Class. Quantum Grav. **15** (1998) 2645–2655. Printed in the UK PII: S0264-9381(98)94994-2

Three-dimensional topology-independent methods to look for global topology

Boudewijn F Roukema†§ and Vincent Blanloeil‡‖

† Observatoire de Strasbourg, 11, rue de l'Université, Strasbourg F-67000, France
and
Nicolaus Copernicus Astronomical Center, ul. Bartycka 18, 00-716 Warsaw, Poland
‡ Institut de recherche mathématique avancée, Université Louis Pasteur et CNRS, 7 rue René-Descartes, F-67084 Strasbourg Cedex, France

Received 10 June 1998

Abstract. The spacelike hypersurface of the Universe at the present cosmological time is a three-dimensional manifold. A non-trivial global topology of this spacelike hypersurface would imply that the apparently observable universe (the sphere of particle horizon radius) could contain several images of the single, physical Universe. Recent three-dimensional techniques for constraining and/or detecting this topology are reviewed. Initial applications of these techniques using x-ray bright clusters of galaxies and quasars imply (weak) candidates for a non-trivial topology.

PACS numbers: 9880E, 0420G, 0240, 9854

1. Introduction

If the physical Universe is smaller than the 'observable Universe', i.e. if the fundamental polyhedron of the Universe is smaller than the sphere of horizon radius in the universal covering space, then some (or many) regions of space will be observable at several (or many) different 'look-back' times (de Sitter 1917, Lemaître 1958). The word 'Universe' can be taken here to refer either theoretically to the spacelike hypersurface at the present cosmological time, or observationally to the observed past time cone considered in comoving coordinates. Since spacelike hypersurfaces are three dimensional, use of three-dimensional (3D) information on astrophysical objects known to exist in the covering space provides a straightforward way to search for or constrain the global topology of the Universe.

The reader is referred to Lachièze-Rey and Luminet (1995) and to other contributions to this workshop for an introduction to cosmological topology and to Luminet (1998) for an interesting historical introduction.

A brief mathematical description of the relationship between a universal covering space X, a compact 3-manifold M and its fundamental polyhedron P is provided in section 2. For a fuller introduction to 3D geometry and topology, see Thurston (1982, 1997).

Three-dimensional topology detection techniques are based on the required existence of multiple topological images of single physical objects. Techniques applicable to objects observed to successively larger scales are described in section 4.1 ('cosmic crystallography',

§ E-mail address: roukema@iap.fr
‖ E-mail address: blanloei@math.u-strasbg.fr

Lehoucq *et al* 1996), section 4.2 ('brightest x-ray clusters', Roukema and Edge 1997) and section 4.3 ('local isometry detection', Roukema 1996).

A Friedmann–Lemaître–Robertson–Walker metric (implying constant curvature of any spatial hypersurface) is assumed throughout this paper. Comoving coordinates are used, i.e. positions of objects observed in our past time cone are projected to the (3D) spacelike hypersurface at the present epoch, $t = t_0$. Spectroscopic redshifts, denoted by z, are used to obtain radial distance estimates (termed 'proper distances' by Weinberg (1972), equation (14.2.21)†) in the $t = t_0$ spacelike hypersurface.

2. Topology of compact 3-manifolds: covering spaces and fundamental polyhedra

If we assume the Universe to have an FLRW metric and a trivial topology, then it is one of the three 3-manifolds S^3, the 3-sphere, E^3, Euclidean 3-space, or H^3, the hyperbolic 3-space (negatively curved). This apparent space is called the 'universal covering space', which we call X.

This can be related to the real 3-manifold of the Universe, which we call M and which can be thought of physically in terms of a fundamental polyhedron P as follows.

One can construct a complete geometric 3-manifold by choosing Γ, a discrete subgroup of the group of isometries of X, acting freely on X (i.e. the set $\{g \in \Gamma\colon gx = x\}$ is trivial for all x in X) and take the quotient X/Γ. Of course, at the present we know neither X nor Γ for the real Universe.

Observations favour $X = E^3$ ($\Omega_0 + \lambda_0 = 1$ in terms of standard metric parameters, where $\lambda_0 \equiv \Lambda c^2/(3H_0^2)$) and $X = H^3$ ($\Omega_0 + \lambda_0 < 1$), though as long as $\Omega_0 + \lambda_0 = 1$ remains reconcilable with observations, $X = S^3$ is likely to remain a possibility where $\Omega_0 + \lambda_0 = 1 + \epsilon$ and $\epsilon \ll 1$ if Ω_0 and λ_0 are only measured by traditional, local physical properties such as density. (An interesting global method is the effect predicted by Barrow *et al* (1985, p 937), who showed that compactness and a small amount of rotation of the observable Universe could, in principle, enable the case $\Omega_0 + \lambda_0 = 1$ to be distinguished from the case $\Omega_0 + \lambda_0 = 1 \pm \epsilon$ ($\epsilon \ll 1$) by the presence of a 'spiralling effect' which could be seen in the cosmological microwave background.)

Then, *a convex fundamental polyhedron* for the discrete group Γ of the group of isometries of X is a convex polyhedron P in X such that its interior P° fulfils:

(i) the members of $\{gP^\circ\colon g \in \Gamma\}$ are mutually disjoint;
(ii) $X = \cup\{gP\colon g \in \Gamma\}$;
(iii) $\{gP\colon g \in \Gamma\}$ is a locally finite family of subsets of X.

To find a fundamental polyhedron, for all $g \neq 1$ in Γ and for x in X, we define $H_g(x) = \{y \in X\colon d(x, y) < d(x, gy)\}$. Then the *Dirichlet domain* $D(x)$, with centre x, for Γ is $D(x) = \cap\{H_g(x)\colon g \neq 1 \in \Gamma\}$ when Γ is non-trivial and $D(x) = X$ when Γ is trivial. The closure $\bar{D}(x)$ is a convex fundamental polyhedron for Γ.

Using a fundamental polyhedron P in X, one can build a 3-manifold M by gluing together the sides of P. Poincaré's fundamental polyhedron theorem proves that the inclusion of P in X induces an isometry from M to X/Γ, where Γ is the discrete subgroup of the group of isometries of X such that (S, R) is a group presentation for Γ with S the set of sides of P and R the set of relations determined by the gluing.

When Γ is not trivial this construction gives a non-trivial topology.

† The 'proper distance' should not be confused with the quantity that Weinberg (1972, p 485) calls 'proper motion distance' and that Peebles (1993, p 321, equation (13.36)) calls 'angular size distance'. The proper distance and proper motion distance are identical for zero curvature, but not otherwise.

Let us be more precise. Choose a base point x_0 of X and let $a\colon S^1 \to X/\Gamma$ be a loop based at x_0; let $b\colon [0,1] \to X$ be a lift of a starting at x_0 and ending at $g_a\, x_0$ (note that g_a is unique since Γ acts freely). The map $l\colon \pi_1(X/\Gamma) \to \Gamma$ defined by $l(a) = g_a$ is a homomorphism which is obviously surjective (i.e. onto). Suppose for $a \in \pi_1(X/\Gamma)$ we have $l(a) = 1$. Then a lift of a in $\pi_1(X)$ is equal to 1 since X is simply connected, hence $a = 1$ and l is injective (i.e. 1:1). Therefore, $l\colon \pi_1(X/\Gamma) \to \Gamma$ is an isomorphism.

Since Γ is a non-trivial group, the fundamental group $\pi_1(X/\Gamma)$, which could be thought of as the group of non-shrinkable loops of X/Γ, and hence of M, is non-trivial. That is, M has a non-trivial topology and can be referred to as multi-connected.

For an example of theoretical ideas for the physical meaning of Γ, see e Costa and Fagundes (1998). Here we merely consider observational detection of Γ.

Characterizing the ‘size’ of fundamental polyhedra of 3-manifolds in a way useful observationally requires at least two parameters. Here we adopt the ‘injectivity radius’, r_{inj}, i.e. half of the smallest distance from an object to one of its topological images and the out-radius, r_+, which is the radius of the smallest sphere (in the covering space) which totally includes the fundamental polyhedron (Cornish *et al* 1998a). We refer here to $2r_{\mathrm{inj}}$ as the injectivity diameter and $2r_+$ as the out-diameter. For a discussion of these and related size parameters, see Cornish *et al* (1998a), and note that $2r_{\mathrm{inj}}$ and $2r_+$ are similar to the parameters α and β adopted by Lachièze-Rey and Luminet (1995, section 10.3.3).

While both parameters represent in some sense the ‘size’ of the fundamental polyhedron, it should particularly be remembered that many hyperbolic compact 3D manifolds can have $r_{\mathrm{inj}} \ll r_+$. Since we live in the plane of a disc galaxy—which obscures most astronomical observations within several degrees of the plane—it would be difficult to measure $2r_{\mathrm{inj}}$ if it is the size of a geodesic at an angle ‘close’ to the Galactic plane.

3. Multiple topological images of observable objects

In a multi-connected universe, the covering space, or ‘apparent’ universe, is tiled by copies of the fundamental polyhedron (Dirichlet domain). So the basic principle of detecting multi-connectedness is to find multiple ‘topological images’†.

In a flat covering space, the particle horizon radius, R_H, does not geometrically constrain r_{inj} and r_+. In a hyperbolic covering space, the two quantities are both directly related to the curvature radius, R_C (at least in the orientable cases), by Mostow’s rigidity theorem which states that a homotopy equivalence between two orientable hyperbolic 3-manifolds is homotopic to an isometry. Hence, r_{inj} and r_+ are bounded below by R_C and hence by R_H to within a few orders of magnitude (depending on which of the many compact hyperbolic manifolds applies to the universe).

So, if physical conditions in the very early Universe (quantum epoch) tend to minimize the volume and the ‘present-day’ Universe is negatively curved, then the fundamental polyhedron may well be small enough that multiple topological images of points of space are likely to exist within the present-day observable sphere. Zero curvature provides no such constraint.

Since the Universe is of finite age, information (photons) can only be received from within a sphere (in the covering space) of finite radius around the observer. This sphere may contain several copies of the fundamental polyhedron, in which multiple topological images of single physical objects can be found. However, objects which are seen further

† Also called ‘topological clones’. The terminology ‘ghosts’ is not preferred since it implies that some images are less physically real than others.

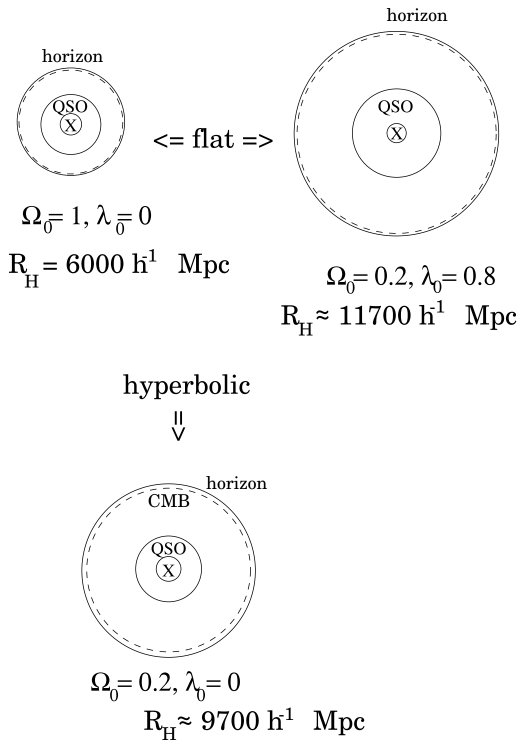

Figure 1. Relative sizes of observable domains of the universal covering space, shown in comoving coordinates on a radially linear (proper distance) scale depending on different options for the metric parameters Ω_0, λ_0. The horizon radius R_H is defined by the age of the Universe. The inner circles for each choice of metric indicate from the centre outwards redshifts of: $z = 0.5$ (roughly that to which we can see x-ray emission from the richest clusters of galaxies); $z = 3$ (to which there is a high quasar density) and $z = 1000$ (roughly the redshift of the cosmic microwave background, broken circle). In the flat models, the tangential distance scale is constant and the same as the radial scales; in the hyperbolic model, the tangential distance scale increases (more Gpc mm^{-1} on the page) as a function of increasing radius.

towards this 'horizon' are seen earlier in the history of the Universe, so are seen at different stages of the transition between a relatively smooth (to $\sim 0.001\%$) material to the formation of high-density objects such as quasars, clusters of galaxies and galaxies. So any set of observable objects can only be seen inside a certain sub-sphere (or a spherical shell) which is smaller than the observable sphere.

Depending on the metric parameters (Ω_0, λ_0) of the Universe, the fraction of the comoving observable sphere covered by a set of observable objects varies.

Figure 1 shows some characteristic distances to which different types of objects have so far been observed in significant quantities for a range of the metric parameters covering the values consistent with a wide range of observational cosmological tests. The redshifts used are indicative only.

The proper distance to a redshift z can be evaluated in general as

$$d(z) = \frac{c}{H_0} \int_{1/(1+z)}^{1} \frac{\mathrm{d}a}{a\sqrt{\Omega_0/a - \kappa_0 + \lambda_0 a^2}} \qquad (1)$$

where $\kappa_0 \equiv \Omega_0 + \lambda_0 - 1$†. This can be expressed in terms of the curvature radius

$$R_C \equiv \frac{c}{H_0\sqrt{|\kappa_0|}} \tag{2}$$

and the proper motion distance $d_{pm}(z)$ as

$$d(z) = \begin{cases} R_C \sinh^{-1}[d_{pm}(z)/R_C], & \kappa_0 < 0 \\ d_{pm}(z), & \kappa_0 = 0 \\ R_C \sin^{-1}[d_{pm}(z)/R_C], & \kappa_0 > 0. \end{cases} \tag{3}$$

If $\Omega_0 > 0$ and $\lambda_0 = 0$, then the closed expression

$$d_{pm}(z) = \frac{c}{H_0} \frac{2\left[z\Omega_0 + (\Omega_0 - 2)\left(\sqrt{\Omega_0 z + 1} - 1\right)\right]}{\Omega_0^2(1+z)} \tag{4}$$

can be used (Weinberg 1972, p 485). Or, for $\kappa_0 \neq 0$, equations (1) and (2) can be combined to give

$$\frac{d(z)}{R_C} = \sqrt{|\kappa_0|} \int_{1/(1+z)}^{1} \frac{\mathrm{d}a}{a\sqrt{\Omega_0/a - \kappa_0 + \lambda_0 a^2}}. \tag{5}$$

Only very bright galaxy clusters are seen as far as $z = 0.5$, and systematic 'all-sky' surveys for galaxy clusters (e.g. to an x-ray luminosity of $L_X(0.1\text{–}2.4\ \text{keV}) \gtrsim 10^{45}$ erg s^{-1}) are only presently being carried out to $z \sim 0.1$. Quasars are seen to redshifts higher than $z \approx 3$, but drop quickly in number density (e.g. Shaver *et al* 1998).

The fraction of the horizon distance covered by the sphere to $z = 3$ only decreases slightly between the models with different metric parameters, from 50% in the $\Omega_0 = 1$, $\lambda_0 = 0$ (Einstein–de Sitter) model to 39% in the $\Omega_0 = 0.2$, $\lambda_0 = 0$ model. If multiple copies of the fundamental polyhedron are to be detected in a radial direction, the relative efficiencies of 3D methods using objects (e.g. quasars) visible to these redshifts does not change much. Due to the negative curvature, however, the number of copies of a fundamental polyhedron which could be placed alongside one another in a tangential direction around a sphere makes a search to $z \lesssim 3$ much less efficient relative to a CMB (cosmic microwave background) search for an $\Omega_0 = 0.2$, $\lambda_0 = 0$ universe relative to an Einstein–de Sitter universe.

Therefore, if the Universe is as negatively curved as to give $\Omega_0 = 0.2$, $\lambda_0 = 0$, then 3D searches for topological images are going to be most efficient for multi-connected manifolds which have $r_{\text{inj}} \ll r_+$. Since this is the case for many of the hyperbolic compact manifolds, a 3D method which can be applied to objects seen to $z \lesssim 3$ may be capable of detecting non-trivial topology of the Universe, i.e. at least one of the generators $g \in \Gamma$, in this case.

On the other hand, the flat universe model dominated by a cosmological constant ($\Omega_0 = 0.2$, $\lambda_0 = 0.8$) shows nearly identical relative efficiencies of $z \lesssim 3$ and CMB methods as for the flat, $\lambda_0 = 0$ model, but the relative usefulness of objects seen to $z \lesssim 0.5$ is much lower than in the model with a cosmological constant than in the model without.

4. Previous 3D constraints and new 3D methods

The out-diameter, $2r_+$, is strongly bounded below by about 60 h^{-1} Mpc to 150 h^{-1} Mpc by the absence of secondary topological images of the Coma cluster of galaxies (Gott 1980) and by the existence of 'large-scale structure' ('great walls' and filaments formed by galaxies, de Lapparent *et al* 1986, Geller and Huchra 1989, da Costa *et al* 1993, Deng *et al*

† For readers of the popular Peebles (1993), Ω_0, λ_0 and κ_0 correspond to Peebles' Ω, Ω_Λ and $-\Omega_R$, respectively.

1996, Einasto *et al* 1997). Equivalently, this is a constraint that $2r_+ \gtrsim R_H/100$. Although mathematically not strictly excluded, it would seem difficult for the injectivity diameter, $2r_{\mathrm{inj}}$, to be as small as $2r_{\mathrm{inj}} \lesssim R_H/100$ in a way that would fit the spatial distribution and physical properties of observed objects.

At distances greater than $R_H/10$, the formation and evolution of astrophysical objects over corresponding time scales becomes much more serious than at small scales.

In addition, catalogues of observed objects are limited to either wide-angle surveys to small radial distances or 'deep' surveys over small solid angles. The two types of surveys have complementary advantages for detecting topology, though the use of the wide-angle surveys is simpler. The increase in the characteristic radial distances and solid angles of these surveys will increase rapidly over the next few decades.

To avoid the problems of evolution, methods based primarily on the 3D positions of the objects (rather than their physical properties) are needed. The methods of 'cosmic crystallography' (section 4.1) and of 'local isometry searching' (section 4.3) were created for use in catalogues of objects which are subject to evolutionary effects on the individual astrophysical objects. If the evolutionary effects are not too strong (and if viewing angle is not a problem), the former method is applicable. If many objects have only a subset of their topological images visible due to such effects, the latter method is necessary.

Nevertheless, the existence of a 'unique' object (e.g. much more brilliant than all others of its class) at a large radial distance could still be useful in finding a lower bound to r_+ (particularly if a systematic survey over 4π sr were available). This idea can be used by consideration of the 'richest' galaxy cluster found by the x-ray emission emitted by its hot $T \sim 10^7$ K gas (section 4.2).

For simplicity, most of the discussion below is presented in the context of a flat, $\Omega_0 = 1$, $\lambda_0 = 0$ universe, but it should be kept in mind that physical (Cornish *et al* 1996) and geometrical arguments favour a hyperbolic universe. For detailed discussions of the hyperbolic case, see the work of Fagundes (1985, 1989, 1996). For numerical representations of compact hyperbolic manifolds and visualization software, SNAPPEA and GEOMVIEW (http://www.geom.umn.edu/) are recommended.

For completeness, it should also be mentioned that attempts have been made to use the essentially two-dimensional information in the CMB to observationally bound r_{inj} from below, by making assumptions on the distributions of amplitudes and phases of temperature fluctuations at the epoch of the CMB, by accepting foreground corrections as valid and by considering either particular cases of flat geometries (Stevens *et al* 1993, Starobinsky 1993, Jing and Fang 1994, de Oliveira Costa and Smoot 1995, Levin *et al* 1998) or individual cases of hyperbolic geometries (Bond *et al* 1998). If the caveats of these techniques are accepted as correct, then r_{inj} seems not much smaller than the horizon size. Indeed, the latter authors find a candidate multi-connected hyperbolic manifold, for $\Omega_0 = 0.8$, which is 'preferable to standard CDM' relative to the observed CMB (Bond *et al*, section 4.3), but the volume of its fundamental polyhedron is slightly larger than that of the observable sphere.

It should also be noted that non-trivial topology is usually adopted in N-body simulations of the formation of galaxies and large-scale structure, but for numerical rather than physical reasons (see Farrar and Melott (1990) for an explicit analysis).

4.1. Cosmic crystallography

In a catalogue of objects in which multiple topological images of single physical objects are often seen, a histogram of object–object pair separations (for all $N(N+1)/2$ pairs in a set of N objects) should show sharp peaks due to pairs of topological images separated by multiples of the vectors which generate the fundamental polyhedron from the covering space. Lehoucq *et al* (1996) used simulations to show that this method should be efficient and independent of topology, at least for the case of zero curvature. The application of this method to the classical Abell and ACO cluster catalogues, to $z \approx 0.25$, did not show any obvious topological signal (but see also Fagundes and Gausmann 1997). The method was devised to also work in the cases of non-zero curvature, but its practical application under astronomical conditions has yet to be carried out.

The Abell and ACO catalogues will soon be superseded by x-ray selected 'all-sky'† catalogues which suffer less serious systematic biases, but only to $z \sim 0.1$ in programmes already in progress for finding rich clusters.

The intrinsically brightest of these clusters should be found to a factor of several higher in redshift, though in numbers too small for the histogram method to show any peaks. In this case, the following method can be applied.

4.2. X-ray clusters as 'standard candles'

If a small number of the brightest objects of a given class (in particular, galaxy clusters selected in x-rays) are known out to a given redshift, these can be considered to be unique objects if their evolutionary properties are simple enough.

In the case of the richest galaxy clusters, which are dominated by hot gas, these objects are unlikely to become any less luminous as time increases, though they are likely to increase somewhat in luminosity. This is because galaxy clusters are the largest gravitationally bound objects which have had time to collapse over the age of the Universe, and are dominated by their hot (monatomic) hydrogen gas which is in approximate kinetic equilibrium. Within cosmologically available time scales, it is difficult to see how a high enough fraction of this gas could either cool, escape or turn into galaxies in order for a secondary topological image at a lower redshift (i.e. at a more recent epoch) to be invisible in x-rays.

Roukema and Edge (1997) noticed that RX J1347.5-1145, which is probably the brightest x-ray cluster known (in the 0.1–2.4 keV frequency band) is quite distant from us (with respect to other known galaxy clusters). So, if we could be sure that there were no topological images of this cluster closer to us in any direction, then the distance to this cluster would give the lower limit $r_+ \gtrsim 1100 \pm 100\ h^{-1}$ Mpc. (The uncertainty is due to the observational uncertainty in the metric parameters; the range shown in figure 1 is adopted.) However, since galactic obscuration is important, this is strictly speaking a weaker limit, i.e. $2r_+ \gtrsim 1100 \pm 100\ h^{-1}$ Mpc.

Because the object is unique, the only topological image pair geodesics which are excluded are those extending from the known image to inside the borders of the observed volume. Small closed geodesics, e.g. one running from RX J1347.5-1145, at a galactic longitude and latitude ($l^{\rm II} = 324^\circ, b^{\rm II} = 49^\circ$) and $z = 0.451$, to the point ($l^{\rm II} = 324^\circ, b^{\rm II} = 19^\circ, z = 0.451$), would not contradict the existence of RX J1347.5-1145 as an (apparently) unique object. Hence, this method constrains $2r_+$ more easily than $2r_{\rm inj}$.

Serendipitously, a candidate for the topology was noticed by comparing the 3D positions of the several bright clusters listed by Roukema and Edge (1997). Three (of the seven clusters studied) form a right angle (to 2% accuracy) with side lengths equal within 1%. This is just what would be expected in the case of a $T^2 \times X$ manifold where X is unknown. There is no obvious physical motivation for this case to be favoured, though it is commonly

† 'All-sky' can mean as little as $\frac{2}{3}$, though usually more, of 4π sr, due to obscuration by the Galaxy.

used for pedagogical purposes, as in several of the early attempts at trying to constrain topology using the CMB as observed by the COBE satellite.

Roukema and Edge (1997) list several arguments against topological identity of the three clusters, but the cleanest observational test would be to verify or refute the existence of further implied topological images. While further implied topological images could exist at high redshifts, the cluster might not have formed at those early epochs. So implied images at low enough redshifts that the cluster is guaranteed to be in existence with a minimum luminosity should be considered.

Specific predictions of this (weak) candidate 3-manifold containing two known generators g_1, g_2 and one unknown generator g_3 are as follows. If this candidate as suggested by Roukema and Edge (1997) were correct, then the object seen by ROSAT, RX J203150.4-403656, should be a galaxy cluster at $0.38 < z < 0.40$, and the Arp–Madore galaxy cluster AM 0750-490 (which would be the low-redshift topological image of MS 1054-0321) would be at $0.23 < z < 0.26$.

4.3. Local isometry searches

The previous methods can only be applied to objects whose evolution is relatively simple. The objects which can be easily seen to $z \sim 3$ in significant numbers, quasars, have evolutionary properties which are not well understood. They are likely to have short lifetimes, either with recurrent bursts related to (maybe) mergers of galaxy dark matter haloes or differing lifetimes depending on individual quasar properties. In addition, according to the ‘unified model’ of active galactic nuclei (AGN), a quasar seen from a very different angle appears much fainter, as a Seyfert galaxy, for example, with a very small chance of having already been observed in high-redshift galaxy searches.

One approach to using quasars is to consider special cases. Fagundes and Wichoski (1987) searched for images of the Galaxy seen as quasars in directions separated by 180° or 90°.

A more general method is to accept the problem of multiple topological image visibility and search in a large catalogue for the rare cases in which *several* objects in two different topological images of a single physical 3D region are both visible. In other words, one searches for an isometry between two regions of the covering space, each of a few hundred h^{-1} Mpc in size.

If several such isometries are found, then these should be used to generate the full set of transformations from the covering space to the fundamental polyhedron. These would then be confirmed (or refuted) by the predictions of the 3D positions of multiple topological images of other quasars, or by comparison with the CMB.

This method was first presented by Roukema (1996) and applied to a catalogue of $N \approx 5000$ quasars at $z > 1$. Two isometries, i.e. two pairs of quasar quintuplets separated by more than 300 h^{-1} Mpc, were found. Due to the number density distribution of quasars, this is not necessarily due to topological imaging. Simulated catalogues showed that there is about a 30% chance of finding two similar coincidences in a universe of trivial topology with the same observational selection criteria for finding quasars.

At best, these two isometries could be considered as defining a weak candidate for the 3-manifold in which we live. This candidate would be non-orientable.

The technique as presented by Roukema (1996) was only an initial implementation of the basic principle. The parameters chosen may not necessarily be optimal for obtaining a detection.

For example, although the number of non-topological isometries of n-tuplets is higher when n is lower, it may well be possible that the signal (due to topological isometries) may increase faster than the noise (due to the number density distribution of the catalogue for a simply connected universe) when n is lowered, so that the signal-to-noise ratio would increase.

Another use of the technique using the same size data set would be to test a series of universe models with incrementally different metric parameters (e.g. $\Omega_0 = 0.20, 0.21, 0.22, \ldots, 1.10$), in particular, the negatively curved models. The increment should be chosen as a function of the uncertainty regarding quasars' (3D) positions.

Alternatively, one could allow for a scaling factor when comparing n-tuplets, in which case isometries due to any value of the curvature would automatically be detected. However, this would also increase the number of chance coincidences of n-tuplets, and is not strictly correct—it would distort the local geometrical relations if the n-tuplets are not small enough.

In the method of adopting successive values of the metric parameters, the successive signal-to-noise ratios (numbers of isometric n-tuplet pairs due to topology compared to numbers due to chance) are S_i/N_i where

$$S_i \begin{cases} = 0, & (i \not\approx i^*) \\ > 0, & (i \approx i^*) \end{cases} \tag{6}$$

and $(\Omega_0, \lambda_0)_{i^*}$ are the metric parameters of the real Universe. The signal-to-noise ratio S_i/N_i is then zero except when the metric parameters are close to correct, i.e. $i \approx i^*$, and has a maximum of S_{i^*}/N_{i^*}. (The values of the metric parameters would then be quite tightly constrained.)

If a scaling factor is used instead, then the calculation should be much faster (since a single set of metric parameters is adopted), but the signal-to-noise ratio becomes

$$\begin{aligned} \frac{\sum_i S_i}{\sum_i N_i + \sum_{i \neq j} N_{ij}} &= \frac{\sum_{i \approx i^*} S_i}{\sum_i N_i + \sum_{i \neq j} N_{ij}} \\ &\approx \frac{S_{i^*}}{\sum_i N_i + \sum_{i \neq j} N_{ij}} \\ &\ll S_{i^*}/N_{i^*}, \end{aligned} \tag{7}$$

where N_{ij} are numbers of chance coincidences for scaling factors which imply inconsistent curvature estimates. So, the signal-to-noise ratio is much lower if scaling is used. Even if the scaling is done requiring consistent values of the curvature between n-tuplets at different redshifts (which would probably slow down the calculation), the signal-to-noise ratio would still be lower than for the successive metric value method. Additionally, distortions due to the greater than zero size of n-tuplets would make S_{i^*} in the scaling method slightly lower than for the successive metric method.

The other development of this method is that since quasars are visible to about $R_H/2$, future quasar surveys will provide a more thorough sampling of this volume, so that the 'rare' cases searched for will increase in number and the chances of detection (if the topology is detectable) will increase.

5. Conclusion

Several methods have been developed in the last few years to either detect or constrain the topology of the spatial part of the Universe. The relative efficiency (in terms of fundamental polyhedron crossings) of 3D to 2D methods depends moderately on the precise values of the metric parameters Ω_0, λ_0. Objects seen to about $z \sim 3$ would cross half the horizon distance for any presently accepted metric parameters, and in a cosmological constant dominated universe, objects seen to $z \sim 0.1$–0.5 would cover many fewer copies of the fundamental polyhedron than would the CMB.

Initial applications of 3D methods to existing observational catalogues or individual observations indicate several (weak) candidates for the 3-manifold in which we live (or more precisely, for some of the generators of the 3-manifold). These candidates are falsifiable with moderate observational investment in telescope time.

Moreover, further development of the local isometry search method is presently possible for application to existing observational quasar catalogues.

Looking to the future, new catalogues of objects made over the next few years, in particular all-sky surveys of quasars, will possibly allow the topology of the Universe to be detected to a high significance by the local isometry search method. Alternatively, the 'circles method' of Cornish *et al* (1998b) applied to the observations by either MAP or Planck (planned CMB satellites) is likely to either reveal or constrain the topology of the Universe.

Within a decade, we should know whether or not the topology of the Universe is detectable, and if so what it is.

Acknowledgments

This research has been partially supported by the Polish Council for Scientific Research grant KBN 2 P03D 008 13.

References

Barrow J D, Juskiewicz R and Sonoda D H 1985 *Mon. Not. R. Astron. Soc.* **213** 917
Bond J R, Pogosyan D and Souradeep T 1998 *Class. Quantum Grav.* **15** 2671
Cornish N J, Spergel D N and Starkman G D 1996 *Phys. Rev. Lett.* **77** 215
——1998a *Phys. Rev.* D **57** 5982
——1998b *Class. Quantum Grav.* **15** 2657
da Costa L N *Cosmic Velocity Fields* ed F Bouchet and M Lachièze-Rey (Gif-sur-Yvette: Editions Frontières) p 475
de Lapparent V, Geller M J and Huchra J P 1986 *Astrophys. J.* **302** L1
de Oliveira Costa A and Smoot G F 1995 *Astrophys. J.* **448** 477
de Sitter W 1917 *Mon. Not. R. Astron. Soc.* **78** 3
Deng X-F, Deng Z-G and Xia X-Y 1996 *Chin. Astron. Astrophys.* **20** 383
e Costa S S and Fagundes H V 1998 *Preprint* gr-qc/9801066
Einasto J *et al* 1997 *Nature* **385** 139
Fagundes H V 1985 *Astrophys. J.* **291** 450
——1989 *Astrophys. J.* **338** 618
——1996 *Astrophys. J.* **470** 43
Fagundes H V and Gausmann E 1997 *Phys. Lett.* A **238** 235
Fagundes H V and Wichoski U F 1987 *Astrophys. J.* **322** L5
Farrar K and Melott A L 1990 *Comput. Phys.* **4** 185
Geller M J and Huchra J P 1989 *Science* **246** 897
Gott J R I 1980 *Mon. Not. R. Astron. Soc.* **193** 153
Jing Y-P and Fang L-Z 1994 *Phys. Rev. Lett.* **73** 1882
Lachièze-Rey M and Luminet J-P 1995 *Phys. Rep.* **254** 136
Lehoucq R, Luminet J-P and Lachièze-Rey M 1996 *Astron. Astrophys.* **313** 339
Lemaître G 1958 *La Structure et l'Evolution de l'Univers, Onzième Conseil de Physique Solvay* ed R Stoops (Brussels: Stoops) p 1
Levin J, Scannapieco E and Silk J 1998 *Preprint* astro-ph/9802021
Luminet J-P 1998 Conceptions de l'espace en physique, Ecole des Houches *Acta Cosmologica* (gr-qc/9804006)
Peebles P J E 1993 *Principles of Physical Cosmology* (Princeton, NJ: Princeton University Press)

Roukema B F 1996 *Mon. Not. R. Astron. Soc.* **283** 1147
Roukema B F and Edge A C 1997 *Mon. Not. R. Astron. Soc.* **292** 105
Shaver P A, Hook I M, Jackson C A, Wall J V and Kellermann K I 1998 *Preprint* astro-ph/9801211
Starobinsky A A 1993 *JETP Lett.* **57** 622
Stevens D, Scott D and Silk J 1993 *Phys. Rev. Lett.* **71** 20
Thurston W P 1982 *Bull. Am. Math. Soc.* **6** 357
——1997 *Three-Dimensional Geometry and Topology* ed S Levy (Princeton, NJ: Princeton University Press)
Weinberg S 1972 *Gravitation and Cosmology* (New York: Wiley)