientation preservingly isotopic to \( K_F^{(1)} \), since their Seifert forms are isomorphic.

Recall that \( H_{n-1}(K_0) \) is torsion free by our assumption. Therefore, \( K_T^{(i)} \) are spherical knots for \( i = 0, 1 \). Since \( K_0 \# K_T^{(1)} \) is orientation preservingly isotopic to \( K_F^{(1)} \# K_T^{(0)} \# K_T^{(1)} \), it is also orientation preservingly isotopic to \( K_1 \# K_T^{(0)} \). Hence condition (2) holds. This completes the proof.

\( \square \)

**Remark 4.6.** Let \( K_F \) be the simple fibered \((2n - 1)\)-knot as in Lemma 4.2. Then its Seifert form is skew-symmetric for \( n \) even, and is symmetric for \( n \) odd. Note that unimodular skew-symmetric matrices have even ranks and the congruence class of such a matrix is uniquely determined by its rank. Therefore, when \( n \) is even, the oriented isotopy class of \( K_F \) is determined by its rank, which is even. On the other hand, when \( n \) is odd, unimodular symmetric matrices are not determined by its rank. For details, refer to [9], for example.

### 5. Examples

In the previous sections, we have seen that for two simple fibered \((2n - 1)\)-knots with \( n \geq 3 \), the following implications hold:

- pull back equivalent
  \( \implies \) cobordant after taking connected sums with some spherical knots
  \( \implies \) (simple) homotopy equivalent.

In fact, these implications hold for arbitrary \( m \)-knots with \( m \geq 5 \) which satisfy the connectivity conditions mentioned in Remark 2.3. In this section, we show that the converses of the above two implications do not hold in general by giving several important examples.

**Proposition 5.1.** For every odd integer \( n \geq 3 \), there exists a pair \((K_0, K_1)\) of simple fibered \((2n - 1)\)-knots with the following properties.

1. The knots \( K_0 \) and \( K_1 \) are cobordant.
2. The knots \( K_0 \) and \( K_1 \) are not pull back equivalent.
Proof. Let us consider the following two matrices:
\[
L_0 = \begin{pmatrix} 9 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 25 & 1 \\ -1 & 0 \end{pmatrix}.
\]
Note that they are both unimodular and that
\[
S_0 = L_0 - tL_0 = S_1 = L_1 - tL_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.
\]
Let us show that \( L_0 \) and \( L_1 \) are algebraically cobordant in the sense of [1, (1.2), Definition] for \( \varepsilon = (1)^n = -1 \).

Set \( m = t(5,0,3,0) \) and \( m' = t(0,3,0,5) \). Then it is easy to see that the submodule \( M \) of \( Z^4 \) generated by \( m \) and \( m' \) constitutes a metabolizer for \( L = L_0 \oplus (-L_1) \). Furthermore, \( M \) is pure in \( Z^4 \); in other words, \( M \) is a direct summand of \( Z^4 \). Since \( S_0 = S_1 \) are non-degenerate, we have only to verify the condition c.2 of [1, (1.2), Definition].

Let us consider the following two matrices:
\[
S = S_0 \oplus (-S_1) = L - tL. \quad \text{Let} \quad S^* : Z^4 \rightarrow Z^4, S_0^* : Z^2 \rightarrow Z^2 \quad \text{and} \quad S_1^* : Z^2 \rightarrow Z^2
\]
be the adjoints of \( S, S_0 \) and \( S_1 \) respectively. It is easy to see that Coker \( S_0^* = \text{Coker} S_1^* \) is naturally identified with \( Z_2 \oplus Z_2 \). Furthermore, we have
\[
S^*(m) = 'mS = (0,10,0,-6) \quad \text{and} \quad S^*(m') = 'm'S = (-6,0,10,0).
\]
Therefore, \( S^*(M)^\ominus \), the smallest direct summand of \( Z^4 \) containing \( S^*(M) \), is the submodule of \( Z^4 \) generated by \((0,5,0,-3) \) and \(( -3,0,5,0) \). Hence, for the natural quotient map \( d : Z^4 \rightarrow \text{Coker} S^* = (Z_2 \oplus Z_2) \oplus (Z_2 \oplus Z_2) \), we have
\[
d(S^*(M)^\ominus) = \{(x,x) : x \in \text{Coker} S_0^* = Z_2 \oplus Z_2\},
\]
since \( \text{Im} S_i^* \) is generated by \((2,0) \) and \((0,2) \), \( i = 0,1 \), and \( \text{Im} S^* \) is generated by \((2,0,0,0),(0,2,0,0),(0,0,2,0) \) and \((0,0,0,2) \). Therefore, we conclude that the unimodular matrices \( L_0 \) and \( L_1 \) are algebraically cobordant.

Now, there exists a simple fibered \((2n - 1)\)-knot \( K_i \) which realizes \( L_i \) as its Seifert form with respect to the fiber, \( i = 0,1 \) (see [3, 4]). By [1, Theorem 3], \( K_0 \) and \( K_1 \) are cobordant.

Let us now show that \( K_0 \) and \( K_1 \) are not pull back equivalent. By Corollary 3.4, we have only to show that for any spherical simple fibered \((2n - 1)\)-knots \( \Sigma_0 \) and \( \Sigma_1 \) in \( S^{2n+1} \), \( K_0 \sharp \Sigma_0 \) is never orientation preservingly isotopic to \( K_i \sharp \Sigma_i \).

Since \( K_i \sharp \Sigma_i \) is a fibered knot, we can consider the monodromy on the \( n \)th homology group of the fiber, \( i = 0,1 \). Let us denote by \( H_i \) the monodromy matrix of \( K_i \sharp \Sigma_i \) and by \( L_i \) its Seifert matrix with respect to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for \( K_i \) and that for \( \Sigma_i \). It is known that \( H_i = (-1)^{n+1}L_i^{-1}(tL_i) \) (for example, see [3]). Therefore, we have
\[
H_0 = \begin{pmatrix} -1 & 0 \\ 18 & -1 \end{pmatrix} \oplus H_0' \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 \\ 50 & -1 \end{pmatrix} \oplus H_1',
\]
where \( H_i' \) is the monodromy matrix of \( \Sigma_i, \ i = 0,1 \).
Let us consider $\text{Ker}((I + H_i)^2)$, where $I$ is the unit matrix, $i = 0, 1$. Since $\Sigma_i$ are spherical knots, the monodromy matrices $H_i$ cannot have the eigenvalue $-1$. Therefore, $\text{Ker}((I + H_i)^2)$ corresponds exactly to the homology of the fiber of $K_i$.

Suppose that $K_0 \simeq K_0$, where $K_0$ is orientation preservingly isotopic to $K_1$. Then the Seifert form of $K_0$ restricted to $\text{Ker}((I + H_0)^2)$ should be isomorphic to that of $K_1$ restricted to $\text{Ker}((I + H_1)^2)$. This means that $L_0$ should be congruent to $L_1$. However, this is a contradiction, since there exists an element $x \in \mathbb{Z}^2$ such that $xL_0x = 9$, while such an element does not exist for $L_1$.

Thus, we conclude that $K_0$ and $K_1$ are not pull back equivalent.

Note that the simple fibered knots $K_0$ and $K_1$ constructed above are special; however, $H_{n-1}(K_i)$, $i = 0, 1$, are not torsion free.

**Remark 5.2.** In fact, we can find infinitely many examples as in the above proposition. For example, we could use the matrices

$$
\begin{pmatrix}
p^2 & 1 \\
-1 & 0
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
q^2 & 1 \\
-1 & 0
\end{pmatrix}
$$

for arbitrary relatively prime odd integers $p$ and $q$. Or we could also use $K_0 \# K'$ and $K_1 \# K'$, instead of $K_0$ and $K_1$, for any simple fibered $(2n - 1)$-knot $K'$ whose monodromy does not have the eigenvalue $-1$.

As has been remarked in Remark 2.3, under a certain connectivity condition, if two $m$-knots $K_0$ and $K_1$ with $m \geq 5$ are pull back equivalent, then they are cobordant after taking connected sums with some spherical knots. The above example shows that the converse is not true in general.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition, which is a slight modification of Lemma 4.5 and is implicitly proved in the proof of Theorem 4.4.

**Proposition 5.3.** Let $K_0$ and $K_1$ be simple fibered $(2n - 1)$-knots with $n \geq 3$. If $K_0 \# \Sigma_0$ and $K_1 \# \Sigma_1$ are cobordant for some spherical knots $\Sigma_0$ and $\Sigma_1$, then the Seifert forms of $K_0$ and $K_1$ restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.

**Remark 5.4.** In fact, the above proposition is implicitly proved also in [11]. Based on this, Vogt proves the following. The usual $(2n - 1)$-dimensional spherical knot cobordism group $C_{2n-1}$ acts on the cobordism semi-group of simple $(2n - 1)$-knots with torsion free homologies by connected sum. The orbit space of the action inherits a natural semi-group structure. Then this orbit space contains infinitely many free generators as a commutative semi-group for $n \geq 3$.

Vogt [11] also proves that the action of $C_{2n-1}$ on the cobordism semi-group of simple $(2n - 1)$-knots is fixed point free for $n \geq 3$. This can also be proved by using...
[1, (5.1), Proposition]. In fact, for an arbitrary spherical simple \((2n - 1)\)-knot \(\Sigma\) whose Alexander polynomial is nontrivial and irreducible, \(K_0^*\Sigma\) is never cobordant to \(K\) for any simple \((2n - 1)\)-knot \(K\), since the Alexander polynomials of \(K_0^*\Sigma\) and \(K\) do not satisfy a Fox–Milnor type relation necessary to be cobordant (see [1, (5.1), Proposition]).

The following example answers the question raised at the beginning of Sec. 1 negatively.

**Example 5.5.** Let us consider the following unimodular matrices:

\[
L_0 = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^{n+1} & 0 & 0 & 1 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}.
\]

Then, for every integer \(n \geq 3\), there exist simple fibered \((2n - 1)\)-knots \(K_i\) in \(S^{2n+1}\) whose Seifert matrices are given by \(L_i, i = 0, 1\) (see [3, 4]). Note that if we denote their fibers by \(F_i, i = 0, 1\), then \(F_1\) is orientation preservingly diffeomorphic to \(F_0^*(S^n \times S^n)\). In particular, \(K_0\) and \(K_1\) are orientation preservingly diffeomorphic to each other.

It is easy to verify that the Seifert form restricted to \(I(K_1)\) is the zero form, while it is not zero for \(K_0\). Hence, by Proposition 5.3, \(K_0^*\Sigma_0\) is never cobordant to \(K_1^*\Sigma_1\) for any spherical (but not necessarily simple or fibered) knots \(\Sigma_0, \Sigma_1\).

Note that for this example, we have \(H_{n-1}(K_i) \cong \mathbb{Z} \oplus \mathbb{Z}, i = 0, 1\).

Let us give another kind of an example together with another argument.

**Example 5.6.** Let us consider the unimodular matrices

\[
L_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}
\]

and their associated simple fibered \((2n - 1)\)-knots \(K_i, i = 0, 1\), as in the previous example, for every even integer \(n \geq 4\). By a similar argument, we see that \(K_0\) and \(K_1\) are orientation preservingly diffeomorphic to each other.

Suppose that for some spherical \((2n - 1)\)-knots \(\Sigma_i, i = 0, 1\), \(K_0^*\Sigma_0\) is cobordant to \(K_1^*\Sigma_1\). We may assume that \(\Sigma_0\) and \(\Sigma_1\) are simple. We see easily that the Alexander polynomials of \(K_0\) and \(K_1\) are given by

\[
\Delta_{K_0}(t) = \det(tL_0 + t^*L_0) = t^2 + t + 1
\]

and

\[
\Delta_{K_1}(t) = \det(tL_1 + t^*L_1) = -(t^4 + t^3 - t^2 + t + 1)
\]
respectively. Note that the both polynomials are irreducible over \( \mathbb{Z} \). If \( K_0 \Sigma_0 \) is cobordant to \( K_1 \Sigma_1 \), then by [1, (5.1), Proposition], we see that

\[
\Delta_{K_0}(t) \Delta_{\Sigma_0}(t) \Delta_{K_1}(t^{-1}) \Delta_{\Sigma_1}(t^{-1}) = t^3 f(t) f(t^{-1})
\]

for some \( \lambda \in \mathbb{Z} \) and \( f(t) \in \mathbb{Z}[t, t^{-1}] \) (Fox–Milnor type relation), where \( \Delta_{\Sigma_i}(t) \) is the Alexander polynomial of \( \Sigma_i \), \( i = 0, 1 \).

Note that we have \( |\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3 \) and \( |\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1 \). Since \( \Delta_{K_0}(t) \) is irreducible of degree 2, and \( \Delta_{K_1}(t) \) is irreducible of degree 4, the above relation leads to a contradiction.

Hence, \( K_0 \Sigma_0 \) is not cobordant to \( K_1 \Sigma_1 \) for any spherical (not necessarily simple or fibered) \((2n-1)\)-knots \( \Sigma_0, \Sigma_1 \). Note that we have \( H_{n-1}(K_i) \cong \mathbb{Z}_3, \ i = 0, 1 \), for this example.

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