Fibered knots in low and high dimensions

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Abstract. This paper is a survey of cobordism theory for knots. We first recall the classical results of knot cobordism, and next we give some recent results.

We classify fibered knot up to cobordism in all dimensions, and we give several examples of fibered knots which are cobordant and non isotopic. Some related results about surfaces embedded in S^4 are given.

"... the theory of "Cobordisme" which has,
within the few years of its existence,
led to the most penetrating insights into
the topology of differentiable manifolds."
H. Hopf,
International Congress of Mathematics, 1958.

1. Introduction

1.1. Historic. In the sixties, R. Fox and J. Milnor [**F-M**] were the first to study *cobordism* of embeddings of S^1 in S^3. Next, M. Kervaire [**K2**] and J. Levine [**L2**] studied embeddings of $(2n-1)$-spheres into codimension two spheres, and gave a classification of these embeddings up to cobordism. Moreover, M. Kervaire defined a group structure on the sets C_{2n-1}, of cobordism classes of $(2n-1)$-spheres embedded in S^{2n+1}, and $\widetilde{C}_{2n-1}$, of concordance classes of $(2n-1)$-spheres embedded in S^{2n+1}.

Remark that spherical knots were only studied as codimension two embeddings into spheres, because in the P.L. category E. Zeeman [**Ze**] proved that all spherical knots in codimension greater or equal to three are unknoted, and J. Stallings [**Sta**] proved that it is also true in the topological category as soon as the knots are of dimension greater or equal to two. In the smooth category A. Heafliger [**Ha1**] proved that cobordism of spherical knots in codimension greater or equal to three implies isotopy.

After J. Milnor's work [**M4**] on isolated singularities of complex hypersurfaces, the notion of *fibered knot* became the right topological framework which correspond to algebraic knots. Then the topology of isolated singularities of complex hypersurfaces appears as a motivation to study embeddings of general manifolds into codimension two spheres. In the beginning of the seventies, D.T. Lê [**Lê1**] proved that isotopy and cobordism are equivalent for one dimensional algebraic

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knots. About twenty years later, P. Du Bois and F. Michel [**DB-M**] gave the first examples of cobordant algebraic spherical knots which are not isotopic. Then the classification of fibered knots up to cobordism became an open problem.

1.2. Contents. This survey is organized as follows. In section **2** we give definitions and the classifications of spherical knots and fibered knots of high dimension up to cobordism. The classification of three dimensional knots up to cobordism is given in section **3**. In section **4** we give examples of cobordant knots and explain the pull back relation for knots. The results for even dimensional knots are given in section **5**, in which we explain recent results about embedded surfaces in S^4, and four dimensional manifolds embedded in S^6.

Throughout the paper, we shall work in the smooth category. All the homology and cohomology groups are understood to be with integer coefficients unless otherwise specified.

2. Definitions and classical results

Since we want to study cobordism of codimension two embeddings of more general manifolds than spheres, we define

DEFINITION 2.1. *A closed $(n-2)$-connected oriented $(2n-1)$-dimensional manifold embedded in the $(2n+1)$-sphere S^{2n+1} is called a* $(2n-1)$-knot*, or simply a knot. A $(2n-1)$-knot is* spherical*, if the embedded manifold is abstractly homeomorphic to S^{2n-1}.*

As said before this definition is motivated by the study of the topology of isolated singularities of complex hypersurfaces.

More precisely, let $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ be a holomorphic germ with an isolated singularity at the origin. The orientation preserving homeomorphism class of the pair $(D_\varepsilon^{2n+2}, f^{-1}(0) \cap D_\varepsilon^{2n+2})$ does not depend on the choice of ε small, it is the *topological type* of f. The diffeomorphism class of the oriented pair $(S_\varepsilon^{2n+1}, K_f)$ where $K_f = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is the *algebraic knot* associated to f.

In [**M4**], J. Milnor proved that algebraic knots associated to isolated singularities of germs $f : \mathbf{C}^{n+1}, 0 \to \mathbf{C}, 0$ are some $(2n-1)$-dimensional closed, oriented and $(n-2)$-connected submanifolds of the sphere S^{2n+1}. Moreover the complementary of an algebraic knot K_f in the sphere S^{2n+1} admits a fibration over the one dimensional sphere S^1, and the closure of the fibers are some $2n$-dimensional oriented and closed submanifolds of S^{2n+1} which admits K_f as boundary. This motivates the following definition

DEFINITION 2.2. *We say that a $(2n-1)$-knot K is* fibered*, if there exists a fibration $\phi : S^{2n+1} \smallsetminus K \to S^1$ with ϕ being trivial on $U \smallsetminus K$, where U is a small open tubular neighborhood of K, such that the closure of each fiber is a manifold which has K as boundary.*

For high dimensional fibered knots isotopy classes are well known since we have the following theorem

THEOREM 2.3. [**D1**] *Let $n \geq 3$, two $(2n-1)$-dimensional fibered knots are isotopic if and only if they have isomorphic Seifert forms.*

We will focus on the cobordism classes of knots.

Definition 2.4. *Two* $(2n-1)$*-knots* K_0 *and* K_1*, embedded in* S^{2n+1}*, are* cobordant *if there exists a proper submanifold* X *of* $S^{2n+1} \times [0,1]$ *such that*

(1) X *is diffeomorphic to* $K_0 \times [0,1]$*,*
(2) $\partial(X) = (K_0 \times \{0\}) \cup (K_1 \times \{1\})$*.*

When all the manifolds are oriented, we say that K_0 *and* K_1 *are* oriented cobordant *if they are cobordant and* $\partial(X) = (-K_0 \times \{0\}) \cup (K_1 \times \{1\})$*, where* $-K_0$ *is the manifold* K_0 *with the reversed orientation.*

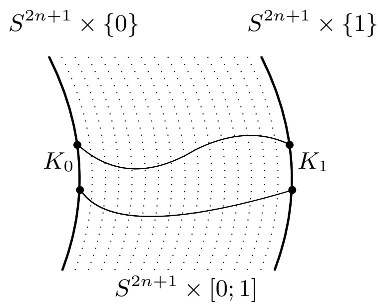

Figure 1: *A cobordism between* K_0 *and* K_1

It is clear that isotopic knots are cobordant, but the converse is not true in general.

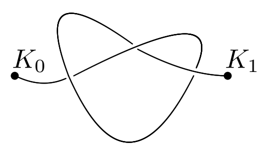

Figure 2: *A cobordism which is not an isotopy*

We also introduce the notion of *concordance* defined for embeddings

Definition 2.5. *We say that two embeddings* $f_i : K \to S^{2n+1}$*, for* $i = 0, 1$*, of a closed* $(2n-1)$*-dimensional and* $(n-2)$*-connected manifold* K*, are* concordant *if there exists a proper embedding* $\Phi : K \times [0,1] \to S^{2n+1} \times [0,1]$ *such that* $\Phi_{|K \times \{i\}} = f_i$ *for* $i = 0, 1$*.*

We say that two embeddings $f_i : K \to S^{2n+1}$*, for* $i = 0, 1$*, of a closed* $(2n-1)$*-dimensional and* $(n-2)$*-connected manifold* K *are* cobordant *if there exists a diffeomorphism* $h : K \to K$ *such that* $f_0 \circ h$ *and* f_1 *are concordant.*

This definition can be extended to embeddings of S^n into S^{n+2}. Let C_n be the set of cobordism classes of n-spheres embedded in S^{n+2}, and $\widetilde{C}_n$ be the set of concordance classes of n-spheres embedded in S^{n+2}. In [**K1**] M. Kervaire showed that the natural surjection $\mathfrak{i} : \widetilde{C}_n \to C_n$ is a group homorphism.

Let us denote by $\mathfrak{O}_n$ the group of h-cobordism classes of differential homotopy n-spheres, and by $b\mathfrak{P}_{n+1}$ the subgroup of $\mathfrak{O}_n$ of h-cobordism classes represented by differential homotopy n-spheres which are boundary of parallelizable manifolds. Then we have the following Theorem

THEOREM 2.6. [**K1**] *For $n \leq 5$ we have $\widetilde{C}_n \cong C_n$, and for $n > 6$ we have the following exact sequence* $0 \to \mathfrak{O}_{\mathfrak{n}}/b\mathfrak{P}_{n+1} \to \widetilde{C}_n \xrightarrow{\mathfrak{i}} C_n \to 0$

In the following we will study cobordism classes of knots. We will give classifications of knots up to cobordism using *Seifert forms*. First, recall that for every $(2n-1)$-knots K, there exists a compact oriented $2n$-dimensional submanifold F of S^{2n+1} having K as boundary. Such a manifold F is called a *Seifert manifold* for K. See [**R1**] for a construction in the case one dimensional knots, for higher dimensional knots it is a classical result which comes from obstruction theory. For fibered knots the closure of a fiber is a Seifert manifold associated with the knot.

DEFINITION 2.7. *Suppose that F is a compact oriented $2n$-dimensional submanifold of S^{2n+1}, and let G be the quotient of $H_n(F)$ by its $\mathbf{Z}$-torsion. The* Seifert form *associated with F is the bilinear form $A : G \times G \to \mathbf{Z}$ defined as follows. For $(x, y) \in G \times G$, we define $A(x, y)$ to be the linking number in S^{2n+1} of ξ and η_+, where ξ and η are n-cycles in F representing x and y respectively, and η_+ is the n-cycle η pushed off F into the positive normal direction to F in S^{2n+1}.*

By definition a Seifert form for a $(2n-1)$-knot K is the Seifert form associated with a Seifert manifold for K.

Let us consider the case of the trefoil knot. The Seifert manifold F considered for this knot is a surface such that the rank of the first homology group is two. If we denote by η and ξ the 1-cycles which represent the generators of $\mathrm{H}_1(F)$ then the Seifert matrix can be computed as follows.

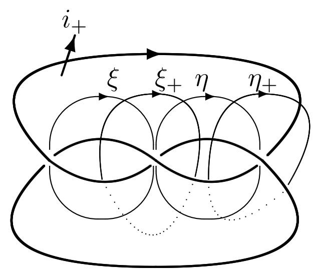

FIGURE 3: *Computation of the seifert matrix for the trefoil knot*

According to the choices of orientation made to draw cycles in Figure 3, the Seifert matrix for the trefoil knot is then $A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$. One can remark that the Seifert matrix is not usually symmetrical. But the matrix $S = A + (-1)^n A^T$ is the matrix of the usual intersection form for F, where F is the Seifert manifold of the $(2n-1)$-knot $K = \partial F$.

When a knot is fibered, we shall often call the closure of each fiber simply a *fiber*. The Seifert form associated with a fiber is always unimodular because of Alexander's duality (see [**Kau**]). In the following, for a fibered $(2n-1)$-knot, we use the Seifert form associated with a fiber unless otherwise specified.

Definition 2.8. *A $(2n-1)$-knot is* simple*, if it admits an $(n-1)$-connected Seifert manifold. Furthermore, a fibered $(2n-1)$-knot is* simple*, if its fiber is $(n-1)$-connected.*

As a consequence of J. Milnor's results, see [**M4**], we have that algebraic knots are not only fibered but they are simple as well. Remark that when a $(2n-1)$-knot is simple the group $\mathrm{H}_n(F)$ is torsion free because F is a $(n-1)$-connected Seifert manifold.

Recall that for a $(2n-1)$-knot K with a seifert manifold F, when $n \geq 3$ we have the following exact sequence

$$(*)\quad 0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F,K) \to \mathrm{H}_{n-1}(K) \to 0$$

the homomorphism S_* is induced by the inclusion. But if we denote by $\widetilde{\mathfrak{P}}$ the isomorphism $\mathrm{H}_n(F,K) \overset{\widetilde{\mathfrak{P}}}{\cong} \mathrm{Hom}_{\mathbf{Z}}(\mathrm{H}_n(F);\mathbf{Z})$ given by the Poincaré duality and the universal coefficient isomorphism, then $S_* : \mathrm{H}_n(F) \to \mathrm{H}_n(F,K)$, and S^* the adjoint of $S = A + (-1)^n A^T$, are related together by $S^* = S_* \circ \widetilde{\mathfrak{P}}$.

Since cobordant knots are diffeomorphic, to have a cobordism we need topological information of the knot, i.e., the boundary of the Seifert manifold. Considering the previous exact sequence $(*)$ we will use the kernel and the cokernel of the homomorphism S^* to get these data.

It is clear that in the case of spherical knots, these considerations are not necessary since S_* and S^* are isomorphisms. First we will concentrate on the case of spherical knots.

2.1. Spherical knots. The main tool to study cobordism of spherical knots is the Seifert form associated to a Seifert manifold of a knot.

First, in [**L3**] J. Levine described modifications on the Seifert form of a spherical simple knot after changing Seifert manifold.

Following J. Levine, an *enlargement* of a square matrix A to a matrix A' is defined as follows

$$A' = \begin{pmatrix} A & \mathcal{O}_1 & \mathcal{O}_2 \\ \alpha & 0 & 0 \\ \mathcal{O}_3 & 1 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} A & \beta & \mathcal{O}_1 \\ \mathcal{O}_2 & 0 & 1 \\ \mathcal{O}_3 & 0 & 0 \end{pmatrix}$$

where $\mathcal{O}_i$ are row or column vectors of 0 for $i = 1,2,3$, and α is a column vector of scalars, and β is a row vector of scalars.

Then A is a *reduction* of A'.

Two square matrices are *S-equivalent* if they are related by enlargement and reduction operations. The equivalence classes defined are called *S-equivalence classes.* J. Levine proved

Theorem 2.9. [**L3**] *Isotopic spherical knots have S-equivalent Seifert forms.*

Remark 2.10. *One can generalize this result to non spherical knots (see* [**D1**]*).*

M. Kervaire showed that there exists a group structure on the set of cobordism classes of spherical knots. The connected sum is the operation and the inverse of a spherical knot is given by its mirror image. Then two spherical knots are cobordant if their connected sum is cobordant to the trivial knot. We say that a $(2n-1)$-knot $K \subset S^{2n+1}$ is *null-cobordant* when it is cobordant to the trivial knot, i.e., when there exists a $2n$-ball D embedded into the $(2n+2)$-ball B^{2n+2} such that $\partial D = K$ and $\partial B^{2n+2} = S^{2n+1}$.

In the case of spherical knots M. Kevaire and J. Levine used the following equivalence relation on the set $\mathcal{A}$ of integral bilinear forms defined on free $\mathbf{Z}$-modules of finite rank

DEFINITION 2.11. *Let $A : G \times G \to \mathbf{Z}$ be a bilinear form in $\mathcal{A}$. The form A is Witt associated to* 0*, if the rank m of G is even and there exists a pure submodule M of rank $m/2$ in G such that A vanishes on M. Such a submodule M is called a* metabolizer *for A.*

In [**K2**] and [**L2**] one can find proofs of the following Theorem

THEOREM 2.12. [**K2, L2**] *For $n \geq 3$, a spherical $(2n-1)$-knot is null-cobordant if and only if its Seifert form is Witt associated to* 0.

The hypothesis $n \geq 3$ is a technical condition which comes from the proof of the sufficiency. This is because in order to construct an embedded $2n$-disk with K as boundary it is necessary to be able to do embedded surgeries a $(2n+2)$-ball on a Seifert manifold associated to K. When $n = 1, 2$ this is not possible in general.

For two spherical knots K_0 and K_1 with Seifert forms A_0 and A_1 respectively, the oriented connected sum $K = K_0 \sharp (-K_1)$ has $A = A_0 \oplus -A_1$ as Seifert form asociated with the oriented connected sum along the boundaries of the Seifert manifold associated to K_0 and $-K_1$. Hence, as a consequence of Theorm 2.12 we have that the two spherical knots K_0 and K_1 are cobordant if and only if the Seifert form $A = A_0 \oplus -A_1$ is Witt-associated to 0, then we sometimes say that A_0 and A_1 are *Witt equivalent*.

2.2. Cobordism of fibered knots. In this section we will give the classification of fibered knots up to cobordism. This cannot be done by a direct generalization of the results proved by M. Kervaire and J. Levine. We will have to consider the topological data contained in the kernel and the cokernel of the intersection form (see Equation $(*)$).

First, we will explain why we restrict our study to fibered knots. Let K be a $(2n-1)$ dimensional knot with ≥ 1 with a Seifert manifold denoted by F, then we have the following long exact sequence

$$\ldots \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to \ldots$$

where maps are induced by inclusions.

As we can guess after Theorem 2.12, the existence of a metabolizer will be necessary to have knot cobordism. But More precisely, we will need to control the contribution to the metabolizer given by the cycles coming from the homology of the boundary of the fibers. Hence in the long exact sequence above we must have the following short exact sequence

$$0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F, K) \to \mathrm{H}_{n-1}(K) \to 0.$$

One possibility is to only study knots if they have a Seifert manifold for which we have this short exact sequence, for instance simple knots, and use only these Seifert manifolds.

But in the case of fibered knots this condition is automatic.

Proposition 2.13. *Let $n \geq 1$. Let K be a fiber knot of dimension $2n-1$ and let F be a Seifert manifold associated to the fibration, then we have the following short exact sequence*

$$0 \to H_n(K) \to H_n(F) \xrightarrow{S_*} H_n(F,K) \to H_{n-1}(K) \to 0.$$

Proof. Let F be a Seifert manifold given by the fibration. Then $S^{2n+1} \setminus F$ has the same homotopy type as F. Now by Alexander duality we get the following isomorphisms $\mathrm{H}_k(F) \cong \mathrm{H}^{2n-k}(S^{2n+1} \setminus F) \cong \mathrm{H}^{2n-k}(F)$, for $k > 0$.

Moreover by Poincaré duality we have $\mathrm{H}_k(F,K) \cong \mathrm{H}^{2n-k}(F)$, and this implies

$$(\star) \quad \mathrm{H}_k(F,K) \cong \mathrm{H}_k(F).$$

Now consider the long exact sequence

$$\ldots \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F,K) \to \ldots$$

since K is $(n-2)$-connected then we have the following short exact sequence

$$0 \to \mathrm{H}_{n+1}(F) \xrightarrow{\alpha} \mathrm{H}_{n+1}(F,K) \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \to$$

$$\mathrm{H}_n(F,K) \to \mathrm{H}_{n-1}(K) \to \mathrm{H}_{n-1}(F) \xrightarrow{\beta} \mathrm{H}_{n-1}(F,K) \to 0$$

Since α is a monomorphism with $(\star)$ it is an isomorphism, and since β is an epimorphism with $(\star)$ it is an isomorphism. Finally we get the desired short exact sequence

$$0 \to \mathrm{H}_n(K) \to \mathrm{H}_n(F) \xrightarrow{S_*} \mathrm{H}_n(F,K) \to \mathrm{H}_{n-1}(K) \to 0$$

□

Then for high dimensional fibered knots we have a complete characterization of cobordism classes using their Seifert forms as follows.

Theorem 2.14. [**B3**] *For $n \geq 3$, two fibered $(2n-1)$-knots are cobordant if and only if their Seifert forms are algebraically cobordant.*

The condition on the integer n is only used for the sufficiency, and we have the following theorem

Theorem 2.15. [**B3**] *Two cobordant fibered knots have algebraically cobordant Seifert forms.*

To define the algebraic cobordism we first need to fix some notations and definitions. Let $\mathcal{A}$ be the set of all bilinear forms defined on free $\mathbf{Z}$-modules G of finite rank. Set $\varepsilon = (-1)^n$. For $A \in \mathcal{A}$, let us denote by A^T the transpose of A, by S the ε-symmetric form $A + \varepsilon A^T$ associated with A, by $S^* : G \to G^*$ the adjoint of S with G^* being the dual $\mathrm{Hom}_{\mathbf{Z}}(G, \mathbf{Z})$ of G, and by $\overline{S} : \overline{G} \times \overline{G} \to \mathbf{Z}$ the ε-symmetric non degenerate form induced by S on $\overline{G} = G/\mathrm{Ker}\, S^*$. A submodule M of G is said to be *pure*, if G/M is torsion free, or equivalently if M is a direct summand of G. For a submodule M of G, let us denote by $M^\wedge$ the smallest pure submodule of G which contains M. We denote by $\overline{M}$ the image of M in $\overline{G}$ by the natural projection map.

Definition 2.16. [**B3**] *Let* $A_i : G_i \times G_i \to \mathbf{Z}$, $i = 0, 1$, *be two bilinear forms in* $\mathcal{A}$. *Set* $G = G_0 \oplus G_1$ *and* $A = A_0 \oplus -A_1$. *The form* A_0 *is said to be* algebraically cobordant *to* A_1, *if there exists a metabolizer* M *for* A *such that* $\overline{M}$ *is pure in* $\overline{G}$, *an isomorphism* $\psi : \operatorname{Ker} S_0^* \to \operatorname{Ker} S_1^*$, *and an isomorphism* $\theta : \operatorname{Tors}(\operatorname{Coker} S_0^*) \to \operatorname{Tors}(\operatorname{Coker} S_1^*)$ *which satisfy the following two conditions:*

(c1) $$M \cap \operatorname{Ker} S^* = \{(x, \psi(x)); x \in \operatorname{Ker} S_0^*\} \subset \operatorname{Ker} S_0^* \oplus \operatorname{Ker} S_1^* = \operatorname{Ker} S^*,$$

(c2) $$\begin{aligned} d(S^*(M)^\wedge) &= \{(y, \theta(y)); y \in \operatorname{Tors}(\operatorname{Coker} S_0^*)\} \\ &\subset \operatorname{Tors}(\operatorname{Coker} S_0^*) \oplus \operatorname{Tors}(\operatorname{Coker} S_1^*) = \operatorname{Tors}(\operatorname{Coker} S^*), \end{aligned}$$

where d *is the quotient map* $G^* \to \operatorname{Coker} S^*$ *and "*Tors*" means the torsion subgroup. In the above situation, we also say that* A_0 *and* A_1 *are* algebraically cobordant with respect to ψ and θ.

Note that the algebraic cobordism is an equivalence relation, see [**B-M**, Theorem 1].

Theorem 2.17. [**B-M**] *Algebraic cobordism is an equivalence relation on the set* $\mathcal{A}$.

If A_i are Seifert forms associated with $(n-1)$-connected Seifert manifolds F_i of *simple* $(2n-1)$-knots K_i, $i = 0, 1$, then S_{i_*} is the intersection form of F. Hence $\operatorname{Ker} S_{i_*}$ and $\operatorname{Coker} S_{i_*}$ are naturally identified with $H_n(K_i)$ and $H_{n-1}(K_i)$ respectively, since we have the exact sequence

$$\begin{aligned} 0 = H_{n+1}(F_i, K_i) \to H_n(K_i) \to H_n(F_i) \xrightarrow{S_{i_*}} H_n(F_i, K_i) \\ \to H_{n-1}(K_i) \to H_{n-1}(F_i) = 0, \end{aligned}$$

where we identify $H_n(F_i, K_i)$ with the dual of $H_n(F_i)$, see $(*)$.

As remarked before, in the case of a spherical knot K we have $H_n(K) = H_{n-1}(K) = 0$, so the intersection form is an isomorphism. Hence the algebraic cobordism of Seifert forms associated to spherical knots is reduced to Witt equivalence, and Theorem 2.14 gives the classification of M. Kervaire and J. Levine.

Let $K_0 = \partial F_0$ and $K_1 = \partial F_1$ be two $(2n-1)$-knots with $n \geq 3$. Denote by A_0 and A_1 the Seifert forms associated with F_0 and F_1 respectively.

To prove Theorem 2.14 (see [**B3**]), we first suppose that K_0 and K_1 are cobordant and then construct a metabolizer for $A_0 \oplus -A_1$ which fulfills the definition of algebraic cobordism to have the necessity.

For sufficiency we suppose that A_0 and A_1 are algebraically cobordant. Then we prove that we can do embedded surgeries, in a $(2n+2)$-ball, on the connected sum of Seifert manifolds associated to the knots. The result of these surgeries is a submanifold W of B^{2n+2} such that $\partial W = K_0 \coprod K_1$ and $\mathrm{H}_*(W; K_i) = 0$ for $i = 0; 1$. Then with h-cobordism theorem $W \cong K_0 \times K_i$ gives the cobordism between K_0 and K_1. This is where we need to have high dimensional knots, because these technics are not valid for knots of dimensions one and three.

3. Three dimensional knots

In this section we will deal with three dimensional knots. This case is much more difficult than the case of high dimensional knots because the dimension of the Seifert manifolds is four. The topology of four dimensional manifolds is very particular, and the usual technics used in the case of higher dimensional manifolds are not available anymore.

The algebraic cobordism of Seifert forms is necessary for the cobordism of simple fibered knots in any dimension (see Theorem 2.15). But there exists some non isotopic three dimensional fibered knots which have diffeomorphic fibers and congruent Seifert matrices (see Example 3.1). Hence, in the case of three dimensional knots isotopy classes, and then cobordism classes must be characterized by new equivalence relations on Seifert forms. Isotopy classes were studied in [**S4**], for cobordism classes we will define a new equivalence relation for Seifert forms. To do that we will need Spin structures.

Recall that a *Spin structure* on a manifold X means the homotopy class of a trivialization of its tangent bundle over the 2-skeleton $X^{(2)}$. Note that X admits a spin structure if and only if its second Stiefel-Whitney class $w_2(X)$ vanishes and that if it admits, then the set of all spin structures on X is in one-to-one correspondence with $H^1(X;\mathbf{Z}_2)$.

Let K be a 3-knot, with Seifert manifold F, embedded in S^5. All the manifolds are oriented. Then K has a natural normal 2-framing $\nu = (\nu_1, \nu_1)$ in S^5 such that the first normal vector field ν_1 is obtained as the inward normal vector field of $K = \partial F$ in F. The homotopy class of this 2-framing does not depend on the choice of the Seifert manifold F. Then K carries a tangent 3-framing on its 2-skeleton $K^{(2)}$ such that the juxtaposition with the above 2-framing gives the standard framing of S^5 restricted to $K^{(2)}$ up to homotopy. This means that K carries a natural Spin structure, which is determined uniquely up to homotopy. Furthermore, this Spin structure coincides with that induced from the Seifert manifold F, which is endowed with the natural spin structure induced from S^5.

In the case of 3-knots the Spin structure must be considered as the following example shows.

Example 3.1. *Set $\mathcal{M} = S^1 \times \Sigma_g$, where Σ_g is the closed connected orientable surface of genus $g \geq 2$. Let K_0 and K_1 be the simple fibered $\mathcal{M}$-knots constructed in* [**S4**, Proposition 3.8]. *They have the property that their Seifert forms are isomorphic, but that there exists no diffeomorphism between K_0 and K_1 which preserves their Spin structures and they are not isotopic.*

In order to study cobordism of 3-knots we will use some results only valid for three dimensional manifolds without torsion on the first homology group, hence we define

Definition 3.2. [**B-S1**] *We say that a 3-knot K is free if $H_1(K)$ is torsion free over* $\mathbf{Z}$.

Moreover by considering free knots we do not need to consider the condition (c2) in the definition of the algebraic cobordism (see Definition 2.16), which is always a necessary condition for knot cobordism (see Theorem 2.15).

Definition 3.3. *Consider two simple 3-knots K_0 and K_1. Let A_0 and A_1 be the Seifert forms of K_0 and K_1 respectively with respect to* 1*-connected Seifert manifolds. We say that A_0 and A_1 are* Spin cobordant*, if there exists an orientation preserving diffeomorphism $h : K_0 \to K_1$ such that*

(1) *h preserves their spin structures,*

(2) *A_0 and A_1 are algebraically cobordant with respect to $h_* : H_2(K_0) \to H_2(K_1)$ and $h_*|\mathrm{Tors}\, H_1(K_0) : \mathrm{Tors}\, H_1(K_0) \to \mathrm{Tors}\, H_1(K_1)$, where we identify $H_2(K_i)$ and $H_1(K_i)$ with $\mathrm{Ker}\, S_i^*$ and $\mathrm{Coker}\, S_i^*$ respectively for $i = 0, 1$*

Note that if K_0 and K_1 are free 3-knots, then we do not need to consider the condition (c2) of Definition 2.16 and hence the isomorphism $h_*|\mathrm{Tors}\, H_1(K_0)$ in the above definition.

Spin cobordism of Seifert forms with respect to 1-connected Seifert manifolds is an equivalence relation on the set of pairs of a Seifert manifold of a simple 3-knot and its Seifert form.

Remark 3.4. *In Example 3.1 above, the Seifert forms of K_0 and K_1 are algebraically cobordant, but are not Spin cobordant. Hence they cannot be cobordant by Proposition 3.6. Example 3.1 shows that Spin structures are essential in the theory of cobordism for* 3*-knots.*

In [**B-S1**] we prove

Theorem 3.5. [**B-S1**] *Two simple fibered free* 3*-knots are cobordant if and only if their Seifert forms with respect to* 1*-connected fibers are Spin cobordant.*

Let us see that Spin cobordism is necessary to have knot cobordism. Let K_0 and K_1 be two cobordant free 3-knots with fibers F_0 and F_1 respectively. Denote by $\mathcal{C} \cong K_0 \times [0,1]$ the submanifold of $S^5 \times [0,1]$ which gives the cobordism between K_0 and K_1, and set $N = F_0 \cup \mathcal{C} \cup (-F_1)$. By classical obstruction theory the closed oriented 4-manifold $N \subset S^5 \times [0,1]$ is the boundary of a compact oriented 5-submanifold W of $S^5 \times [0,1]$. Using a normal 2-framing of C in $S^5 \times [0,1]$ induced from the inward normal vector field along $N = \partial W$ in W, we see that the diffeomorphism h between K_0 and K_1 induced by C preserves their Spin structures.

Moreover, in [**B-M**], it has been shown that the two forms A_0 and A_1, associated with cobordant fibered knots, are algebraically concordant with respect to $h_* : H_2(K_0) \to H_2(K_1)$ and $h_*|\mathrm{Tors}\, H_1(K_0) : \mathrm{Tors}\, H_1(K_0) \to \mathrm{Tors}\, H_1(K_1)$.

Finaly we get the following result, in which the knots are not necessary simple.

Proposition 3.6. [**B-S1**] *If two simple fibered* 3*-knots are cobordant, then their Seifert forms with respect to* 1*-connected fibers are Spin cobordant.*

Using the 5-dimensional stable h-cobordism theorem due to T. Lawson [**La**] and F. Quinn [**Q**], and S. Boyer's work [**Bo**] we also have the following Theorem, in which the 3-knots are simple and free, but may not be fibered.

Theorem 3.7. [**B-S1**] *Consider two simple free* 3*-knots. If their Seifert forms with respect to* 1*-connected Seifert manifolds are spin concordant, then the* 3*-knots are concordant.*

The proof of this Theorem is very technical and difficult, we refer to [**B-S1**] for details. Finally Proposition 3.6 and Theorem 3.7 imply Theorem 3.5.

4. Examples, and pull-back relation for knots

First we construct concordant, but not isotopic non spherical 3-knots.

Example 4.1. [**B-S1**]

Let us call a stabilizer *a simple fibered spherical* 3*-knot whose fiber* F *is diffeomorphic to* $(S^2 \times S^2)\sharp(S^2 \times S^2) \smallsetminus \operatorname{Int} D^4$*. Such a stabilizer does exist, see* [**S2**, §4]*. Moreover, we denote by* K_S *a stabilizer with Seifert matrix (see* [**S1**, p. 600] *or* [**S4**, §10]*)*

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$$

with respect to a basis of $H_2(F)$ *denoted by* a_1, a_2, a_3, a_4*.*

Since A *is not congruent to the zero form,* K_S *is a non-trivial 3-knot.*

Moreover, the submodule generated by a_1 *and* a_3 *is a metabolizer for* A*, and one can do embedded surgeries on the two cycles* a_1 *and* a_3*, represented by two embedded 2-spheres in* F*. The result of this embedded surgery in* D^6 *is a four dimensional disc* D *properly embedded in* D^6 *with* K_S *as boundary. Thus* K_S *is null-cobordant, i.e., cobordant to the trivial spherical 3-knot.*

Then consider any simple fibered 3-knot K *which is not spherical. The two simple fibered 3-knots* $K\sharp K_S$ *and* K *are not isotopic since the ranks of the second homology groups of their fibers are distinct. However, these knots are cobordant.*

In the following example we construct non spherical cobordant but not isotopic simple fibered knots of dimension $2n-1$ with $n \geq 3$. These knots are not related together by stabilization, i.e., by connected sum of a given knot with null cobordant spherical knots.

Example 4.2. [**B2**]

Let K_i*, with* $i = 0, 1$*, be the algebraic knots of dimension* $2n-1$*,* $n \geq 3$*, associated with the isolated singularity at* 0 *of the germs of holomorphic functions* $h_i : (\mathbf{C}^{n+1}, 0) \to (\mathbf{C}, 0)$ *defined by:*

$$h_i(x_0, \dots, x_n) = g_i(x_0, x_1) + x_2^p + x_3^q + \sum_{k=4}^{n} x_k^2$$

with

$$g_0(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{s+6} - 4x_1x_0^{(s+9)/2}\Big) \\ \Big((x_0^2 - x_1^5)^2 - x_1^{r+10} - 4x_0x_1^{(r+15)/2}\Big),$$

and

$$g_1(x_0, x_1) = (x_0 - x_1)\Big((x_1^2 - x_0^3)^2 - x_0^{r+14} - 4x_1x_0^{(r+17)/2}\Big) \\ \Big((x_0^2 - x_1^5)^2 - x_1^{s+2} - 4x_0x_1^{(s+7)/2}\Big).$$

where $s \geq 11$ *and* $s \neq r+8$ *are odd, and,* p *and* q *are distinct prime numbers which not divide the product* $\epsilon = 330(30+r)(22+s)$*, see [DB-M] p.166. We denote by* A_i*,* $i = 0, 1$ *the Seifert form associated to* K_i *defined on a free* $\mathbf{Z}$*-module of finite rank* H_i*.*

Let L be the algebraic knot of dimension $2n-1$ associated to the isolated singularity at 0 defined by the germ:

$$\begin{array}{rccc} f: & (\mathbf{C}^{n+1}, 0) & \to & (\mathbf{C}, 0) \\ & (x_0, \ldots, x_n) & \mapsto & \displaystyle\sum_{k=0}^{n} x_k^2 \end{array}$$

according to [D] prop. 2.2 p.50 this algebraic knot has $A = \Big((-1)^{n(n+1)/2}\Big)$, defined on a free $\mathbf{Z}$-module of rank one G, as Seifert matrix.

We construct L_i the connected sum of L and K_i for $i = 0, 1$. The Seifert form for L_i is the integral bilinear form $A \oplus A_i$ defined on a free $\mathbf{Z}$-module $G_i = G \oplus H_i$ of finite rank. The links L_i are simple fibered since $A \oplus A_i$ is unimodular (see [K-W] chap. V §3 p. 118).

The knots L_0 and L_1 are cobordant by Theorem 2.14, but they cannot be isotopic since K_0 and K_1 are not isotopic.

According to [A], th. 4 p. 117, the knots L_0 and L_1, which are the connected sum of two algebraic links, are not algebraic.

As in Example 4.1, stabilization is a natural way to construct high dimensional cobordant but non isotopic knots. For that, start with a knot K and do the connected sum $K \sharp K_S$ where K_S is null-cobordant spherical knot. Then K and $K \sharp K_S$ are cobordant but not isotopic.

Hence, for non spherical knots, the following question naturally arises.

($\mathcal{Q}$) ***If two non-spherical knots are simple homotopy equivalent as abstract manifolds, then are they cobordant after taking connected sums with some spherical knots?***

According to the codimension two surgery theory [**Ma1**], this is true provided that the relevant knots satisfy some connectivity conditions and that one of them is obtained as the inverse image of the other one by a certain degree one map between the ambient spheres. We define

Definition 4.3. [**B-M-S**] *Let K_0 and K_1 be m-knots in S^{m+2}. We say that K_0 is a* pull back *of K_1 if there exists a degree one smooth map $g : S^{m+2} \to S^{m+2}$ with the following properties:*

(1) *g is transverse to K_1,*
(2) *$g^{-1}(K_1) = K_0$,*
(3) *$g|_{K_0} : K_0 \to K_1$ is an orientation preserving simple homotopy equivalence.*

In this case, we write $K_0 \succ K_1$.

The following are some direct consequences of the previous definition.

(1) $K \succ K$ for any m-knot K.
(2) $K_0 \succ K_1$ and $K_1 \succ K_2$ imply $K_0 \succ K_2$ for any m-knots K_0, K_1 and K_2.
(3) $K_0 \succ K_1$ and $K_0' \succ K_1'$ imply $K_0 \sharp K_0' \succ K_1 \sharp K_1'$ for any m-knots K_0, K_0', K_1 and K_1'.

Furthermore, if we restrict ourselves to spherical m-knots, then it is not difficult to see that the *trivial m-knot K_U* is the minimal element, i.e., $K \succ K_U$ for every spherical m-knot K, where K_U is defined to be the isotopy class of the boundary of an $(m+1)$-dimensional disk embedded in S^{m+2}.

Here are some basic results on the pull back relation for fibered knots.

Theorem 4.4. [**B-M-S**] *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with fibers F_0 and F_1 respectively, where $n \geq 3$. Suppose* $\operatorname{rank} H_n(F_0) = \operatorname{rank} H_n(F_1)$. *If $K_0 \succ K_1$, then K_0 and K_1 are orientation preservingly isotopic.*

Corollary 4.5. [**B-M-S**] *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. If $K_0 \succ K_1$ and $K_1 \succ K_0$, then K_0 is orientation preservingly isotopic to K_1. In other words, the relation "$\succ$" defines a partial order for simple fibered $(2n-1)$-knots in S^{2n+1} for $n \geq 3$.*

Theorem 4.6. [**B-M-S**] *Let K_0 and K_1 be simple fibered $(2n-1)$-knots in S^{2n+1} with $n \geq 3$. Then $K_0 \succ K_1$ if and only if there exists a spherical simple fibered $(2n-1)$-knot Σ in S^{2n+1} such that K_0 is orientation preservingly isotopic to the connected sum $K_1 \sharp \Sigma$.*

Let K_0 and K_1 be two simple fibered $(2n-1)$-knot, with $n \geq 3$. If K_0 is pull back equivalent to K_1, then they are cobordant after taking connected sums with some spherical knots. With the following Proposition, we show that the converse is not true in general.

Proposition 4.7. [**B-M-S**] *For every odd integer $n \geq 3$, there exists a pair (K_0, K_1) of simple fibered $(2n-1)$-knots with the following properties.*

(1) *The knots K_0 and K_1 are cobordant.*
(2) *The knots K_0 and K_1 are not pull back equivalent.*

Proof. Let us consider the following two matrices:

$$L_0 = \begin{pmatrix} 9 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 25 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that they are both unimodular and that

$$S_0 = L_0 - {}^tL_0 = S_1 = L_1 - {}^tL_1 = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}.$$

Let us show that L_0 and L_1 are algebraically cobordant in the sense of [**B-M**, (1.2) Definition] for $\varepsilon = (-1)^n = -1$.

Set $m = {}^t(5,0,3,0)$ and $m' = {}^t(0,3,0,5)$. Then it is easy to see that the submodule M of $\mathbf{Z}^4$ generated by m and m' constitutes a metabolizer for $L = L_0 \oplus (-L_1)$. Furthermore, M is pure in $\mathbf{Z}^4$: in other words, M is a direct summand of $\mathbf{Z}^4$. Since $S_0 = S_1$ are non-degenerate, we have only to verify the condition c.2 of [**B-M**, (1.2) Definition].

Set $S = S_0 \oplus (-S_1) = L - {}^tL$. Let $S^* : \mathbf{Z}^4 \to \mathbf{Z}^4$, $S_0^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ and $S_1^* : \mathbf{Z}^2 \to \mathbf{Z}^2$ be the adjoints of S, S_0 and S_1 respectively. It is easy to see that $\operatorname{Coker} S_0^* = \operatorname{Coker} S_1^*$ is naturally identified with $\mathbf{Z}_2 \oplus \mathbf{Z}_2$. Furthermore, we have

$$S^*(m) = {}^tmS = (0,10,0,-6) \quad \text{and} \quad S^*(m') = {}^tm'S = (-6,0,10,0).$$

Therefore, $S^*(M)^\wedge$, the smallest direct summand of $\mathbf{Z}^4$ containing $S^*(M)$, is the submodule of $\mathbf{Z}^4$ generated by $(0,5,0,-3)$ and $(-3,0,5,0)$. Hence, for the natural quotient map $d : \mathbf{Z}^4 \to \operatorname{Coker} S^* = (\mathbf{Z}_2 \oplus \mathbf{Z}_2) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_2)$, we have

$$d(S^*(M)^\wedge) = \{(x,x) \,:\, x \in \operatorname{Coker} S_0^* = \mathbf{Z}_2 \oplus \mathbf{Z}_2\},$$

since $\operatorname{Im} S_i^*$ is generated by $(2,0)$ and $(0,2)$, $i = 0,1$, and $\operatorname{Im} S^*$ is generated by $(2,0,0,0)$, $(0,2,0,0)$, $(0,0,2,0)$ and $(0,0,0,2)$. Therefore, we conclude that the unimodular matrices L_0 and L_1 are algebraically cobordant.

Now, there exists a simple fibered $(2n-1)$-knot K_i which realizes L_i as its Seifert form with respect to the fiber, $i = 0,1$ (see [**D1, 2**]). By [**B-M**, Theorem 3], K_0 and K_1 are cobordant.

Let us now show that K_0 and K_1 are not pull back equivalent. By Theorem 4.6, we have only to show that for any spherical simple fibered $(2n-1)$-knots Σ_0 and Σ_1 in S^{2n+1}, $K_0 \sharp \Sigma_0$ is never orientation preservingly isotopic to $K_1 \sharp \Sigma_1$.

Since $K_i \sharp \Sigma_i$ is a fibered knot, we can consider the monodromy on the n-th homology group of the fiber, $i = 0,1$. Let us denote by H_i the monodromy matrix of $K_i \sharp \Sigma_i$ and by $\widetilde{L}_i$ its Seifert matrix with respect to the same basis. Here, we choose a basis which is the union of a basis of the homology of the fiber for K_i and that for Σ_i. It is known that $H_i = (-1)^{n+1} \widetilde{L}_i^{-1} ({}^t\widetilde{L}_i)$ (for example, see [**D1**]). Therefore, we have

$$H_0 = \begin{pmatrix} -1 & 0 \\ 18 & -1 \end{pmatrix} \oplus H_0' \quad \text{and} \quad H_1 = \begin{pmatrix} -1 & 0 \\ 50 & -1 \end{pmatrix} \oplus H_1',$$

where H_i' is the monodromy matrix of Σ_i, $i = 0,1$.

Let us consider $\operatorname{Ker}((I + H_i)^2)$, where I is the unit matrix, $i = 0,1$. Since Σ_i are spherical knots, the monodromy matrices H_i' cannot have the eigenvalue -1. Therefore, $\operatorname{Ker}((I + H_i)^2)$ corresponds exactly to the homology of the fiber of K_i.

Suppose that $K_0 \sharp \Sigma_0$ is orientation preservingly isotopic to $K_1 \sharp \Sigma_1$. Then the Seifert form of $K_0 \sharp \Sigma_0$ restricted to $\operatorname{Ker}((I + H_0)^2)$ should be isomorphic to that of $K_1 \sharp \Sigma_1$ restricted to $\operatorname{Ker}((I + H_1)^2)$. This means that L_0 should be congruent to L_1. However, this is a contradiction, since there exists an element $x \in \mathbf{Z}^2$ such that ${}^t x L_0 x = 9$, while such an element does not exist for L_1.

Thus, we conclude that K_0 and K_1 are not pull back equivalent. $\square$

EXAMPLE 4.8. *In fact, we can find infinitely many examples as in the above proposition. For example, we could use the matrices*

$$\begin{pmatrix} p^2 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} q^2 & 1 \\ -1 & 0 \end{pmatrix}$$

for arbitrary relatively prime odd integers p and q.

We can also use $K_0 \sharp K'$ and $K_1 \sharp K'$, instead of K_0 and K_1, for any simple fibered $(2n-1)$-knot K' whose monodromy does not have the eigenvalue -1.

Let us now give some examples of pairs of knots which are diffeomorphic but not cobordant even after taking connected sums with (not necessarily simple or fibered) spherical knots. For this, we use the following proposition.

PROPOSITION 4.9. [**B-M-S**] *Let K_0 and K_1 be simple fibered $(2n-1)$-knots with $n \geq 3$. If $K_0 \sharp \Sigma_0$ and $K_1 \sharp \Sigma_1$ are cobordant for some spherical knots Σ_0 and Σ_1, then the Seifert forms of K_0 and K_1 restricted to $I(K_0)$ and $I(K_1)$, respectively, are isomorphic to each other.*

REMARK 4.10. *The usual $(2n-1)$-dimensional spherical knot cobordism group C_{2n-1} acts on the cobordism semi-group of simple $(2n-1)$-knots with torsion free* *homologies by connected sum. The orbit space of the action inherits a natural semi-group structure. Then this orbit space contains infinitely many free generators as a commutative semi-group for $n \geq 3$.*

Moreover, for an arbitrary spherical simple $(2n-1)$-knot Σ whose Alexander polynomial is nontrivial and irreducible, $K\sharp\Sigma$ is never cobordant to K for any simple $(2n-1)$-knot K, since the Alexander polynomials of $K\sharp\Sigma$ and K do not satisfy a Fox-Milnor type relation necessary to be cobordant (see [**B-M**, (5.1) Proposition]*).*

In the following example we give a pair of knots for which their connected sum with any spherical knots are never cobordant. This answers the question $(\mathcal{Q})$ mentioned before negatively.

Example 4.11. [**B-M-S**] *Let us consider the following unimodular matrices:*

$$L_0 = \begin{pmatrix} 0 & 1 \\ (-1)^{n+1} & 0 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ (-1)^{n+1} & 0 & 0 & 1 \\ 0 & (-1)^{n+1} & 0 & 0 \end{pmatrix}.$$

Then, for every integer $n \geq 3$, there exist simple fibered $(2n-1)$-knots K_i in S^{2n+1} whose Seifert matrices are given by L_i, $i = 0, 1$ (see [**D1, 2**]*). Note that if we denote their fibers by F_i, $i = 0, 1$, then F_1 is orientation preservingly diffeomorphic to $F_0\sharp(S^n \times S^n)$. In particular, K_0 and K_1 are orientation preservingly diffeomorphic to each other.*

It is easy to verify that the Seifert form restricted to $I(K_1)$ is the zero form, while it is not zero for K_0. Hence, by Proposition 4.9, $K_0\sharp\Sigma_0$ is never cobordant to $K_1\sharp\Sigma_1$ for any spherical (but not necessarily simple or fibered) knots Σ_0, Σ_1.

Note that for this example, we have $H_{n-1}(K_i) \cong \mathbf{Z} \oplus \mathbf{Z}$, $i = 0, 1$.

Let us give another kind of an example together with an argument using the Alexander polynomial.

Example 4.12. [**B-M-S**] *Let us consider the unimodular matrices*

$$L_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad L_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and their associated simple fibered $(2n-1)$-knots K_i, $i = 0, 1$, with $n \geq 4$ even. As in Example 4.11 we see that K_0 and K_1 are orientation preservingly diffeomorphic to each other.

Now, suppose that for some spherical $(2n-1)$-knots Σ_i, $i = 0, 1$, $K_0\sharp\Sigma_0$ is cobordant to $K_1\sharp\Sigma_1$. We may assume that Σ_0 and Σ_1 are simple. The Alexander polynomials of K_0 and K_1 are respectively

$$\Delta_{K_0}(t) = \det(tL_0 + {}^tL_0) = t^2 + t + 1$$

and

$$\Delta_{K_1}(t) = \det(tL_1 + {}^tL_1) = -(t^4 + t^3 - t^2 + t + 1).$$

Both of these polynomials are irreducible over $\mathbf{Z}$. If $K_0\sharp\Sigma_0$ is cobordant to $K_1\sharp\Sigma_1$, then by [**B-M**, (5.1) Proposition]*, we must have a Fox-Milnor type relation*

$$(\mathfrak{R}) \quad \Delta_{K_0}(t)\Delta_{\Sigma_0}(t)\Delta_{K_1}(t^{-1})\Delta_{\Sigma_1}(t^{-1}) = t^{\lambda} f(t) f(t^{-1})$$

for some $\lambda \in \mathbf{Z}$ and $f(t) \in \mathbf{Z}[t, t^{-1}]$, where $\Delta_{\Sigma_i}(t)$ denote the Alexander polynomial of Σ_i, $i = 0, 1$.

Note that we have $|\Delta_{K_0}(1)| = |\Delta_{K_1}(1)| = 3$ and $|\Delta_{\Sigma_0}(1)| = |\Delta_{\Sigma_1}(1)| = 1$. Since $\Delta_{K_0}(t)$ is irreducible of degree 2, and $\Delta_{K_1}(t)$ is irreducible of degree 4, the relation $(\mathfrak{R})$ leads to a contradiction.

Hence, $K_0 \sharp \Sigma_0$ is not cobordant to $K_1 \sharp \Sigma_1$ for any spherical (not necessarily simple or fibered) $(2n-1)$-knots Σ_0, Σ_1. In this example we have $H_{n-1}(K_i) \cong \mathbf{Z}_3$, for $i = 0, 1$.

The last example gives non cobordant 3-knots with isomorphic Seifert forms.

Example 4.13. *Let $\mathcal{M}$ be a nontrivial orientable S^1-bundle over the closed connected orientable surface of genus $g \geq 2$. Note that $H_1(\mathcal{M})$ is not torsion free in general. Let $K_1, K_2, \dots, K_n$ be the simple fibered $\mathcal{M}$-knots constructed in* [**S4**, Theorem 3.1]*. They have the property that their Seifert forms are isomorphic to each other, but that any such isomorphism restricted to $H_2(K_i)$ cannot be realized by a diffeomorphism. Thus, the Seifert forms of K_i are algebraically concordant to each other, but are not spin concordant. Hence they are not concordant by Proposition 3.6, which is valid also for non-free simple fibered 3-knots.*

5. Even dimensional knots

We want to consider now cobordism classes of codimension two embeddings of even dimensional manifolds which are not necessary spheres. So we define

Definition 5.1. *A closed $(n-1)$-connected oriented $2n$-dimensional manifold embedded in the $(2n+2)$-sphere S^{2n+2} is called a $2n$*-knot.

Recall that in [**K1**] M. Kervaire showed that C_{2n}, the group of cobordism classes of $2n$-spheres embedded in S^{2n+2}, is trivial; and when $n \geq 3$ R. Vogt [**V2**] proved that the embeddings of a $2n$-manifold, which is closed and $(n-1)$-connected, in S^{2n+2} are cobordant all together. The restriction of the dimension comes from the difficulty to do embedded surgeries in low dimensions, and the impossibility to use the h-cobordism theorem.

To prove that C_{2n} is trivial M. Kervaire showed that an embedded n-sphere in $S^{n+2} = \partial(B^{n+3})$ is the boundary of a $(n+1)$-ball embedded in B^{n+3}.

5.1. Cobordism of surfaces.

In the case of non spherical $2n$-knots, we can prove that any connected and closed surface (oriented or not) embedded in S^4 is the boundary of an embedded handlebody in the ball B^5. More precisely, in [**B-S2**] we characterize the connected closed surfaces embedded in S^4 which are the boundary of an embedded handlebody in B^5. To do that we need to consider Pin^- structure on manifolds.

A Pin^- structure on a manifold X is the homotopy class of a trivialization of $TX \oplus \det TX \oplus \varepsilon^N$ over the 2-squeleton $X^{(2)}$ where TX denotes the tangent bundle, $\det TX$ denotes the orientation line bundle and ε^N is a trivial bundle of dimension N sufficiently big. A Pin^- structure is equivalent to a Spin structure when X is orientable.

The important fact is that when M is a closed surface embedded in S^4, since M is characteristic (i.e., as a submanifold of S^4 it represents the mod 2 homology class dual to the second Stiefel-Whitney class of S^4) the unique Spin structure on $S^4 \setminus M$ induce a unique Pin^- structure on M (see [**Ki-T1**]).

We denote by H_g the orientable handlebody of dimension 3; which is obtained by gluing g orientable 1-handles on the 3-ball. The boundary of H_g is the orientable closed surface of genus g. And we denote by I_g the non-orientable handlebody of dimension 3; which is obtained by gluing g non-orientable 1-handles. The boundary of I_g is the closed non-oriented surface of non-orientable genus $2g$. In the following we will denote by K_g the handlebody H_g or I_g.

DEFINITION 5.2. [**B-S2**] *Let M be a connected closed surface embedded in S^4. Suppose that the genus of M is g if M is orientable and $2g$ if M is non-orientable. Let $\psi : \partial K_g \to M$ a diffeomorphism. We say that ψ is* Pin$^-$ compatible *if the Pin$^-$ structure on ∂K_g induced by ψ extend to K_g.*

From now S^4 will be oriented. When a connected closed surface M is embedded in S^4 we denote by $e(M) \in \mathbf{Z}$ the Euler number (see [**Wh1**]) of the normal bundle of M in S^4. When M is oriented this number is always equal to 0. But when M is non-orientable, according to [**Wh1**] the set of value of $e(M)$ is equal to $\{-2g, 4-2g, 8-2g, \ldots, 2g\}$.

When M is orientable there always exists a submanifold V of S^4 such that $\partial V = M$ (see [**E**]). Similary to the case of odd dimensional knots we call such a manifold V a *Seifert manifold* for M. If M is non-orientable a Seifert manifold exists for M if and only if $e(M) = 0$ (see [**Go-L**]), then remark that the unique Spin structure on S^4 induces a Pin$^-$ structure on V and this Pin$^-$ structure induces a Pin$^-$ structure on M, which is exactly the same described before (see [**F**]).

In [**B-S1**] we proved the following theorem

THEOREM 5.3. [**B-S1**] *Let M be a connected closed surface embedded in $S^4 = \partial B^5$, and let $\psi : \partial K_g \to M$ a diffeomorphism, where K_g denotes the 3 dimensional handlebody with g 1-handles. There exists an embedding $\widetilde{\psi} : K_g \to B^5$ such that its restriction to the boundary coincide with ψ if and only if $e(M) = 0$ and ψ is Pin$^-$ compatible.*

REMARK 5.4. *Since every 3-dimensional manifolds admits a Heegaard spliting of genus g, then as a consequence of Theorem 5.3 we have a new proof of Rohlin's Theorem on embeddings of 3-dimensional manifols in $\mathbf{R}^3$ (see [**B-S2**]).*

The proof is based on the construction of the handlebody by embedded surgeries in the ball B^5 on a Seifert manifold V for M. To do that we first do the gluing $V' = V \cup_\partial K_g$ of V and K_g along their boundaries. Then we use the nullity of the group $\Omega_3^{Pin^-}$ to have the existence of an abstract manifold W with V' as boundary. And then we give a handle decomposition of a manifold $W' \cong W$ such that $\partial W = \partial W'$, with only handle of index 2. This can be done by removing handles of index 1 and 3 using a modification described by A. Wallace [**Wa**]. Then we can do the corresponding embedded surgeries since we only have handle of index equal to 2.

As a corollary of the Theorem 5.3 we can prove the following.

COROLLARY 5.5. [**B-S1**] *Let M be a connected closed surface embedded in $S^5 = \partial B^5$. There exists a 3-dimensional handlebody embedded in B^5 such that its boundary coincides with M if and only if $e(M) = 0$.*

As M. Kervaire did for spherical knots, using Theorem 5.3 we can describe cobordism classes of connected closed surfaces embedded in S^4.

THEOREM 5.6. [**B-S1**] *Let M_0 and M_1 be two connected closed surfaces embedded in S^4. Then they are cobordant if and only if they are diffeomorphic as abstract manifolds and they have same Euler number.*

To prove this Theorem, we first remark that $(\partial K_g \smallsetminus \mathrm{Int}B^2)\times[0,1] \cong K_{2g}$. Then we construct $\Sigma = (S^4 \smallsetminus \mathrm{Int}B^4)\times[0,1] \cong B^5$, such that $M_0 \cap B^4 = M_1 \cap B^4 = B^2$ and (B^4, B^2) is the standard pair. Then we find a Pin^- compatible diffeomorphism between $(M_0 \smallsetminus B^2) \cup (\partial B^2 \times [0,1]) \cup (M_1 \smallsetminus B^2) = M_0 \sharp M_1 \subset \Sigma$ and $\partial\Big((\partial K_g \smallsetminus \mathrm{Int}B^2)\times[0,1]\Big) \cong \partial K_{2g}$. So we can apply Theorem 5.3 to embed K_{2g} in Σ. The cobordism between M_0 and M_1 is obtained by gluing back $\mathrm{Int}B^4 \times [0,1]$ to Σ, and then replace K_{2g} by $K_g \times [0,1]$ with boundary $M_0 \coprod M_1$.

In the case of oriented surfaces, two oriented closed surfaces embedded in S^4 are cobordant if and only if they have same genus. Hence the monoide of cobordism classes of connected closed oriented surfaces embedded in S^4 is isomorphic to the monoide of the non negative integers.

For non-orientable surfaces, first remark that by adding the cobordism class of an embedding of S^2 into S^4 to the associative magma of cobordism classes of non-orientable surfaces embedded in S^4, we get a monoide denote by $\mathfrak{M}$. We can also describe the monoide structure of $\mathfrak{M}$. Let RP^2_- the projective plane trivialy embedded in S^4 with Euler number equals to -2, and RP^2_+ the projective plane trivialy embedded in S^4 with Euler number equals to 2. For a couple of positive integers (k,l) such that $k+l \geq 1$, let $M_{k,l}$ be the surface embedded in S^4 obtained by doing the connected sum of k copies of RP^2_+ and l copies of RP^2_-. Then we have $e(M_{k,l}) = 2(k-l)$ and the genus of $M_{k,l}$ is equal to $k+l$. Hence the surfaces $M_{k,l}$ are some complete representant of the cobordism classes of connected closed non-orientable surfaces embedded in S^4, and $\mathfrak{M}$ is then isomorphic to the monoide of pairs of non negative integers. If we denote by $[M]$ the cobordism class of the surface M embeded in S^4, and by $g(M)$ the genus of M, the isomorphism is given by

$$\begin{array}{ccc} \mathfrak{M} & \to & \mathbf{N}\times\mathbf{N} \\ [M] & \mapsto & \left(\frac{2g(M)+e(M)}{4}, \frac{2g(M)-e(M)}{4}\right) \end{array}$$

5.2. concordance of surfaces. As a consequence of Theorem 5.3 we have

THEOREM 5.7. [**B-S2**] *Let Σ be a connected closed surface. Two embeddings of Σ into S^4 are concordant if and only if the* Pin^- *structures induced by these embeddings coincide and the Euler numbers of these embeddings are equal.*

In the case of spherical knots the two notions of cobordism and concordance coincide since every diffeomorphism of S^2 which preserve the orientation is isotopic to identity. But if Σ_g denotes the oriented closed surface of genus g, then for every embedding $f : \Sigma_g \to S^4$ there exists an orientation preserving diffeomorphism $h : \Sigma_g \to \Sigma_g$ which do not preserve the Pin^- structure induced by f. Henceforth the embeddings $f \circ h$ and f are not concordant.

The group of orientation preserving diffeomorphism of an oriented connected and closed surface acts transitively on the set of Pin^- structure with trivial Brown invariant (see [**B-S2**]). This imply that the number of concordance classes of the embeddings of an oriented connected closed surfaces is equal to the number of Pin^- sturcture with trivial Brown invariant. If we denote by A_g the number of concordances classes in the case of the oriented connected closed surfaces of genus g, then we have

$$A_g = 2^{g-1}(2^g + 1)$$

Let us denote by C_g the number of concordances of the embeddings of the non orientable connected closed surface N_g of non orietable genus g. According to [**M**] if M is a connected closed non orientable surface of non orientable genus g embedded in S^4, then $e(M) \in \{-2g, 4-2g, 8-2g, \dots, 2g\}$ and all the value in this set can be realized. Hence $C_g = \sum_{i=0}^{g} C_{g,-2g+4i}$, where $C_{g,-2g+4i}$ is the number of concordance classes of embeddings of N_g in S^4 with Euler number equal to $-2g+4i$. Moreover by [**Ki-T1**] $C_{g,-2g+4i}$ is equal to the number of Pin$^-$ structure with Brown invariant equal to $-g+2i$ modulo 8.

β	g: impair	g: pair
0	0	$2^{(g-2)/2}(2^{(g-2)/2}+1)$
1	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0
2	0	2^{g-2}
3	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
4	0	$2^{(g-2)/2}(2^{(g-2)/2}-1)$
5	$2^{(g-3)/2}(2^{(g-1)/2}-1)$	0
6	0	2^{g-2}
7	$2^{(g-3)/2}(2^{(g-1)/2}+1)$	0

TABLE 1: Number of Pin$^-$structure on N_g with Brown invariant $\beta \in \mathbf{Z}_8$

With the values given in Table 1, we get

$$C_g = \begin{cases} 2^{g-2}(g+1) & \text{if } g \text{ is odd,} \\ 2^{g-2}(g+1) + 2^{(g-2)/2} & \text{if } g \text{ is even.} \end{cases}$$

5.3. Cobordism of 4-knots. In the study of cobordism of embeddings, the only case which remains is for 4 dimensional manifolds. In [**B-S3**] we prove the following Theorem

THEOREM 5.8. [**B-S3**] *Let M be an oriented, closed and simply connected 4 dimensional manifold. All the embeddings of M into S^6 are concordant.*

REMARK 5.9. *Since concrodance imply cobordism these embeddings are also cobordant.*

The proof is an adaptation of the proof of Theorem 5.3.

6. Questions

To conclude this paper we want to list some open questions.

($\mathcal{Q}1$) Is the multiplicity of complex holomorphic germs of functions with an isolated singularity a cobordism invariant?

($\mathcal{Q}2$) Is Spin cobordism of Seifert forms associated with non free 3-knots a sufficient condition of cobordism?

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