An integrability condition for fields of nilpotent endomorphisms

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Abstract. We give a necessary and sufficient condition on the 1-jet of a field of nilpotent endomorphisms to be integrable. Together with the well known corresponding condition for an almost complex structure, the nullity of its Nijenhuis tensor, this gives an integrability condition for any field of endomorphisms.


Key words: integrability, equivalence, nilpotent endomorphism, Nijenhuis tensor.

It is a classical question to ask whether (the germ of) an almost complex structure $J$ is (the germ of) a complex structure i.e. if it is integrable: does it exist local coordinates in which $J$ becomes a constant matrix, namely \(
\begin{pmatrix}
0 & I \\
-I & 0
\end{pmatrix}
\)? A well known necessary and sufficient condition on the 1-jet of $J$ is that the Nijenhuis tensor $N_J$ of $J$ vanishes [6]. We address here the same question for a field of nilpotent endomorphisms $A$: instead of \(J^2 = -\text{Id}\), we take \(A^n = 0\) for some $n$. More precisely, we suppose that $A$ is conjugate, at every point, to some fixed nilpotent endomorphism — this is necessary to hope integrability.

Immediately, the nullity of the Nijenhuis tensor $N_A$ is necessary. Simple examples show that this condition is not sufficient, see section 3, see also [4]. We show here that it becomes sufficient, together with the additional condition that each distribution of the flag \((\ker A_p)_{p=1}^{n-1}\) is involutive. The proof, unlike that of [6], follows essentially from the Cauchy-Lipschitz theorem and some standard differential calculus.

In combination with the integrability condition for complex structures, this immediately gives an integrability condition for any smooth endomorphism field $M$: $M$ is integrable if and only if it has constant invariant factors, $N_M = 0$ and $\ker(P(M))$, for each invariant factor $P$ of $M$, is involutive.

A general viewpoint on this type of problems, that we do not use here, is given in [1].

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Everywhere, $A$ is a germ of endomorphism field of $T\mathbb{K}^d$ around 0 in $\mathbb{K}^d$, with $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, i.e. a smooth (holomorphic if $\mathbb{K} = \mathbb{C}$) section of $\text{End}_\mathbb{K}(T\mathbb{K}^d)$ on a neighbourhood $\mathcal{V}$ of 0. All objects: coordinates, tangent bundles etc. are real if $\mathbb{K} = \mathbb{R}$ and complex if $\mathbb{K} = \mathbb{C}$.

Section 1 recalls the requisite definitions and states the results, section 2 gives the proofs and section 3 provides some additional results, comments and examples.

1 Definitions and results

We recall the two following definitions.

1.1 Definition The Nijenhuis torsion tensor of $A$ is the vector valued 2-form defined by:

\[
\]

We let the reader check that it is a tensor, see e.g. [3], ch. 1 prop. 3.12, where the torsion tensor $S_{A,B}$ of some couple $(A, B)$ of fields of endomorphisms is introduced. Our $N_A$ is equal to $\frac{1}{2}S_{A,A}$. The fact that $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ plays no role here.

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1.2 Definition The field $A$ is called integrable if there exists, on a neighbourhoud $V$ of the origin, a coordinate system in which $\text{Mat}(A)$ is constant i.e. a diffeomorphism, or a biholomorphism $\varphi : V \to U \subset \mathbb{K}^d$ such that $\varphi_*A$ is the restriction to $U$ of a linear transformation of $\mathbb{K}^d$.

Here we show the following result.

1.3 Theorem Let $A$ be a germ of field of nilpotent endomorphisms of order $n \geq 1$ on $\mathbb{K}^d$. If $\mathbb{K} = \mathbb{C}$, we take $A$ holomorphic. If $\mathbb{K} = \mathbb{R}$, we take $A$ of class $C^\omega$, $C^\infty$ or $C^r$ with $r \geq n - 1$.

Then $A$ is integrable if and only if the three following conditions are satisfied:
- the invariant factors of $A$ are constant,
- $N_A = 0$,
- each distribution $\ker A^p$, for $p \in \mathbb{N}$, is involutive — hence integrable.

If $A$ is analytic or of class $C^\infty$, the integral coordinates have the same regularity. If $A$ if of class $C^r$ with $n - 1 \leq r < \infty$, they are at least, and possibly not more than, of class $C^{r-n+2}$. If $A$ satisfies the three conditions but is not of class $C^{n-1}$, it is non integrable in general.

1.4 Remark The regularity condition “class $C^{n-1}$”, though minor, has to be mentioned. In other equivalence problems of $G$-structures of order 1 (see [1]), with $G$ reductive, and solved as P.D.E. problems, such a strong regularity condition does not seem to appear (see e.g. [5] or Theorem II of [8]). Here the group $G$ is the commutant of $A$ in $\text{GL}_d(\mathbb{K})$, which is not reductive as soon as $A \neq 0$. The regularity condition seems to be linked to that fact.

The present coordinates are not the solution of an elliptic P.D.E., see Remark 3.3. Instead, they arise naturally as the solution of O.D.E.’s integrated by induction. In that sense the proof of Theorem 1.3 is similar to that of the Frobenius criterion given in [2], C.1.1.

Together with the classical integrability condition for complex structures, the present result gives easily the following corollary.

1.5 Corollary If $A$ is any field of endomorphisms of class $C^\infty$ on $\mathbb{R}^d$, with constant invariant factors, then $A$ is integrable if and only if the three following conditions are realised:
- the invariant factors of $A$ are constant,
- $N_A = 0$,
- the distribution $\ker(P(A))$, for each invariant factor $P$ of $A$, is involutive.

Of course, the minimal regularity condition in general is that $A$ is of class $C^{n-1}$ along each integral leaf of $\ker(P(A))$, with $P = Q^n$, $Q$ irreducible. Eventually, a little remark, proven in section 2, is worth to be pointed out autonomously.

1.6 Remark If $A$ is nilpotent, the nullity of $N_A$ implies the integrability of each distribution $\text{Im} A^p$, but not that of the kernel distributions $\ker A^p$.

2 Proof of the results

If $A$ is integrable, it is conjugate, at any point, to some fixed nilpotent matrix, so it has constant invariant factors. So the first condition of Theorem 1.3 and of Corollary 1.5 is the 0-order integrability condition for $A$, and is necessary. From now on we suppose it holds.

We introduce the following technical torsion-related tensor, and one of its properties.
2.1 Definition If $B$ is another endomorphism field on $\mathbb{V}$ and if $A$ and $B$ commute, we introduce:


The reader may check it is a tensor; the sum $S_{A,B} = N'_{A,B} + N'_{B,A}$ is the torsion of $(A,B)$ cited in Def. 1.1, well-defined even if $AB \neq BA$. So here $N_A = N'_{A,A} = \frac{1}{2}S_{A,B}$.

2.2 Proposition All $N'_{A,p,q}$ for $p,q \in \mathbb{N}^*$ depend only on $N_A$, through both following relations:

(i) for all couple $(X,Y)$ of vectors, $N'_{A,p,q}(X,Y) = \sum_{k=1}^{q} A^q-kN_A(X,A^{k-1}Y)$, 

(ii) for all couple $(X,Y)$ of vectors, $N'_{A,p,q}(X,Y) = \sum_{k=1}^{p} A^p-kN'_{A,p,q}(X,A^{k-1}Y)$.

In particular, if $N_A = 0$, then all $N'_{A,p,q}$ and all $N_{A,p}$ also vanish.

Proof. As $N'_{A,p,q}(X,Y) = -N'_{A,q,p}(Y,X)$, (i) is a special case of (ii). Let us prove (ii) by induction on $p$. It is trivial for $p = 1$. Suppose it holds for some $p$.

$$N'_{A,p+1,q}(X,Y) = [A^{p+1}X, A^qY] - A^{q}[A^{p+1}X, Y] - A^{p+1}[X, A^qY] + A^{p+1}[X, A^{q+1}Y]$$

$$= [A^{p+1}X, A^qY] - A^{q}[A^{p+1}X, Y] - A^{p+1}[X, A^qY] + A^{p+1}[X, A^{q+1}Y]$$

$$+ A[A^{p}X, A^qY] - A^{q+1}[A^{p}X, A^qY] - A^{p}[X, A^qY] + A^{p+1}[X, A^{q+1}Y]$$

$$= N'_{A,p,q}(A^{p}X, Y) + A(N'_{A,p,q}(X,Y)),$$

hence it holds for $p + 1$. \[\square\]

Proof of Remark 1.6. Now we can prove Remark 1.6. As $N_A = 0$, each distribution Im $A^p$ is integrable. Let us prove it is involutive, the conclusion follows by the Frobenius criterion. Let us take $X$ and $Y$ any vector fields and show: $[A^{p}X, A^{p}Y] \in \text{Im} A^p$. By Proposition 2.2, $N_A(X,Y) = 0$, so $[A^{p}X, A^{p}Y] = A^p[X, A^{p}Y] + A^p[A^{p}X, Y] - A^{p}[X, A^{p}Y]$ and we are done. Besides, example 3.6 gives a counterexample the integrability of $\text{ker} A^p$.

2.3 Notation If $A$ satisfies the three conditions of Theorem 1.3, using Remark 1.6, we denote respectively by $\mathcal{I}^p$ and $\mathcal{K}^p$ the integral foliation of the distribution $\text{Im} N^p$, respectively $\text{ker} N^p$, for any $p$. We shortly denote $\mathcal{I}^1$ by $\mathcal{I}$, and denote by $\pi$ the projection $\mathcal{V} \to \mathcal{V}/\mathcal{I}$. Each time it is needed, $\mathcal{V}$ is implicitly replaced by a smaller neighbourhood of 0.

To prove the theorem, we already prove the following, which is a part of it.

2.4 Lemma If the three conditions of Theorem 1.3 are realised, then there exist sections $\sigma$ of $\pi$, of class $C^{r+1}$ in case $\mathbb{K} = \mathbb{R}$, holomorphic in case $\mathbb{K} = \mathbb{C}$, which respect the foliations $\mathcal{K}^1 \subset \ldots \subset \mathcal{K}^{n-1} \subset \mathcal{K}^n \subset \mathcal{V}$ i.e. such that:

$$\text{for all } p \in \mathbb{N}, \; d\sigma(d\pi(\text{ker} A^p)) \subset \text{ker} A^p.$$

Note. The lemma expresses the fact that, under our assumptions, along one fibre of $\pi$, the leaves of $\mathcal{K}^p$ have the same image by $\pi$.

Proof. First, we use Proposition 2.2 to show the following stronger version of Remark 1.6: any of the distributions $\ker A^p + \text{Im} A^p$ is involutive. Take $(X, X')$ and $(Y, Y')$ two couples of vector fields in $\ker A^p$, respectively in $\text{Im} A^p$. Then: $[X + Y, X' + Y'] = [X, X'] + [Y, Y'] + [X, Y'] + [Y, X']$. As $\ker A^p$, by assumption, and $\text{Im} A^p$, by Remark 1.6, are involutive,
We now build, by induction on $k \in \ker A^p$ and $[Y, Y'] \in \text{Im} A^q$. We are left with showing, for instance, that $[X, Y'] \in \ker A^0 + \text{Im} A^q$ i.e. that $A^0[X, Y'] \in \text{Im} A^{p+q}$. Take a field $Z$ such that $Y' = A^q Z$:


So we may denote by $\mathcal{K}^p + T^q$ the integral foliation of $\ker A^p + \text{Im} A^q$. As $A$ is of class $C^r$, the tangent distribution of this foliation is of class $C^r$ i.e. the foliation is of class $C^{r+1}$. Now we consider local coordinates $(x^1, \ldots, x^n) = ((x^1_i)_{i=1}^{\dim \ker A^p}, (x^2_i)_{i=1}^{\dim \ker A^p-\dim \ker A^q}, \ldots, (x^n_i)_{i=1}^{\dim \ker A^n-\dim \ker A^{n-1}}) \subset \mathcal{V}$, of class $C^{r+1}$ [or holomorphic], such that the plaques of $\mathcal{K}^p$ in $\mathcal{V}$ are the levels $\{(x^{p+1}, \ldots, x^n) = \text{const.}\}$. As $(\ker A^{n-2} + \text{Im} A^q) \cap \ker A^{n-1}$ is integrable, with leaves included in those of $\mathcal{K}^{n-1}$, we may modify the $x^{n-1}$ and replace them by $C^{r+1}$ coordinates $(u^{n-1}, v^{n-1})$ such that:

- as before, the plaques of $\mathcal{K}^{n-2}$ in $\mathcal{V}$ are the levels $\{((u^{n-1}, v^{n-1}), x^n) = \text{const.}\}$,
- additionally, the plaques of $(\mathcal{K}^{n-2} + T) \cap \mathcal{K}^{n-1}$ in $\mathcal{V}$ are the levels $\{((u^{n-1}, v^{n-1}), x^n) = \text{const.}\}$.

To simplify the notation, we set $(u^n, v^n) := (x^n)$ with $(u^n)$ an empty family. Proceeding downwards step by step, we get coordinates $((u^1, v^1), \ldots, (u^n, v^n))$ such that, for all $p$:

- the plaques of $\mathcal{K}^p$ are the levels $\{((u^{p+1}, v^{p+1}), \ldots, (u^n, v^n)) = \text{const.}\}$,
- those of $(\mathcal{K}^{p-1} + T) \cap \mathcal{K}^p$ are the levels $\{((u^{p+1}, (u^{p+2}, v^{p+2}), \ldots, (u^n, v^n)) = \text{const.}\}$.

Finally set $S = \{(u^1, \ldots, u^n) = 0\}$, then $\pi_1 : S \to \mathcal{V}/T$ is a diffeomorphism [a biholomorphism if $\mathbb{K} = \mathbb{C}$]. We take $\sigma = (\pi_1)^{-1}$, this section is as wanted. \qed

\textbf{Note.} The same reasoning may have directly built coordinates such that each plaque of any foliation $\mathcal{K}^p + T^q$ is a level of an adequate sub $k$-tuple of them, a property satisfied by the coordinates finally given by Theorem 1.3. We have not done it, stating only in the present lemma the minimum needed by the following proof of the theorem.

\textbf{Proof of the theorem.} If $A$ is integrable, the integrability of $\ker A^p$ and the nullity of $N_A$ are immediate. Let us prove the converse. It is implicit in the following that:

- in case $\mathbb{K} = \mathbb{C}$, the reader shall replace “of class $C^k$” by “holomorphic” everywhere,
- the flow $(\Phi^t)_{t \in \mathbb{K}, |t| < \varepsilon}$ of a vector field is obtained by the Cauchy-Lipschitz theorem, in case $\mathbb{K} = \mathbb{R}$ and $r \neq \omega$, and by the Cauchy-Kovalevskaya theorem, in case $\mathbb{K} = \mathbb{R}$ and $r = \omega$, or $\mathbb{K} = \mathbb{C}$.

We set $d_a = \dim (d\pi(\ker A^a)/d\pi(\ker A^{a-1}))$. \textbf{Remark:} with this notation, the invariant factors of $A$ are $((X^a)^{d_a}_{k=1})_{a=1}^n$ i.e. $(X, \ldots, X, X^2, \ldots, X^2, \ldots, X^n, \ldots, X^n)_{d_1 \text{ times}}$.

Proving that $A$ is integrable amounts to building a field of basis $\beta$ on $\mathcal{V}$ such that:

\begin{enumerate}
\item $\text{Mat}_{\beta}(A)$ is constant,
\item any two vector fields of $\beta$ commute (in other terms, the field $\beta$ is integrable).
\end{enumerate}

We now build, by induction on $k$, basis fields $\beta^{(k)}$ satisfying (i), and satisfying (ii) modulo $\text{Im} A^{k+1}$. At step $n - 1$, the field $\beta = \beta^{(n-1)}$ will satisfy (ii) modulo $\text{Im} A^n = \{0\}$ i.e. really satisfy (ii). We will be done.

In the following, the passages in \textit{[italic between brackets]} track the regularity questions in case $A$ is only of class $C^r$, $r < \infty$. They may be fully passed in a first reading.
Figure 1: Defining the fields \( Z_{i,j}^{(0)} \), from the coordinates \( \sigma_{i,j} \) on \( \mathcal{V} \). Then \( \beta^{(0)} = (A^n Z_{i,j}^{(0)})_{a,i,j} \).

**Initiating the induction : building the field \( \beta^{(0)} \).**

Take a section \( \sigma \) of \( \pi \) as given by Lemma 2.4 and denote by \( \mathcal{S} \) its image. Then we choose \((\tau_{n,j})_{j=1}^{d_n}\) any local coordinate system of \( \pi(\mathcal{V})/\pi(K^{n-1}) \); we pull it back on \( \pi(\mathcal{V})/\pi(K^{n-2}) \) by the natural projection and complete it by additional coordinate functions so that \((\tau_{n,j})_{j=1}^{d_n}, (\tau_{n-1,j})_{j=1}^{d_{n-1}}\) is a local coordinate system of \( \pi(\mathcal{V})/\pi(K^{n-2}) \). Iterating this process, we build a local system \((\tau_{n,j})_{j=1}^{d_n}, (\tau_{n-1,j})_{j=1}^{d_{n-1}}, \ldots, (\tau_{1,j})_{j=1}^{d_1}\) of coordinates of \( \pi(\mathcal{V}) \). Denoting by \((\tau_{n,j})_{j=1}^{d_n}, (\tau_{n-1,j})_{j=1}^{d_{n-1}}, \ldots, (\tau_{1,j})_{j=1}^{d_1}\) the corresponding coordinate vectors, each \( \tau_{i,j} \) is in \( d\pi(\ker(A^n) \setminus \ker(A^{n-1})) \). For any \((i,j)\), we set \( Z_{i,j} = d\sigma Z_{i,j} \), they are vector fields in \( T_{\mathcal{S}} \). As \( \sigma \) is of class \( C^{r+1} \), the \( Z_{i,j} \) are of class \( C^r \) along \( \mathcal{S} \). We extend them, arbitrarily, in \( \pi \)-basic fields [of class \( C^r \)] defined on \( \mathcal{V} \), which we denote by \( Z_{i,j}^{(0)} \). See figure 1. We set \( \beta^{(0)} = (A^n Z_{i,j}^{(0)})_{a,i,j} \). We notice the following set of facts (\( \mathbf{H}_0 \)):

\[
\begin{align*}
(\mathbf{H}_0) & \quad 
\begin{cases}
(1) & Z_{i,j}^{(0)} = Z_{i,j} \text{ along } \mathcal{S} \text{ (by definition of the } Z_{i,j}^{(0)}), \\
(2) & \text{for any } (a,b) \in \mathbb{N}^2, \text{ for any } (i,j) \text{ and } (i',j'), \ [A^n Z_{i,j}^{(0)}, A^b Z_{i',j'}^{(0)}] \in \text{Im } A, \\
(3) & \text{for any } (a,b) \in (\mathbb{N}^*)^2, \text{ for any } (i,j) \text{ and } (i',j'), \ [A^n Z_{i,j}^{(0)}, A^b Z_{i',j'}^{(0)}] \in \text{Im } A^2.
\end{cases}
\end{align*}
\]

For the second line: the \( Z_{i,j}^{(0)} \) are \( \pi \)-basic, while \( d\pi A^n Z_{i,j}^{(0)} = 0 \) as soon as \( a > 0 \), so the \( A^n Z_{i,j}^{(0)} \) are “\( \pi \)-null”, thus also \( \pi \)-basic. The third one holds as \( N_A = 0 \). Indeed:

\[
\begin{align*}
[A^n Z_{i,j}^{(0)}, A^b Z_{i',j'}^{(0)}] & = A\left[ A^{a-1} Z_{i,j}^{(0)}, A^b Z_{i',j'}^{(0)} \right] + A\left[ A^n Z_{i,j}^{(0)}, A^{b-1} Z_{i',j'}^{(0)} \right] - A^2\left[ A^{a-1} Z_{i,j}^{(0)}, A^{b-1} Z_{i',j'}^{(0)} \right] \\
& \in \text{Im } A^2.
\end{align*}
\]

**Propagating the induction : building the field \( \beta^{(k+1)} \), once given the field \( \beta^{(k)} \).**

We now prove the following fact by induction on \( k \) (the wanted field \( \beta^{(k)} \) is the field \( (A^n Z_{i,j}^{(k)})_{a,i,j} \)).

\[\begin{align*}
(\mathbf{H}_k) & \quad \text{There exist vector fields } Z_{i,j}^{(k)} \text{ [of class } C^{r-k} \text{] on } \mathcal{V} \text{ such that:}
\end{align*}\]

\[
\begin{align*}
(1) & \quad Z_{i,j}^{(k)} = Z_{i,j} \text{ along } \mathcal{S}, \\
(2) & \text{for any } (a,b) \in \mathbb{N}^2, \text{ for any } (i,j) \text{ and } (i',j'), \ [A^n Z_{i,j}^{(k)}, A^b Z_{i',j'}^{(k)}] \in \text{Im } A^{k+1}, \\
(3) & \text{for any } (a,b) \in (\mathbb{N}^*)^2, \text{ for any } (i,j) \text{ and } (i',j'), \ [A^n Z_{i,j}^{(k)}, A^b Z_{i',j'}^{(k)}] \in \text{Im } A^{k+2}.
\end{align*}
\]
The proof of \((H_0)\) just above initiates the induction. Let us suppose that \((H_k)\) is true and prove \((H_{k+1})\). To alleviate the following, we re-index temporarily all the \(A^a Z_i^{(k)}\) for \(a > 0\), in an arbitrary order, denoting them as the \(Z_i^{(k'), j}\), for \(i\) going from 1 to some \(N\). We denote by \((\Phi_1^j)_{1 \leq k, k < l}\) the flow of \(Z_i^{(k')}\). \([As the Z_i^{(k')} are of class C^{r-k} by (H_k), \Phi_1^j is of class C^{r-k} as a function of \((t, m) \in \mathbb{R} \times \mathcal{V}\). By (H_k) (1) and by construction of the \(Z_{i,j}\) along \(S, (Z_i^{(k')}_{i=1}^N)\) is a linearly independent family spanning a direct complement of \(TS\) in \(TV\). Therefore, the application \(\Phi : (m, (t_j^N)_{j=1}^N) \mapsto \Phi_N^t \circ \cdots \circ \Phi_1^m\) is a diffeomorphism \([of class C^{r-k}]\) from \(S \times B_{\mathcal{V}}(0, \varepsilon)\) on a tubular neighbourhood of \(S\) in \(\mathcal{V}\), for some small \(\varepsilon > 0\). So we may define the fields \(Z_{i,j}^{(k+1)}\) \([as vector fields of class C^{r-(k+1)}]\) in this neighbourhood of \(S\) by:

\[
\begin{align*}
Z_{i,j}^{(k+1)} &= Z_{i,j} \text{ along } S, \\
Z_{i,j}^{(k+1)}(\Phi(m, t_1, \ldots, t_N)) &= d(\Phi_N^t \circ \cdots \circ \Phi_1^m)(m).Z_{i,j}(m).
\end{align*}
\]

Then \((H_{k+1})\) (1) holds by definition of the \(Z_{i,j}^{(k+1)}\). In particular the \(Z_{i,j}^{(k+1)}\) commute along \(S\). So, as by \((H_k)\) (3) the fields \(Z_i^{(k')}\) commute modulo \(\text{Im } A^{k+2}\), the \(Z_{i,j}^{(k+1)}\) commute modulo \(\text{Im } A^{k+2}\) on their whole definition domain: by definition of the \(Z_{i,j}^{(k)}\), \([Z_i^{(k)}, Z_{i,j}^{(k+1)}] = 0\) along \(\Phi(S, \mathbb{K} \times \{0\}^N)\). You obtain \([Z_i^{(k)}, Z_{i,j}^{(k+1)}] \equiv 0\) by pushing this bracket by the \(\Phi_1^j\) for \(j > i\) and applying the Jacobi identity. For the same reason, the \(Z_{i,j}^{(k+1)}\) commute modulo \(\text{Im } A^{k+2}\) with the \(Z_i^{(k)}\) i.e. with the \(A^a Z_i^{(k)}\) for \(a > 0\). That is to say, we get, for any \((i, j)\) and \((i', j')\):

\[
[Z_{i,j}^{(k+1)}, Z_{i',j'}^{(k+1)}] \in \text{Im } A^{k+2} \quad (a) \quad \text{and: } a > 0 \Rightarrow [Z_{i,j}^{(k+1)}, A^a Z_{i',j'}^{(k)}] \in \text{Im } A^{k+2} \quad (b).
\]

It follows that, for any \((i, j)\):

\[
Z_{i,j}^{(k+1)} \equiv Z_{i,j}^{(k)} [\text{Im } A^{k+1}]  \quad (c) \quad \text{and: } (a > 0, b > 0) \Rightarrow A^a Z_{i,j}^{(k+1)} = A^b Z_{i,j}^{(k)} [\text{Im } A^{k+2}]  \quad (d).
\]

Indeed, \(Z_{i,j}^{(k+1)} = Z_{i,j}^{(k)}\) along \(S\) and, by \((H_k)\) (2), \([Z_i^{(k)}, A^a Z_{i,j}^{(k)}] \in \text{Im } A^{k+1}\). Together with \((b)\), this gives \((c)\). Point \((d)\) follows immediately.

Then, point \((d)\) enables to replace \(A^a Z_{i,j}^{(k)}\) by \(A^a Z_{i',j'}^{(k+1)}\) in \((b)\). Indeed, by \((H_k)\) (3), the \(A^a Z_{i,j}^{(k)}\), for \(a \geq k + 2\), span \(\text{Im } A^{k+2}\) locally around \(S\), so \((d)\) may be written as:

\[
A^a Z_{i,j}^{(k)} = A^a Z_{i,j}^{(k+1)} + \sum_{a=k+2} A^a \alpha_{a,i,j} A^a Z_{i,j}^{(k)}\]

with some functions \(\alpha_{a,i,j}\). Replace in \((b)\) and re-use \((b)\), you get:

\[
a > 0 \Rightarrow [Z_{i,j}^{(k+1)}, A^a Z_{i',j'}^{(k+1)}] \in \text{Im } A^{k+2} \quad (e).
\]

Similarly, \((d)\) enables also to replace the \(A^a Z_{i',j'}^{(k)}\) by \(A^a Z_{i',j'}^{(k+1)}\) in \((H_k)\) (3), giving:

\[
(a > 0, b > 0) \Rightarrow [A^a Z_{i,j}^{(k+1)}, A^b Z_{i',j'}^{(k+1)}] \in \text{Im } A^{k+2} \quad (f).
\]

Together, points \((a)\), \((e)\), and \((f)\) imply \((H_{k+1})\) (2), which is now proved. Finally the nullity of \(\mathcal{N}_A\) gives \((H_{k+1})\) (3). Take \(a\) and \(b\) in \(\mathbb{N}^*\); \(\mathcal{N}_A(A^a Z_{i,j}^{(k+1)}, A^b Z_{i',j'}^{(k+1)}) = 0 \quad i.e.:

\[
\begin{align*}
[A^a Z_{i,j}^{(k+1)}, A^b Z_{i',j'}^{(k+1)}] &= A[A^{a-1} Z_{i,j}^{(k+1)}, A^b Z_{i',j'}^{(k+1)}] + A[A^a Z_{i,j}^{(k+1)}, A^{b-1} Z_{i',j'}^{(k+1)}] \\
&\quad \in \text{Im } A^{k+2} \quad \text{by \((H_{k+1})\) (2)} \\
&\quad - A^2 [A^{a-1} Z_{i,j}^{(k+1)}, A^{b-1} Z_{i',j'}^{(k+1)}] \\
&\quad \in \text{Im } A^{k+2} \quad \text{by \((H_{k+1})\) (2)} \\
&\quad \in \text{Im } A^{k+3}.
\end{align*}
\]
The induction is proven. In case $A$ is of class at least $C^\infty$, the basis field $\beta := \beta^{(n-1)}$ fulfills our requirements and ends the proof.

[In case $A$ is only of class $C^r$, $r < \infty$, the end of the proof must be slightly modified. Carrying on the induction up to $(H_{n-1})$ would provide some $C^{n-r+1}$ fields $Z_{i,j}^{(n-1)}$, but as possibly $n - r + 1 = 0$, this is useless: commuting class $C^0$ but not $C^1$ do not provide corresponding coordinate functions, in general. Instead, we use directly the $C^{r-n+2}$, diffeomorphism $\Phi : (m, (t_i)_{i=1}^N) \mapsto \Phi^N_{1} \circ \ldots \circ \Phi^N_{1} (m)$ of this $(n-1)^{th}$ step of the induction. As the fields $A^a Z_{i,j}^{(n-2)}$, with $a > 0$, commute, and parametrising $m \in S$ by its coordinates $\sigma (\overline{z_i})$, $\Phi$ is nothing but a local parametrisation of $\mathcal{V}$ by a system of coordinates of class $C^{n-r+2}$, with coordinate vectors all the $A^a Z_{i,j}^{(n-1)}$ with $a \geq 0$. These coordinate vectors form a basis field in which $\text{Mat}(A)$ has a constant Jordan form. We are done.]

Eventually, the condition that $A$ is of class $C^{n-1}$ is necessary, and the given regularity of the integral coordinates is optimal: this follows from Example 3.4 in the next section. $\square$

**Proof of Corollary 1.5.** The integrability of the characteristic subspaces of $A$ amounts to their involutivity, through the Frobenius criterion. In turn this is implied by the nullity of $N_A$. First, let us build integral coordinates on the integral leaf of each characteristic subspace, through the origin. On each characteristic subspace, take $A = S + N$ the “semi-simple + nilpotent” decomposition of $A$.

On the integral leaf of the spaces relative to some real eigenvalue $\lambda$, $S = \lambda \text{Id}$, so applying Theorem 1.3 to the nilpotent part $N$ gives the coordinates.

On the integral leaf of the other spaces, $S = \lambda \text{Id} + \mu J$ for some $J$ with $J^2 = -\text{Id}$. For any commuting endomorphisms $U$ and $V$, $N_{U+V} = N_U + N_V + N_{U,V} + N_{N,U}$, so using Proposition 2.2, we get that, for any $P \in \mathbb{K}[X]$, $N_{P(A)} = 0$ as soon as $N_A = 0$. So here $N_J = N_N = 0$, $J$ is integrable by the integrability condition for complex structures, and $N$, viewed as a complex endomorphism, is integrable by Theorem 1.3.

Finally, take the unique “product” coordinate system extending the coordinates built above, on $\mathbb{R}^d$: it is integral for $A$. Indeed, for each characteristic subspace $E$ of $A$, you may take $Q \in \mathbb{R}[X]$ such that $Q(A)|_E = A_E$ on $E$ if $A_E$ is invertible, $Q(A)|_E = A_E + \text{Id}_E$ on $E$ if $A_E$ is nilpotent, and $Q(A)|_F = 0$ on the sum $F$ of the other characteristic subspaces. To prove that the matrix of $A$ is constant in our coordinates, we must check that $(LV) (X) = 0$, i.e. that $[Y, AX] = 0$, for any coordinate vector fields $X$ tangent to $E$ and $Y$ tangent to $F$. Now, as $N_{Q(A),A} = 0$: $Q(A)[Y, AX] = [Q(A)Y, AX] - A [Q(A)Y, X] + Q(A)A[Y, X] = 0$. As $[Y, AX] \in E$ and as $Q(A)|_E$ is invertible, we are done. $\square$

### 3 Some additional results and examples

**3.1 Proposition** [A higher partial regularity of the coordinates, when $\mathbb{K} = \mathbb{R}$] In restriction to each integral leaf $T_k^\infty$ of $\text{Im} A^k$, for each $k \leq n - 1$, the coordinates built by Theorem 1.3 are of class $C^{r-n+2+k}$, and in general not more. Besides, The coordinates that are constant along the leaves of $T_k^\infty$ are of class $C^{r+2-k}$, and in general not more.

**Proof.** The optimality: “in general not more” follows from Example 3.4 below. To prove the announced regularities, recall that the $(A^a Z_{i,j}^{(n-1)})_{a,i,j}$ are the coordinate vectors finally obtained in Theorem 1.3. In the proof of Theorem 1.3, each vector field $A^a Z_{i,j}^{(n-1)}$ is well-determined modulo $\text{Im} A^{a+k+1}$ from the moment that $A^a Z_{i,j}^{(k)}$ is defined i.e. $A^a Z_{i,j}^{(k)} \equiv$
\[
A^a Z_{ij}^{(k+1)} \equiv \ldots \equiv A^a Z_{ij}^{(n-1)} [\text{Im } A^{a+k+1}]. \text{ In particular:}
\]

(i) The \((A^a Z_{ij}^{(n-1)})_{a \geq n-k-1}\) are well-determined from step \(k\) of the induction \(i.e. A^a Z_{ij}^{(n-1)} = A^a Z_{ij}^{(k+1)}\). But the \(Z_{ij}^{(k)}\) are of class \(C^{r-k}\), so the \((A^a Z_{ij}^{(k)})_{a \geq n-k-1}\) are of class \(C^{r-k}\).

(ii) As the \(Z_{ij}^{(k)}\) are of class \(C^{r-k}\), and as for any \(a \geq 0\), \(A^a Z_{ij}^{(n-1)} = A^a Z_{ij}^{(k-1)} [\text{Im } A^k]\) then the \([A^a Z_{ij}^{(n-1)}] \mod [\text{Im } A^{a-k}]\) are all of class \(C^{r+1-k}\).

Now, the \(A^a Z_{ij}^{(n-1)}\) with \(a \geq n - k - 1\) are the coordinate vectors along the leaves of \(T^{n-k-1}\). So by (i), the coordinates are of class \(C^{r-k+1}\) along those leaves, the first claim.

For the last claim, denote the coordinates given by Theorem 1.3 by \((y_i)_{i=1}^n = ((y_{ij})_{j=1}^n)_{i=1}^n\), on such a way that the plaques of \(T^k\) are the levels of the \(N\)-tuple \((y_i)_{i>n-k}\). Then the \((\frac{\partial}{\partial y_i})_{i>n-k}\) are the \((A^a Z_{ij}^{(n-1)})_{i,j<k}\) and we have to show that the \((y_i)_{i>n-k}\) are of class \(C^{r+2-k}\) for any \(k \geq 1\). Take any coordinate system \((y_i)_{i=1}^n = ((y_{ij})_{j=1}^n)_{i=1}^n\) of class \(C^{r+1}\) such that the plaques of \(T^k\) are the levels of the \(N\)-tuple \((y_i)_{i>n-k}\). As the \((y_i)_{i=1}^n\) share the same property, the matrix \(M = (dy_i(\frac{\partial}{\partial y_j}))_{i,j=1}^{i>n-k\ j \leq n-k}\) is upper block triangular, as well as

\[
\text{Mat} \left( \frac{\partial}{\partial y_j} \right)_{i=1}^{i>n-k\ j \leq n-k} = M^{-1}. \text{ Thus, for each } k \geq 1, \left( dy_i \left( \frac{\partial}{\partial y_j} \right) \right)_{i>n-k\ j \leq n-k} = 0 \text{ and:}
\]

\[
\left. \left( dy_i \left( \frac{\partial}{\partial y_j} \right) \right)_{i,n-k\ j \leq n-k} = \text{Mat} \left( \frac{\partial}{\partial y_j} \right)_{i>n-k} \left( \frac{\partial}{\partial y_i} \text{ mod [Im } A^k] \right)_{i>n-k}^{-1} \right. .
\]

By (ii), the matrix on the right side is of class \(C^{r+1-k}\) so the \((dy_i)_{i>n-k}\) are of class \(C^{r+1-k}\).

Theorem 1.3 Proposition [Uniqueness of the integral coordinates] Let \(A\) be an integrable field of nilpotent endomorphisms. Let \(I\) be the integral foliation of the distribution \(\text{Im } A\), let \(\mathcal{K}^p\) be that of each \(\ker A^p\) for \(p \in \mathbb{N}\) and let \(\pi\) be the projection \(V \rightarrow V/I\). Then a system of integral coordinates for \(A\), in which Mat(\(A\)) is a constant Jordan matrix, is uniquely given by the independent choice of:

- a section \(\sigma\) of \(\pi\), respecting the foliations \(\mathcal{K}^1 \subset \ldots \subset \mathcal{K}^{n-1}\) as defined in Lemma 2.4,
- a system \((\mathcal{Z}_1, \ldots, \mathcal{Z}_n) = ((\mathcal{Z}_{i,j})_{j=1}^{d^1}, \ldots, (\mathcal{Z}_{n,j})_{j=1}^{d^n})\) of coordinates of \(\pi(V)\) respecting the foliations \(\pi(\mathcal{K}^1) \subset \ldots \subset \pi(\mathcal{K}^{n-1})\) i.e. such that each leaf of each \(\pi(\mathcal{K}^p)\) is a level \(\{((\mathcal{Z}_{p+1}), \ldots, (\mathcal{Z}_n)) = \text{cst.}\}\).

More precisely, there is a unique “Jordan” coordinate system \(((z'), (z_1), \ldots, (z_n))\) for \(A\), characterised by the fact that:

- \(((z_1), \ldots, (z_n)) = \pi^* (\mathcal{Z}_1, \ldots, \mathcal{Z}_n)\), (the levels of this \(k\)-tuple are the plaques of \(I\)),
- the coordinates \((z')\) are determined by the fact that \(\{((z') = 0)\}\) is the image of \(\sigma\) and that the \(k\)-tuple \((\frac{\partial}{\partial z_i})_{i=1}^{i=n}\) is equal to that of the non null \((A^a \frac{\partial}{\partial z_{n,j}})_{j=1}^{j=n}\). (The coordinates \((z')\) parametrise the leaves of \(I\)). Explicitly, the fields of coordinate vectors are the \(k\)-tuple:

\[
\left( (A^{n-1} \frac{\partial}{\partial z_{n,j}})_{j=1}^{j=n}, (A^{n-2} \frac{\partial}{\partial z_{n-1,j}})_{j=1}^{j=n-1}, \ldots, (A^0 \frac{\partial}{\partial z_{n-i,j}})_{j=1}^{j=n-i}, (\frac{\partial}{\partial z_{n-i,j}})_{j=1}^{j=n-i} \right)_{i=1}^{i=n}.
\]

Proof. Back to the proof of Theorem 1.3, we have to check that, once \(S\) and the fields \(Z_{i,j}\) along it are chosen, the extension of the \(Z_{i,j}^*\) such that all \(A^a Z_{i,j}\) commute is unique. Suppose that \(Z_{i,j}^*\) is another such extension. As \(Z_{i,j} = Z_{i,j}^*\) along \(S\) and as both fields are pushed by
the flow of fields spanning \( \text{Im} A \), it comes that \( Z_{i,j}' \equiv Z_{i,j} \) \( [\text{Im} A] \) on \( \mathcal{V} \). Now suppose that we have shown that \( Z_{i,j}' \equiv Z_{i,j} \) \( [\text{Im} A^k] \) on \( \mathcal{V} \). Then for all \( a > 0 \), \( A^a Z_{i,j}' \equiv A^a Z_{i,j} \) \( [\text{Im} A^{k+1}] \) on \( \mathcal{V} \). As the \( Z_{i,j}' \) are pushed, from \( \mathcal{S} \), by the flow of the \( A^a Z_{i,j}' \), and the \( Z_{i,j} \) are pushed by that of the \( A^a Z_{i,j} \), the uniqueness of the solutions of O.D.E. gives that \( Z_{i,j}' \equiv Z_{i,j} \) \( [\text{Im} A^{k+1}] \) on \( \mathcal{V} \). By induction on \( k \), we are done. \( \square \)

3.3 Remark Let us consider the particular case of an endomorphism field \( A \), constant in the natural coordinates of the compact manifold \( T = \mathbb{R}^d / \mathbb{Z}^d \). It follows from Proposition 3.2 that the space of (global) integral coordinates for \( A \) is infinite dimensional. This shows that such coordinates are not the solution of an elliptic problem. Instead, they appear naturally as the solution of a system of O.D.E., with an “initial condition” arbitrarily chosen in some infinite dimensional function space. This holds as soon as the minimal polynomial of \( A \) contains a factor \( (\alpha - p) \). We may check that the system, with the initial conditions, is equivalent to:

\[
\begin{cases}
\text{if:} \\
\forall i, k \in [1, n-2], \ \frac{\partial \alpha_i}{\partial x_k} = \frac{\partial \alpha_{i+1}}{\partial x_{k+1}} \text{ and } \frac{\partial \alpha_i}{\partial x_1} = 0.
\end{cases}
\]

We assume that this condition is satisfied, so Theorem 1.3 applies. Notice that then, the knowledge of \( \alpha_{n-1} \) determines all the other \( \alpha_i \), up to an additive constant. Let us build the integral coordinates \( (y_i)_{i=1}^n \), determined by an arbitrary choice of \( \sigma \) and by the choice \( \tau_n = x_n \) i.e. by \( y_n = x_n \) (see Proposition 3.2). Notice that necessarily, \( y_n = y_n(x_n) \), as the levels of both \( x_n \) and \( y_n \) are the integral leaves of \( \ker A^{n-1} \). So, a reparametrisation \( y_n(x_n) \) of the last coordinate amounts to multiply all the \( \alpha_i \) by \( 1/y_n'(x_n) \), thus if \( \alpha_{n-1} \) cannot be made independent of \( x_n \) by a multiplication by some function of \( x_n \) — we now assume this —, we do not lose any generality by taking directly \( y_n = x_n \). Similarly, a different choice of \( \sigma \) amounts to add to each \( x_i \) with \( i \leq n-1 \), some function \( f(x_n) \). This lets all the \( \frac{\partial \alpha_i}{\partial x_n} \) unchanged and adds some linear combination of them to \( \frac{\partial \alpha_i}{\partial x_n} \). In turn this lets the \( \alpha_i \) unchanged, up to additive constants. So we do not lose generality.

Now the coordinates \( y_i \) are determined by the above initial condition and the system:

\[
M \cdot A(\alpha) \cdot M^{-1} = A(0, \ldots, 0, 1) \quad \text{with } M = \left( \frac{\partial y_i}{\partial x_j} \right)_{i,j=1}^n.
\]

As the \( y_i \) must respect the foliations \( \mathcal{I}^p \), notice that \( y_k = y_k(x_k, \ldots, x_n) \). We let the reader check that the system, with the initial conditions, is equivalent to:

\[
(*) \begin{cases}
\text{if:} \\
y_n = x_n \text{ and for } i \leq n-1, \ y_i = 0 \text{ along } \{x_1 = \ldots = x_{n-1} = 0\}, \\
\frac{\partial y_{n-k}}{\partial x_{n-1}} = \sum_{i=1}^k P_i \frac{\partial y_{n-k+i}}{\partial x_{n-1}} \text{ for all } k \in [1, n-1], \\
\frac{\partial y_i}{\partial x_{n-1}} = \frac{\partial y_{i-1}}{\partial x_{n-1}} \text{ for all } k \in [1, n-2] \text{ and } i \in [2, n-1],
\end{cases}
\]

where the \( P_i \) are the rational fractions in the \( \alpha_i \) inductively defined by: \( P_1 = \frac{1}{\alpha_{n-1}} \) and \( P_i = -\sum_{j=1}^{i-1} \frac{\alpha_{n-1-j} + 1}{\alpha_{n-1}} P_j \). This system is overdetermined but, by Theorem 1.3, and as we
have assumed that \( \mathcal{N}_A = 0 \), we now it is holonomic i.e. it admits a (here unique) solution. This solution is determined by the relation \( y_{n-1} = P_1 = \frac{1}{\alpha_{n-1}} \) and, by induction on \( k \), by the equations, directly given by integration of \((*)\):

\[
(**) \quad y_{n-k} = \int_0^{x_{n-1}} \sum_{i=1}^k (P_i \frac{\partial y_{n-k+i}}{\partial x_n})(x_1, \ldots, x_{n-2}, t, x_n) dt + \sum_{i=2}^{n-1} \int_0^{x_{n-i-1}} \frac{\partial y_{n-k+i-1}}{\partial x_{n-1}}(x_1, \ldots, x_{n-i-1}, t, 0, \ldots, 0, x_n) dt.
\]

We have announced an effective example, so let us provide a simple one. Take \( \alpha_{n-1} = 1/(1 + x_{n-1}\theta(x_n)) \) with \( \theta(t) = t^{r+1} \) for \( t \geq 0 \) and else \( \theta(t) = 0 \). This \( \alpha_{n-1} \) is of class \( C^r \) and not of class \( C^{r+1} \). This gives: \( y_{n-1} = 1 + x_{n-1}\theta(x_n) \) and, by induction left to the reader:

- for \( k \leq r + 1 \), \( y_{n-k} = (1 + \frac{x^k}{2}) \theta^{(k-1)}(x_n) + z_{n-k} \), with \( z_{n-k} \) of class \( C^{r-k+2} \),
- for \( k \geq r + 2 \), \( y_{n-k} \) is not defined.

In Theorem 1.3, we want the \( y_i \) to be of class (at least) \( C^1 \) — else writing \( A \) in them makes no sense — so we must require here that \( y_1 \) is well defined and of class \( C^1 \) i.e. that \( r \geq n - 1 \). Moreover, \( y_1 \) is of class \( C^{r-n+2} \) and not of class \( C^{r-n+3} \), so the regularity given in Theorem 1.3 is optimal. Similarly, the example shows also the optimality of Proposition 3.1.

3.5 Remark We may add that if a vector field \( V \) is of class \( C^s \), its flow \( \Phi^{(s)}_t \) is of class \( C^s \) and, for a generic \( V \), is of not of class \( C^{s+1} \). Thus if \( W \) is another vector field, of class \( C^{s'} \) with \( s' \geq s \), its image \( (\Phi^{(s)}_t)_V \) for \( t \neq 0 \) is of class \( C^{r-1} \) and, for a generic \( V \), is of not of class \( C^s \). Used inductively in the proof of Theorem 1.3, this remark shows that, for a generic field \( A \), the vector fields \( Z^{(k)}_{i,j} \) are of class \( C^{r-k} \) and not more. So for a generic \( A \), the coordinates are not more regular than announced in Theorem 1.3 and Proposition 3.1.

The two little counter-examples 3.6 and 3.7 ensure the independance of both last conditions of Theorem 1.3.

3.6 Example Here is a field \( A \) such that \( \mathcal{N}_A \neq 0 \) and \( \ker A \) is non involutive, with minimal nilpotence index of \( A \) (again 2) and ambient dimension (4). In \( \mathbb{K}^4 \) with \( (x_i)_{i=1}^4 \) define \( A \) by \( A(\frac{\partial}{\partial x_1}) = A(\frac{\partial}{\partial x_2}) = 0 \), \( A(\frac{\partial}{\partial x_3}) = \exp(x_2) \frac{\partial}{\partial x_1} \) and \( A(\frac{\partial}{\partial x_4}) = \frac{\partial}{\partial x_1} \). All \([A^s \frac{\partial}{\partial x_1}, A^b \frac{\partial}{\partial x_2}]\) for \( \{a, b\} \subset \{0, 1\} \) vanish except \([A \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}] = -\exp(x_2) \frac{\partial}{\partial x_1} \), hence \( \mathcal{N}_A \neq 0 \). But \( \ker A = \ker \alpha \) with \( \alpha = dx_4 + x_2 dx_3 \), and \( \alpha \wedge d\alpha = dx_2 \wedge dx_3 \wedge dx_4 \neq 0 \) so \( \ker A \) is not involutive.

3.7 Example Here is a field \( A \) such that \( \mathcal{N}_A \neq 0 \) and \( \ker A \) is involutive, with minimal nilpotence index of \( A \) (again 2) and ambient for it (again 4). Similarly, define \( A \) by \( A(\frac{\partial}{\partial x_3}) = A(\frac{\partial}{\partial x_4}) = 0 \), \( A(\frac{\partial}{\partial x_1}) = \exp(x_2) \frac{\partial}{\partial x_1} \) and \( A(\frac{\partial}{\partial x_2}) = \frac{\partial}{\partial x_1} \). All \([A^s \frac{\partial}{\partial x_1}, A^b \frac{\partial}{\partial x_3}]\) for \( \{a, b\} \subset \{0, 1\} \) vanish except \([A \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}] = [A \frac{\partial}{\partial x_3}, A \frac{\partial}{\partial x_1}] = -\exp(x_2) \frac{\partial}{\partial x_1} \). So \( \mathcal{N}_A \neq 0 \) as \( \mathcal{N}_A(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4}) = -\exp(x_2) \frac{\partial}{\partial x_1} \). But \( \ker A = \ker (dx_3) \cap \ker (dx_4) \) is involutive.

3.8 Remark However, in Theorem 1.3, for some similarity types of endomorphisms \( A \), the second condition may be omitted or relaxed, as it is (partially) implied by the first one. For instance, if \( A \) is cyclic, then for every \( p \), \( \ker A^p = \operatorname{Im} A^{n-p} \) is involutive. More generally, if for some \( p \), \( \dim(\ker A^p/\operatorname{Im} A^{n-p}) = 1 \), then \( \ker A^p \) is involutive. Indeed, take \( (Y_i)_i \) a basis field of \( \operatorname{Im} A^{n-p} \) and \( X \) a field such that \( (X, (Y_i)_i) \) spans \( \ker A^p \). As \( \mathcal{N}_A = \operatorname{Im} A^{n-p} \)
0, \([Y_i, Y_j] \in \text{Im } A^n - p \subset \ker A^p\), besides \([X, X] = 0\). Take \(Z_i\) such that \(Y_i = A^{n-p}Z_i\), then \(A^p[X, Y_i] = -N'_{A^p, A^n} (X, Z_i) + [A^p X, A^{n-p}Z_i] - A^{n-p}[A^p X, Z_i] + A^p[X, Z_i] = 0\) so \([X, Y_i] \in \ker A^p\), thus \(\ker A^p\) is involutive.

3.9 Remark If \(A\) is nilpotent, \(N'_{A} = 0\) does not imply that the \(\ker A^p\) are involutive. It gives however a weaker property: if \(X, Y \in \ker A^p\), then \([X, Y] \in \ker A^{2p}\). Indeed, by Proposition 2.2, \(N'_{A} (X, Y) = 0\), so \(A^{2p} [X, Y] = -[A^p X, A^p Y] + A^p [X, A^p Y] + A^p [A^p X, Y] = 0\).

3.10 Remark In Theorem 1.3, if \(A\) is defined on \(V = \mathbb{K}^d\), the integral coordinates may be in fact built on the whole \(\mathbb{K}^d\). Indeed, Theorem 1.3 builds coordinates on some ball \(B(p, R_p)\), around any point \(p\) of \(V\), with \(R_p\) depending only on the coefficients of the matrix \(A\) around \(p\), through the flows \(\Phi_i\) appearing in the proof of the theorem. So on any precompact set of \(V\), this \(R_p\) is bounded from below by a positive constant. Now take any domain of the type \([-\alpha, \alpha]^d\), on which integral coordinates are defined; by what precedes and by the unicity result 3.2, these coordinates may be extended on some \([-\alpha', \alpha']^d\) with \(\alpha' > \alpha\). We are done.

3.11 Example/Remark A consequence of Corollary 1.5 is that, if \((M, \nabla)\) is a manifold with a torsion free affine connection, any parallel endomorphism field \(A\) on \(M\) is integrable.

More generally, an endomorphism field is integrable if and only if it is parallel for some torsion free affine connection \(\nabla\) (compare [5] Th. 6.1). Indeed, if \(A\) is integrable, define \(\nabla\) by \(\nabla \frac{\partial}{\partial v_i} = 0\) in some integral coordinate system \((v_i)_{i=1}^d\). It is torsion free and immediately \(\nabla A = 0\). Conversely, suppose \(\nabla A = 0\) with \(\nabla U V - \nabla V U = [U, V]\) for all vector fields \(U\) and \(V\). Then \(N'_{A} = 0\) and, by the Frobenius criterion, each distribution \(\ker A^p\) is integrable. Besides, \(\nabla A = 0\) implies that \(A\) has constant invariant factors so Corollary 1.5 applies.

I do not know other significant examples where endomorphism fields satisfying naturally the assumptions of Corollary 1.5 appear.

References


