On the pseudo-Riemannian manifolds whose Ricci tensor is parallel

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Abstract: Ricci-parallel Riemannian manifolds have a diagonal Ricci endomorphism Ric and are therefore, at least locally, a product of Einstein manifolds. This fails in the pseudo-Riemannian case. Using, on the one side, a general result in linear algebra due to Klingenberg (see [Kli54]) and on the other side, a theorem on the holonomy of pseudo-Riemannian manifolds (see [Wu67]), this work classifies the different types of pseudo-Riemannian manifolds whose Ricci tensor is parallel.

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Foreword: The first section discusses de Rham – Wu's theorem, its consequences in our problem, and the reasons why the pseudo-Riemannian case allows specific phenomena. The main theorem, that classifies these phenomena, is stated in section 2 and is proven in sections 3 and 4. Section 5 considers the complex case, i.e. complex manifolds with a complex Riemannian structure, which appears as a special case of the main theorem. Finally, section 6 gives some details on certain low-dimensional cases, on the holonomy decomposition as compared with the Ricci-decomposition, on the number of (linearly independant) metrics with the same connection (hence the same Ricci curvature), together with some symmetric examples.

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Notations: If (M,g) is a Riemannian or pseudo-Riemannian manifold, R is its (3,1)-curvature tensor and ric its Ricci tensor, *i.e.* ric is a bilinear symmetric form on each tangent space defined by $\mathrm{ric} : u, v \mapsto \mathrm{tr}\, R(u,.)v$. We will also denote by $< .,. >$ the metric g, and by Ric the g-selfadjoint endomorphism induced by ric, *i.e.* the endomorphism such that $\mathrm{ric}(.,.) =< \mathrm{Ric}\,.,. >$. We denote by H the restricted holonomy group of M, simply called "holonomy group" here, and classical Lie algebras by old german letters: $\mathfrak{o}$, $\mathfrak{so}$.

1 Motivation : a look at the Riemannian case

The main tool of this work is a theorem by Wu, linking decomposition of the holonomy and Riemannian products (see [Wu67]). This is a generalization to pseudo-Riemannian geometry of a well-known theorem due to de Rham. We give in this section only a part of the result (existence theorem), and postpone to section 6.2 all questions about uniqueness.

Theorem (de Rham – Wu) *Let (M,g) be a complete, simply connected pseudo-Riemannian manifold and $x \in M$. Suppose that $T_xM = E_1 \overset{\perp}{\oplus} E_2$ is a holonomy stable decomposition of T_xM. Then there is a unique couple $((M_1, g_{|M_1}), (M_2, g_{|M_2}))$ of submanifolds of M containing x and a unique isometry f such that:*

- $T_xM_1 = E_1$ *and* $T_xM_2 = E_2$,
- $M_1 \times M_2 \xrightarrow{f} M$,
- $f_{|M_1 \times \{x\}} = \mathrm{Id}_{M_1}$ *and* $f_{|\{x\} \times M_2} = \mathrm{Id}_{M_2}$.

Remarks: 1) Here " $M_1 \times M_2$ " stands for the Riemannian product of M_1 and M_2.
2) By hypothesis, E_1 and E_2 are holonomy-stable, mutually orthogonal. So there exist Σ_1 and Σ_2 two mutually orthogonal parallel distributions equal to E_1 and E_2 at x, respectively. M_1 and M_2 are the maximal integral manifolds of Σ_1 and Σ_2, containing x. They are g-non-degenerate. The other maximal integral manifolds of Σ_1 and Σ_2 are isometric to M_1 and M_2, so M_1 and M_2 do not depend of the choice of x, up to isometry.
3) Without requiring the completeness and the simple connectedness, the same result holds with a local isometry.

Corollary *If (M,g) is a complete, simply connected Riemannian manifold with a parallel Ricci tensor* ric*, then M splits canonically into a Riemannian product of Einstein manifolds. That is to say, there is a unique sequence of reals $(\lambda_i)_{i=1}^n$, a unique sequence of Riemannian Einstein manifolds $(M_i)_{i=1}^n$, [such that $\mathrm{ric} = \lambda_i g$ on each M_i], and an isometry f mapping the Riemannian product $\prod_{i=1}^n M_i$ onto M. That isometry f is unique up to composition with a product of isometries of each factor M_i.*

In particular, simply connected Riemannian symmetric spaces are all a canonical product of Einstein factors. Once again, without completeness and simple connectedness, these results hold locally.

The above corollary is a classical and immediate consequence of de Rham – Wu's theorem. Nevertheless, in view of future comparison, we recall the

Proof of the corollary: Ric is g-selfadjoint and g is positive definite, so Ric is diagonalizable. Let us denote by $E_{i,x} = \ker(\mathrm{Ric} - \lambda_i \,\mathrm{Id})$ the eigenspaces of Ric at a point x, then we have an orthogonal product:

$$T_xM = \overset{\perp}{\underset{i}{\oplus}} E_{i,x}.$$

By assumption now, ric is parallel, g too, and thus Ric and the $E_{i,x}$ are parallel. The decomposition of each T_xM is thus holonomy-stable. As it is also orthogonal, de Rham – Wu theorem gives an isometry $f : M \to \prod_i M_i$, where the M_i are maximal integral submanifolds of the parallel distributions generated by the $E_{i,x}$. $\mathrm{Ric}_{|M_i}$ has the only eigenvalue λ_i, and is diagonalizable, thus $\mathrm{ric}_{|M_i} = \lambda . g_{|M_i}$, *i.e.* the M_i are Einstein. The decomposition $T_xM = \oplus E_{i,x}$ is canonical, so the corresponding values of λ_i and the associated M_i too (up to isometry).

The uniqueness of f is not a deep result here but requires a few technical lines. Let us suppose f and h are two isometries mapping $\prod_i M_i$ onto M; let us denote by a the n-uple $(a_i)_{i=1}^n = f^{-1}(x)$ and by b the n-uple $(b_i)_{i=1}^n = h^{-1}(x)$. By definition, $T_a(\prod_i M_i) = \oplus_i T_{a_i} M_i$ is the decomposition of $T_a(\prod_i M_i)$ into the eigenspaces of its Ricci operator. And similarly with h and b. So, necessarily: $\forall i$, $df(T_{a_i} M_i) = E_{i,x}$ and $\forall i$, $dh(T_{b_i} M_i) = E_{i,x}$. Now, for each i:

- $\{a_1\} \times \ldots \times \{a_{i-1}\} \times M_i \times \{a_{i+1}\} \times \ldots \times \{a_n\}$ is by definition the maximal integral submanifold of the parallel distributions generated by $T_{a_i} M_i$,
- M_i is by definition the maximal integral manifold of the parallel distribution generated by $E_{i,x}$.

Since f is an isometry, necessarily, $f(\{a_1\} \times \ldots \times \{a_{i-1}\} \times M_i \times \{a_{i+1}\} \times \ldots \times \{a_n\}) = M_i$, and similarly with h: $h(\{b_1\} \times \ldots \times \{b_{i-1}\} \times M_i \times \{b_{i+1}\} \times \ldots \times \{b_n\}) = M_i$. So, in restriction to the factor M_i, $h^{-1} \circ f$ is an isometry ϕ_i of this factor (mapping b_i on a_i). Finally, $f = h \circ (\phi_1, \ldots, \phi_n)$. □

Notice that the above corollary is not true in the pseudo-Riemannian case.

Indeed, Ric is still a g-selfadjoint operator, but g is no longer positive definite, so Ric is no longer necessarily diagonalizable. In particular, it may have pairs of non-real complex conjugate eigenvalues and it may have a non-trivial nilpotent part. Now [Kli54], [this thesis, next chapter] or [BBB] gives a complete description of what may be an endomorphism S of a vector space [on any field], when S is selfadjoint with respect to a non-degenerate bilinear form h. It gives a normal form for such an S, which is a particular Jordan form. When the field is $\mathbb{R}$, the signature of h and the nilpotence index of the nilpotent part of S are linked, but both obstructions for S to be diagonalizable may occur: non-real eigenvalues and a non-trivial nilpotent part. So a Ricci-parallel pseudo-Riemannian manifold is *a priori* quite different from a product of Einstein manifolds. For example, there exists some pseudo-Riemannian symmetric spaces with a Ricci endomorphism satisfying $\mathrm{Ric} \neq 0$, $\mathrm{Ric}^2 = 0$.

However, some properties of the holonomy group and some consequences of [Kli54] prevent Ric from being too complicated. A good part of the Riemannian result holds again: Einstein factors appear in the Riemannian product, and two other new types of factors. The main theorem below explains it in terms of (conditions on) the minimal polynomial of the operator Ric. We recall that the minimal polynomial is the unique unitary polynomial generating the ideal of all polynomials which annihilates the operator Ric. Notice that in the Riemannian case, Ric is diagonalizable, so in these terms, the corollary above is the geometric counterpart of the fact that the minimal polynomial of Ric is a product of mutually prime polynomials which are irreducible of degree one. In the pseudo-Riemannian case, the minimal polynomial of Ric may have irreducible factors of degree two and also there may be one irreducible factor to the power two. After the proof of the theorem, we will give in section 5 a few more explanations of one of the new factor types and make a few remarks on the low-dimensional cases in section 6.

2 The main result in the pseudo-Riemannian case

The Main Theorem *Let (M, g) be a pseudo-Riemannian manifold with a parallel Ricci tensor* ric, *and let μ be the minimal polynomial of* Ric. *Then:*

(i) $\mu = \prod_i P_i$ *where:*

- $\forall i \neq j$, $P_i \wedge P_j = 1$, *[i.e.* P_i *and* P_j *are mutually prime],*
- $\forall i$, P_i *is irreducible or* $P_i = X^2$.

(ii) *There is a canonical family* $(M_i)_i$ *of pseudo-Riemannian manifolds such that the minimal polynomial of* $\mathrm{Ric}_i = \mathrm{Ric}_{M_i}$ *on each* M_i *is* P_i*, and a local isometry* f *mapping the Riemannian product* $\prod_i M_i$ *onto* M*.* f *is unique up to composition with a product of isometries of each factor* M_i*. If* M *is complete and simply connected,* f *is an isometry.*

That is to say, M splits canonically into a Riemannian product, with factors M_i of one of the four following types — we denote by P_i the minimal polynomial of Ric_i, the Ricci endomorphism of M_i—:

- if $P_i = (X - \alpha_i)^k$ ($\alpha_i \neq 0$), then $k = 1$, *i.e.* M_i is Einstein,
- if $P_i = X^k$, then $k \leq 2$, so
  - ⋆ either $k = 1$, *i.e.* M_i is Ricci-flat, [which is a particular case of Einstein],
  - ⋆ or Ric_i is nilpotent of index 2,
- if $P_i = (X^2 + p_i X + q_i)^k$ (power of an irreducible), then $k = 1$, so Ric_i has no nilpotent part but is not diagonalizable on $\mathbb{R}$. We will see in section 5 that M_i is then a complex Riemannian manifold, which is Einstein for this structure.

The last two types do not appear in the Riemannian case.

Warning: the obtained decomposition is not the holonomy decomposition (see section 6.2).

3 Two algebraic lemmas

Lemma 1 *Let* E *be a real or complex vector space endowed with a non-degenerate bilinear form* $g = <.,.>$*. Let* D *be a totally isotropic subspace of* E*. Let* R *be a* $(3,1)$*-tensor with the algebraic properties of a curvature tensor,* i.e. *:*

- $R(x, y) = -R(y, x)$
- $R(x, y).z + R(y, z).x + R(z, x).y = 0$ *("Bianchi identity"),*
- $< R(x, y).z, t >=< R(z, t).x, y >$ *(which follows from the first two relations).*

By definition, $\mathrm{ric}(u, v) = \mathrm{tr}(x \mapsto R(u, x).v)$*, and* Ric *is the* g*-selfadjoint associated endomorphism,* i.e. *the endomorphism such that* $< \mathrm{Ric}\,.,. >= \mathrm{ric}(.,.)$.
We assume that, for each couple (x, y) *in* E^2*, the endomorphism* $R(x, y)$ *preserves* D*.*
Then we have:

(i) $\forall x \in D$, $\forall y \in D^\perp$, $R(x, y) = 0$
(this is true in particular for x *and* y *in* D *because* $D \subset D^\perp$*),*

(ii) Ric *preserves* D *(i.e.* $\mathrm{ric}(D, D^\perp) = \{0\}$*),*

(iii) *If* $\beta = (e_i)_{i=1}^p$ *is a basis of* D *and* $(e'_i)_{i=1}^p$ *a family such that:* $\forall i, j$, $< e_i, e'_j >= \delta_{i,j}$ *and* $< e'_i, e'_j >= 0$*),*
then $ric(e_i, e'_j) = \mathrm{tr}(R(e'_j, e_i)_{|D})$.

Proof: **(i)** Let us take $x \in D$, $y \in D^{\perp}$ and $z, t \in E$. Then $< R(x,y).z, t > = < R(z,t).x, y >$. By assumption, $R(z,t).x \in D$, thus both terms are zero and **(i)** follows.
(ii) Let us take $(e_i)_{i=1}^p$ a basis of D and $(e'_i)_{i=1}^p$ a "dual" family of $(e_i)_{i=1}^p$ as defined in **(iii)**. Let D' be the vector space generated by the e_i and the e'_i, and $(f_i)_{i=1}^q$ a pseudo-orthonormal basis of $D'^{\perp}$, *i.e.* such that $\forall i, j$, $< f_i, f_j > = \varepsilon_i \delta_{i,j}$, with $\varepsilon_i = \pm 1$. Then, for $i, j \in \{1, .., p\}$, point **(i)** of the Lemma implies:

$$\begin{aligned}
\mathrm{ric}(e_i, e_j) &= \sum_{k=1}^{p} < R(e_i, e_k).e_j, e'_k > + \sum_{k=1}^{q} \varepsilon_k < R(e_i, f_k).e_j, f_k > + \sum_{k=1}^{p} < R(e_i, e'_k).e_j, e_k > \\
&= \sum_{k=1}^{p} < 0.e_j, e'_k > + \sum_{k=1}^{q} \varepsilon_k < 0.e_j, f_k > + \sum_{k=1}^{p} < R(e_j, e_k).e_i, e'_k > \\
&= 0 + 0 + \sum_{k=1}^{p} < 0.e_i, e'_k > = 0
\end{aligned}$$

and for $i \in \{1, .., p\}$ and $j \in \{1, .., q\}$:

$$\begin{aligned}
\mathrm{ric}(e_i, f_j) &= \sum_{k=1}^{p} < R(e_i, e_k).f_j, e'_k > + \sum_{k=1}^{q} \varepsilon_k < R(e_i, f_k).f_j, f_k > + \sum_{k=1}^{p} < R(e_i, e'_k).f_j, e_k > \\
&= \sum_{k=1}^{p} < 0.f_j, e'_k > + \sum_{k=1}^{q} \varepsilon_k < 0.f_j, f_k > + \sum_{k=1}^{p} < R(f_j, e_k).e_i, e'_k > \\
&= 0 + 0 + \sum_{k=1}^{p} < 0.e_i, e'_k > = 0
\end{aligned}$$

For **(iii)**, we will show a little more.

<u>Claim:</u> $\forall i, j, k, l \in \{1, .., p\}$, $< R(e'_i, e_j).e'_k, e_l > = < R(e'_k, e_j).e'_i, e_l >$.

Using again **(i)**, this implies:

$$\begin{aligned}
\mathrm{ric}(e_i, e'_j) &= \sum_{k=1}^{p} < R(e_i, e_k).e'_j, e'_k > + \sum_{k=1}^{q} \varepsilon_k < R(e_i, f_k).e'_j, f_k > + \sum_{k=1}^{p} < R(e_i, e'_k).e'_j, e_k > \\
&= 0 + 0 - \sum_{k=1}^{p} < R(e'_k, e_i).e'_j, e_k > \\
&= -\sum_{k=1}^{p} < R(e'_j, e_i).e'_k, e_k > = \sum_{k=1}^{p} < R(e'_j, e_i).e_k, e'_k > \\
&= \mathrm{tr}\, R(e'_j, e_i)_{|D}
\end{aligned}$$

The initial claim follows from Bianchi identity:

$$\begin{aligned}
< R(e'_i, e_j).e'_k, e_l > &= - < R(e_j, e'_k).e'_i, e_l > - < R(e'_k, e'_i).e_j, e_l > \\
&= < R(e'_k, e_j).e'_i, e_l > - < R(e_j, e_l).e'_k, e'_i > \\
&= < R(e'_k, e_j).e'_i, e_l > - < 0.e'_k, e'_i > \\
&= < R(e'_k, e_j).e'_i, e_l > .
\end{aligned}$$

□

Lemma 2 *Let (E,g) be a vector space endowed with a non-degenerate symmetric bilinear form $g = <.,.>$, let h be another symmetric bilinear form and let S be the g-selfadjoint endomorphism induced by h: i.e. $h(.,.) = <S.,.>$.*
Let P be the minimal polynomial of S and $\prod_i P_i^{n_i}$ its decomposition into a product of powers of prime polynomials P_i and let $E_i = \ker P_i^{n_i}(S)$ be the characteristic subspaces of S.
Then the E_i are both g- and h-orthogonal.

Proof: Let us take $x \in E_i$, $y \in E_j$ $(i \neq j)$, U and V in $\mathbb{R}[X]$ such that: $UP_i^{n_i} + VP_j^{n_j} = 1$.

$$\begin{aligned}\text{Then: } g(x,y) &= g((UP_i^{n_i} + VP_j^{n_j})(S).x, y) \\ &= g(U(S)P_i^{n_i}(S).x, y) + g(x, V(S)P_j^{n_j}(S).y) \\ &= g(0,y) + g(x,0) \ = \ 0,\end{aligned}$$

$$\begin{aligned}\text{and: } h(x,y) &= g((UP_i^{n_i} + VP_j^{n_j})(S).Sx, y) \\ &= g(SU(S)P_i^{n_i}(S).x, y) + g(x, SV(S)P_j^{n_j}(S).y) \\ &= g(0,y) + g(x,0) \ = \ 0.\end{aligned}$$ □

4 Proof of the theorem

4. a First part

We will first prove the following
Claim: Let $\mu = \prod_i P_i^{n_i}$ be the decomposition of μ into a product of powers of mutually prime irreducible polynomials. Then M is locally isometric to a canonical Riemannian product $\prod_i M_i$, where the minimal polynomial of each Ric_{M_i} is $P_i^{n_i}$. This isometry is global if M is moreover complete, simply connected.

Let us apply Lemma 2 to the tangent space (T_xM, g) at each point x of M, with $h = \mathrm{ric}$. It splits T_xM into a g-orthogonal sum of canonical subspaces: $T_xM = \overset{\perp}{\underset{i}{\oplus}} E_{x,i}$, where the $E_{x,i} = \ker P_i^{n_i}$ are now the characteristic subspaces of Ric at the point x. By assumption, ric is moreover parallel, and therefore Ric and the $E_{x,i}$ too. Thus the obtained decomposition of T_xM is holonomy-stable. By de Rham – Wu theorem, M is locally isometric to the Riemannian product $\prod_i M_i$, where the M_i are submanifolds such that: $\forall x \in M_i$, $T_xM_i = E_{x,i}$ (the M_i are integral submanifolds of the parallel distributions $E_{x,i}$).

By definition of the $E_{x,i}$, $\mathrm{Ric}_i = \mathrm{Ric}_{|M_i} = \mathrm{Ric}_{|E_i}$ has a minimal polynomial of the form $P_i^{n_i}$, P_i prime. The uniqueness of the local isometry $M \simeq \prod_i M_i$ follows from that of the decomposition of $T_xM = \oplus_i E_{x,i}$ and from the explanations detailed in the proof of the corollary of section 1. So the first part is proved.

In the following, we will deal with one of the $(M_i, g_{|M_i})$ and forget the index i. That is to say, we deal with a manifold (M,g), the Ricci endomorphism Ric of which has a minimal polynomial of the form P^n, where P is irreducible.

4. b Second part

We will prove here that $n = 1$, except possibly if $P = X$, in which case $n \leq 2$. Points (i) and (ii) of the theorem follow. Let us begin with a few notations.
• If $\deg P = 1$, *i.e.* $P(X) = (X - \alpha)$, let E be the tangent space T_xM of M at a certain point x, and $T = P(\mathrm{Ric})$. Remark: T is a nilpotent endomorphism.

• If $\deg P = 2$, *i.e.* $P(X) = (X-\alpha)(X-\overline{\alpha})$, $\alpha \in \mathbb{C}\setminus\mathbb{R}$, let F be the complexified space of T_xM. Let us denote again by g, R, ric, and Ric the complexified tensors. We take $E = \ker(\mathrm{Ric} - \alpha \mathrm{Id}_F)^n \subset F$ and $T = \mathrm{Ric} - \alpha \mathrm{Id}_E$ on E. T is nilpotent.

Notice that by Lemma 2, $E^\perp = \ker(\mathrm{Ric} - \overline{\alpha}\mathrm{Id})^n$.

First Claim: if $n > 1$, then $\alpha = 0$. [In particular, if $n > 1$, $\deg P = 1$.]

Let us suppose $n > 1$. Let $D = \operatorname{Im} T^{n-1}$. D is totally isotropic: let $x = T^{n-1}u \in D$, $y = T^{n-1}v \in D$, we have $< x, y >=< T^{n-1}u, T^{n-1}v >=< T^{2n-2}u, v >=< 0, v >= 0$. Indeed, $n > 1$, so $2n-2 \geq n =$ nilpotence index of T. Let $d = \dim D$, $(e_i)_{i=1}^d$ a basis of D, $(e'_i)_{i=1}^d$ a dual family of $(e_i)_{i=1}^d$ [in the sense of lemma 1 (iii)]. Applying Lemma 1, (iii) with E, D and R:

$$\mathrm{ric}(e_i, e'_j) = \mathrm{tr}(R(e'_j, e_i)_{|D})$$

But D is endowed with a non-degenerate bilinear form: let $g_1 : (x, y) \mapsto < T^{n-1}x, y >$; $\ker g_1 = \ker T^{n-1}$, so g_1 is well defined, non-degenerate on the quotient space $E/\ker T^{n-1}$. Now, $T^{n-1} : E/\ker T^{n-1} \to D = \operatorname{Im} T^{n-1}$ is an isomorphism, and so the following formula :

$$g'_1(T^{n-1}\cdot, T^{n-1}\cdot) = g_1(\cdot\,,\cdot)$$

defines a non-degenerate symmetric bilinear form g'_1 on D. Now g_1 is holonomy stable, and ric is parallel by hypothesis. As a consequence, all the $R(x, y)$ are g_1- and ric-antiselfadjoint, whether we are dealing with the complexified version of R or not. And so the same statement holds for g'_1 on D. Therefore $\forall x, y \in E, R(x, y)_{|D} \in \mathfrak{o}(g'_1)$. But as g'_1 is non-degenerate, the elements of $\mathfrak{o}(g'_1)$ are trace free. So, $\forall x, y \in E$, $\mathrm{tr}(R(x, y)_{|D}) = 0$.

Now, from lemma 1,(ii), it follows that: $\forall x \in D$, $\forall y \in D^\perp, \mathrm{ric}(x, y) = 0$. Summing up the results,

- $\forall i \in \{1, .., d\}$, $\forall y \in D^\perp$, $\mathrm{ric}(e_i, y) = 0$
- $\forall i \in \{1, .., d\}$, $\forall j \in \{1, .., d\}$, $\mathrm{ric}(e_i, e'_j) = 0$

But $\operatorname{codim} D^\perp = \dim D = d$, so we have reached: $\forall i \in \{1, .., d\}$, $\forall x \in E$, $\mathrm{ric}(e_i, x) = 0$. That is to say, $\{0\} \subsetneq D \subset \ker(\mathrm{ric})$. So ric is degenerate, and $\alpha = 0$.

Second Claim: If $\alpha = 0$, that is if $P = X^n$, then $n \leq 2$.

Here $T = \mathrm{Ric}$. We cut again the proof into two steps: $n \leq 3$, and then $n \neq 3$.

$n \leq 3$: Let us suppose $n \geq 4$, and take $D = \operatorname{Im}\mathrm{Ric}^{n-2}$. We use exactly the same arguments. D is totally isotropic: let $x = \mathrm{Ric}^{n-2}u \in D, y = \mathrm{Ric}^{n-2}v \in D$, we have:

$$< x, y > = < \mathrm{Ric}^{n-2}u, \mathrm{Ric}^{n-2}v > = < \mathrm{Ric}^{2n-4}u, v > = < 0, v >= 0.$$

Indeed, $n \geq 4$, so $2n-4 \geq n =$ nilpotence index of T. Let us define g_2 by the formula: $g_2(x, y) = < \mathrm{Ric}^{n-2}x, y >$. Then g_2 is well defined, non-degenerate on $E/\ker\mathrm{Ric}^{n-2}$. As above, $\mathrm{Ric}^{n-2} : E/\ker\mathrm{Ric}^{n-2} \to D = \operatorname{Im}\mathrm{Ric}^{n-2}$ is an isomorphism, and so the following formula :

$$g'_2(\mathrm{Ric}^{n-2}\cdot, \mathrm{Ric}^{n-2}\cdot) = g_2(\cdot\,,\cdot)$$

defines a non-degenerate symmetric bilinear form g'_2 on D. By the same way: $\forall x, y \in E$, $R(x, y)_{|D} \in \mathfrak{o}(g'_2)$, g'_2 is non-degenerate on D, and so the $R(x, y)$ are tracefree.

By lemma one and the same remarks, this implies that D is Ricci-flat, *i.e.* that $\mathrm{Im}\,\mathrm{Ric}^{n-1} = \mathrm{Ric}(D) = \{0\}$. But the nilpotence index of Ric is n, so it is impossible. So $n \leq 3$.

$\underline{n \neq 3}$: Let us now suppose $n = 3$. $\mathrm{Im}\,\mathrm{Ric}^{n-2}$ is no longer totally isotropic. Let us now take $D = \mathrm{Im}\,\mathrm{Ric} \cap \ker \mathrm{Ric}$, which is totally isotropic.

Now is involved the purely algebraic result which was told about at the beginning. In our case, [Kli54], [this thesis, next chapter] or [BBB] give the existence of some canonical basis β which is both:
(1) Jordan for Ric, i.e.:

$$\mathrm{Mat}_\beta(\mathrm{Ric}) = \begin{array}{c} \begin{array}{cccccc} \overbrace{\quad}^{E_3} & \overbrace{\qquad}^{\mathrm{Ric}(E_2)} & \overbrace{\quad}^{E_2} & \overbrace{\qquad}^{\mathrm{Ric}^2(E_1)} & \overbrace{\qquad}^{\mathrm{Ric}(E_1)} & \overbrace{\quad}^{E_1} \end{array} \\ \left(\begin{array}{c|cc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{n_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}$$

where $n_1 = \dim E_1 = \dim\,[\ker \mathrm{Ric} \cap \mathrm{Im}\,\mathrm{Ric}^2]$,
$n_2 = \dim E_2 = \dim\,[(\ker \mathrm{Ric} \cap \mathrm{Im}\,\mathrm{Ric})/(\ker \mathrm{Ric}\ \cap \mathrm{Im}\,\mathrm{Ric}^2)]$ and
$n_3 = \dim E_3 = \dim\,[\ker \mathrm{Ric}\,/(\ker \mathrm{Ric} \cap \mathrm{Im}\,\mathrm{Ric})]$.

(2) "canonical" for g, in the following sense:

$$\mathrm{Mat}_\beta(g) = \left(\begin{array}{c|cc|ccc} I_{r_3,s_3} & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & I_{r_2,s_2} & 0 & 0 & 0 \\ 0 & I_{r_2,s_2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & I_{r_1,s_1} \\ 0 & 0 & 0 & 0 & I_{r_1,s_1} & 0 \\ 0 & 0 & 0 & I_{r_1,s_1} & 0 & 0 \end{array}\right)$$

where $I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$ and the r_i and s_i are some integers satisfying $r_i + s_i = n_i$. We do not need more here, but to be more precise, the three couples (r_i, n_i) characterize the couple (g, ric) of bilinear forms on E, up to pull back by a isomorphism U of E: $g \rightsquigarrow g(U.,U.)$, $\mathrm{ric} \rightsquigarrow \mathrm{ric}(U.,U.)$.

Remark: To avoid a misunderstanding, it is important to note that the subspaces E_1, E_2 and E_3 are **not** canonical — but however, are not **any** Jordan subspaces.

Let us denote by $(e_i^k)_{i=1}^{n_k}$ the vectors of β which generate each E_k, $k = 1, 2$ or 3. So $\beta = ((e_i^3)_{i=1}^{n_3}, (\mathrm{Ric}\, e_i^2)_{i=1}^{n_2}, (e_i^2)_{i=1}^{n_2}, (\mathrm{Ric}^2\, e_i^1)_{i=1}^{n_1}, (\mathrm{Ric}\, e_i^1)_{i=1}^{n_1}, (e_i^1)_{i=1}^{n_1})$. Then:

$$\pm 1 = < \operatorname{Ric}^2 e_i^1, e_i^1 > = \operatorname{ric}(\operatorname{Ric} e_i^1, e_i^1) = \operatorname{tr}(v \mapsto R(\operatorname{Ric} e_i^1, v).e_i^1) =$$

$$\sum_{j=1}^{n_1} \pm \underbrace{< R(\operatorname{Ric} e_i^1, \operatorname{Ric}^2 e_j^1).e_i^1, e_j^1 >}_{A_{i,j}} + \sum_{j=1}^{n_1} \pm \underbrace{< R(\operatorname{Ric} e_i^1, \operatorname{Ric} e_j^1).e_i^1, \operatorname{Ric} e_j^1 >}_{B_{i,j}}$$

$$+ \sum_{j=1}^{n_1} \pm \underbrace{< R(\operatorname{Ric} e_i^1, e_j^1).e_i^1, \operatorname{Ric}^2 e_j^1 >}_{C_{i,j}} + \sum_{j=1}^{n_2} \pm \underbrace{< R(\operatorname{Ric} e_i^1, \operatorname{Ric} e_j^2).e_i^1, e_j^2 >}_{D_{i,j}}$$

$$+ \sum_{j=1}^{n_2} \pm \underbrace{< R(\operatorname{Ric} e_i^1, e_j^2).e_i^1, \operatorname{Ric} e_j^2 >}_{E_{i,j}} + \sum_{j=1}^{n_3} \pm \underbrace{< R(\operatorname{Ric} e_i^1, e_j^3).e_i^1, e_j^3 >}_{F_{i,j}}.$$

But all terms are zero. Indeed:

- From Lemma 1, **(i)**:
  - $\forall i, j \leq n_1$, $\operatorname{Ric}^2 e_j^1 \in D$ and $\operatorname{Ric} e_i^1 \in D^\perp$, so $R(\operatorname{Ric} e_i^1, \operatorname{Ric}^2 e_j^1) = 0$,
  - $\forall i \leq n_1$, $\forall j \leq n_2$, $\operatorname{Ric} e_j^2 \in D$ and $\operatorname{Ric} e_i^1 \in D^\perp$, so $R(\operatorname{Ric} e_i^1, \operatorname{Ric} e_j^2) = 0$
  - $\forall i \leq n_1$, $\forall j \leq n_3$, $R(\operatorname{Ric} e_i^1, e_j^3) = 0$,

  thus all the $A_{i,j}$, $D_{i,j}$ and $E_{i,j}$ are zero.

- Using Bianchi identity:

$$\begin{aligned} C_{i,j} &=< R(\operatorname{Ric} e_i^1, e_j^1).e_i^1, \operatorname{Ric}^2 e_j^1 > \\ &= - < R(e_j^1, e_i^1). \operatorname{Ric} e_i^1, \operatorname{Ric}^2 e_j^1 > - < R(e_i^1, \operatorname{Ric} e_i^1).e_j^1, \operatorname{Ric}^2 e_j^1 > \\ &= - < \underbrace{R(\operatorname{Ric} e_i^1, \operatorname{Ric}^2 e_j^1)}_{=0 \text{ by lemma 1, } \mathbf{(i)}}.e_j^1, e_i^1 > + < R(e_i^1, \operatorname{Ric} e_i^1). \operatorname{Ric}^2 e_j^1, e_j^1 > \\ &=< R(e_i^1, \operatorname{Ric} e_i^1). \operatorname{Ric}^2 e_j^1, e_j^1 > \end{aligned}$$

  As ric is parallel we have: $\forall x, y$, $\operatorname{ric}(R(x,y)\,.\,,\,.\,) = -\operatorname{ric}(\,.\,, R(x,y)\,.\,)$, and thus its g-selfadjoint associated endomorphism Ric commutes with all the $R(x,y)$. So:

$$\begin{aligned} C_{i,j} &=< R(e_i^1, \operatorname{Ric} e_i^1). \operatorname{Ric}^2 e_j^1, e_j^1 > \\ &=< \operatorname{Ric}^2 R(e_i^1, \operatorname{Ric} e_i^1).e_j^1, e_j^1 > \\ &=< R(e_i^1, \operatorname{Ric} e_i^1).e_j^1, \operatorname{Ric}^2 e_j^1 > \\ &= - < e_j^1, R(e_i^1, \operatorname{Ric} e_i^1). \operatorname{Ric}^2 e_j^1 > \\ &= -C_{i,j} \end{aligned}$$

  and $\forall i, j, C_{i,j} = 0$

- With the previous remark on the commutation of Ric:

$$\begin{aligned} F_{i,j} &=< R(\operatorname{Ric} e_i^1, e_j^3).e_i^1, e_j^3 > \\ &=< R(e_i^1, e_j^3). \operatorname{Ric} e_i^1, e_j^3 > \\ &=< \operatorname{Ric}(R(e_i^1, e_j^3).e_i^1), e_j^3 > \\ &=< R(e_i^1, e_j^3).e_i^1, \operatorname{Ric} e_j^3 > \\ &=< R(e_i^1, e_j^3).e_i^1, 0 > = 0. \end{aligned}$$

- With the same remark, and commuting this time the role of i and j:

$$\begin{aligned} B_{i,j} &=< R(\operatorname{Ric} e_i^1, \operatorname{Ric} e_j^1).e_i^1, \operatorname{Ric} e_j^1 > \\ &=< R(e_i^1, \operatorname{Ric} e_j^1). \operatorname{Ric} e_i^1, \operatorname{Ric} e_j^1 > \\ &=< R(e_i^1, \operatorname{Ric} e_j^1). \operatorname{Ric}^2 e_i^1, e_j^1 > \\ &= C_{j,i} = 0 \end{aligned}$$

Thus the case $n = 3$ is impossible, which completes the proof. □

5 The case where Ric has two complex conjugate eigenvalues

This part is devoted to the fourth case announced in section 2: the case where the minimal polynomial P of Ric is irreducible of degree 2. So, Ric has no nilpotent part but is not diagonalizable on $\mathbb{R}$.

Proposition 1 *Let (M, g) be a pseudo-Riemannian manifold whose Ricci tensor is parallel and such that the minimal polynomial of* Ric *is of the form: $[(X - \alpha)(X - \overline{\alpha})]^k$, $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Let us denote by D the Levi-Civita connection of M. Then $k = 1$, and M, endowed with the same connection D, admits a complex Riemannian structure, the real part of which is the original metric. That is to say, M admits:*

- *For each $x \in M$, an endomorphism $J \in End(T_xM)$, such that $J^2 = -\operatorname{Id}$, integrable,*
- *h a complex non-degenerate bilinear symmetric form (warning: not a hermitian product) such that D is its Levi-Civita connection (*i.e. *$Dh = 0$), and such that $g = \Re h$.*

Moreover, $(M_{\mathbb{C}}, h)$ is Einstein with factor $\frac{\alpha}{2}$ (or $\frac{\overline{\alpha}}{2}$, it depends on the choice of h), i.e. $\operatorname{ric}_{\mathbb{C}} = \frac{\alpha}{2} h$. *Finally, $\Re(\alpha h) = \operatorname{ric}_{\mathbb{R}}$, so M is Einstein for this —real— metric $\Re(\alpha h)$, that admits the same connection D, so the same R and the same* ric*, as g.*

Remark : With such a minimal polynomial for Ric, M is of even dimension $2n$ and any real metric giving this ric has signature (n, n).

Proof: Let $\lambda, \mu \in \mathbb{R}$ be such that $\alpha = \lambda + i\mu$. $\alpha \notin \mathbb{R}$, so $\mu \neq 0$. Let $J = \frac{1}{\mu}(\operatorname{Ric} - \lambda \operatorname{Id})$. Then:

$$\begin{aligned} J^2 &= \frac{1}{\mu^2}((\operatorname{Ric} - \alpha \operatorname{Id}) + i\mu \operatorname{Id})((\operatorname{Ric} - \overline{\alpha} \operatorname{Id}) - i\mu \operatorname{Id}) \\ &= \frac{1}{\mu^2}((\operatorname{Ric} - \alpha \operatorname{Id})(\operatorname{Ric} - \overline{\alpha} \operatorname{Id}) + i\mu\alpha \operatorname{Id} - i\mu\overline{\alpha} \operatorname{Id} + \mu^2 \operatorname{Id}) \\ &= \frac{1}{\mu^2}(0 + 2\Re(i\mu\alpha) \operatorname{Id} + \mu^2 \operatorname{Id}) \\ &= \frac{1}{\mu^2}(-\mu^2 \operatorname{Id}) \\ &= -\operatorname{Id}. \end{aligned}$$

Since Ric is parallel and J is a polynomial in Ric, J is parallel. By Newlander and Nirenberg theorem [NN57], the almost-complex structure induced by J is thus complex.

Let now x be a point in M, let us define on T_xM the following complex bilinear form: $h : u, v \mapsto h(u, v) = g(u, v) - ig(u, J.v)$. J is g-selfadjoint, so $h(u, v) = h(v, u)$. One easily verifies that: $\forall \gamma, \delta \in \mathbb{R}$, $h((\gamma + i\delta)u, v) = \gamma h(u, v) + i\delta h(u, v)$. g is non-degenerate, thus so is h; J being parallel, $Dh = 0$, so D is the Levi-Civita connection of h. By definition, $g = \Re h$.

Useful remark: If $A \in \mathrm{End}_{\mathbb{R}}(T_xM)$ commutes with J, *i.e.* is in $\mathrm{End}_{\mathbb{C}}(T_xM)$ too, then:

$$\operatorname{tr}_{\mathbb{C}} A = \frac{1}{2}(\operatorname{tr}_{\mathbb{R}} A - i \operatorname{tr}_{\mathbb{R}}(J.A)).$$

As a consequence, $\mathrm{ric}_{\mathbb{C}}(u, v) = \operatorname{tr}_{\mathbb{C}} R(u, .)v = \frac{1}{2}[\operatorname{tr}_{\mathbb{R}} R(u, .)v - i \operatorname{tr}_{\mathbb{R}}(J.R(u, .)v)]$. J is parallel, so commutes with all the $R(a, b)$, and so $\operatorname{tr}_{\mathbb{R}}(J.R(u, .)v) = \operatorname{tr}_{\mathbb{R}}(R(u, .)Jv)$. Finally,

$$\mathrm{ric}_{\mathbb{C}}(u, v) = \frac{1}{2}(\mathrm{ric}_{\mathbb{R}}(u, v) - i\, \mathrm{ric}_{\mathbb{R}}(u, J.v)).$$

>From the main theorem one knows that $k = 1$ so Ric is semi-simple and one may find a real basis $\beta = (e_1, e_1', \ldots, e_n, e_n')$ of T_xM such that:

$$\mathrm{Mat}_\beta(\mathrm{Ric}) = \begin{pmatrix} A & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A \end{pmatrix}, \ \mathrm{Mat}_\beta(g) = \begin{pmatrix} K_2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & K_2 \end{pmatrix}$$

$$\text{and thus } \mathrm{Mat}_\beta(\mathrm{ric}) = \begin{pmatrix} A' & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A' \end{pmatrix},$$

where $K_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $A = \begin{pmatrix} \lambda & -\mu \\ \mu & \lambda \end{pmatrix}$ and $A' = \begin{pmatrix} -\mu & \lambda \\ \lambda & \mu \end{pmatrix}$.

Remark: with that basis, $\mathrm{Mat}_\beta(J) = \begin{pmatrix} J_2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & J_2 \end{pmatrix}$, where $J_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Let $\widetilde{\beta} = \frac{1}{\sqrt{2}}((e_1 + e_1'), \ldots, (e_n + e_n'))$. That $\widetilde{\beta}$ is a basis of T_xM as $\mathbb{C}$-vectorspace and, using the formula linking $\mathrm{ric}_{\mathbb{C}}$ and $\mathrm{ric}_{\mathbb{R}}$: $\mathrm{Mat}_{\widetilde{\beta}}(\mathrm{ric}_{\mathbb{C}}) = \frac{\alpha}{2}\,\mathrm{Id}$. As $\mathrm{Mat}_{\widetilde{\beta}}(h) = \mathrm{Id}$, it follows: $\mathrm{ric}_{\mathbb{C}} = \frac{\alpha}{2}h$. Thus, $(M_{\mathbb{C}}, h)$ is Einstein.

Now, if we let $g' = \Re(\alpha h)$, $\mathrm{ric}_{\mathbb{R}} = 2\Re\, \mathrm{ric}_{\mathbb{C}} = 2\Re(\frac{\alpha}{2}h) = g'$, so (M, g') is Einstein. Indeed, g' induces the same connection D as g, so the same Ricci tensor. $\square$

6 Further remarks

6. a A few words about low dimensions

In low dimensions, we have more precise results.

Proposition 2 *Let (M, g) be an indecomposable pseudo-Riemannian manifold of dimension n whose Ricci tensor is parallel.*

- *If $n \leq 3$, then M has constant curvature.*

- *If $n \le 6$, and if the minimal polynomial of* Ric *is a power of an irreducible polynomial of degree 2, then M is complex, locally symmetric. As complex manifold, it has constant curvature.*

Proof: This result is only a technical remark, short to explain with the following tools, which are classical and can be found in for example in [Bes87], pp.47 and 49:

Definition *If a and b are two symmetric bilinear forms, their Kulkarni-Nomizu product is the following 4-tensor:*

$$(a \owedge b)(x,y,z,t) = a(x,z)b(y,t) + a(y,t)b(x,z) - a(x,t)b(y,z) - a(y,z)b(x,t).$$

Lemma 3 *Let (M,g) be a Riemannian, pseudo-Riemannian or complex Riemannian manifold with dimension n. Let us denote by $s = \operatorname{tr}\operatorname{Ric}$ its scalar curvature. As in [Bes87], let us denote (in this lemma) by R the (4,0) tensor associated to the curvature.*
If $n = 2$, then $R = \frac{s}{4} g \owedge g$ and $\operatorname{ric} = \frac{s}{2} g$.
If $n = 3$, then $R = \frac{s}{12} g \owedge g + (\operatorname{ric} - \frac{s}{3} g) \owedge g$.

The first point of the proposition follows directly from the lemma: if $n \le 3$, M is Ricci-parallel $\Leftrightarrow D\operatorname{ric} = 0 \Rightarrow DR = 0 \Leftrightarrow M$ is locally symmetric. M is irreducible, it has then moreover constant curvature. In the case of the third point, proposition 1 and the previous point imply that M is a complex manifold of dimension $p = 2$ or $p = 3$ ($n = 2p$), with a complex Riemannian structure h. Applying again lemma 3, we conclude that M is locally symmetric and that, as complex manifold, it has constant curvature. By proposition 1, the real manifold M is moreover Einstein for a well-chosen metric. □

Examples All Ricci-parallel pseudo-Riemannian manifolds of dimension 6 or less, such that Ric is non-degenerate and non-diagonalizable over $\mathbb{R}$, are complex symmetric spaces. So Berger's classification of the irreducible symmetric spaces (see [Ber57]) give them; they are:

$$SL_2(\mathbb{C}), \text{ of complex dimension } 3,$$

$$SL_2(\mathbb{C})/\mathbb{C}^\star, \text{ of complex dimension } 2.$$

As complex manifolds, they are Einstein for their "natural" metric, respectively h and $\widetilde{h}$. Here h is given by the complex Killing form of $\mathfrak{sl}_2(\mathbb{C})$ [up to some scaling] and $\widetilde{h}$ is deduced from h on the quotient $SL_2(\mathbb{C})/\mathbb{C}^\star$. As real manifolds, they are Einstein for the real metrics $g = \Re h$ and $\widetilde{g} = \Re \widetilde{h}$ respectively, and not Einstein for the $g_\alpha = \Re(\alpha h)$ and $\widetilde{g}_\alpha = \Re(\alpha \widetilde{h})$ respectively, when $\alpha \in \mathbb{C} \setminus \mathbb{R}$.

6. b Ricci decomposition and holonomy decomposition

The (local or global) product decomposition in Main Theorem is unique, but it may be pursued. More precisely, each factor may be a (local or global) Riemannian product of pseudo-Riemannian manifolds, and decomposing in that way, we get at the end only indecomposable factors. This final decomposition is the holonomy decomposition, and it is not unique in general, as indicated in Wu's paper (see [Wu67], theorem 5 in Appendix I, p.390).

On each factor on which Ric is non-degenerate, the decomposition is unique up to ordering of the factors (one may switch isometric factors).

Now, the factor M^0 on which $\mathrm{Ric}^2 = 0$ may have a further holonomy decomposition too: $M^0 = M_0^0 \times \prod_{i\in I} M_i^0 \times \prod_{j\in J} M_j^0$, with a flat M_0^0, irreducible M_i^0 and indecomposable-reducible M_j^0. Then, if M_0^0 is non-trivial and if J is not empty, the way in which the factor M_0^0 may be inbedded in M^0 is not unique. On the other hand, if M_0^0 is only a point or if J is empty (i.e. there is no indecomposable reducible factor), then the holonomy decomposition is unique up to ordering.

6. c Families of metrics with the same connection

Proposition 3 *Let M be an indecomposable pseudo-Riemannian manifold with a parallel, non-degenerate Ricci tensor, and* D *its covariant derivative. Then, the metrics over M associated to the same* D *are:*

- *either* $\{\lambda\,\mathrm{ric}\,/\lambda \in \mathbb{R}^*\}$*,*
- *or* $\{\lambda\,\mathrm{ric} + \mu g/(\lambda, \mu) \in \mathbb{R}^2\backslash(0,0)\}$*, where g is a metric such that the minimal polynomial of* Ric_g *is irreducible of degree 2 (and then M admits a unique complex structure, corresponding to this family of real metrics).*

Proof: By hypothesis, g and ric are D-parallel, and D is torsion-free, so D is the Levi-Civita connection of any non-degenerate pseudo-Riemannian metric $\lambda\,\mathrm{ric} + \mu g$.

Conversely, let g be a metric on (M, g) inducing the same covariant derivative D.
M is irreducible and Ricci is non-degenerate, so, either $\mathrm{Ric} = \lambda\,\mathrm{Id}$ and g is type 1 or $(\mathrm{Ric} - \alpha\,\mathrm{Id})(\mathrm{Ric} - \overline{\alpha}\,\mathrm{Id}) = 0$, $\alpha \in \mathbb{C}\setminus\mathbb{R}$ and g is type 2. Then:

- If all such metrics are of type 1, we are in the first case of the proposition. Then, M does not admit a parallel J with $J^2 = -\mathrm{Id}$, else $h : u, v \mapsto g(u, v) - ig(u, J.v)$ is a complex Riemannian structure of M and all the $g_\beta = \Re(\beta h)$ for $\beta \in \mathbb{C}\setminus\mathbb{R}$ are real metrics of type 2.
- If there exists two metrics g and g' of type 2 (with corresponding α, Ric, J, α', Ric$'$ and J'), they both belong to the family $\{\lambda\,\mathrm{ric} + \mu g\}$ and $J = J'$. Indeed, g and g' give as in Proposition 1 complex Riemannian metrics h and h'. But $\Re(\alpha h) = \Re(\alpha' h') = \mathrm{ric}$, so $\alpha h = \alpha' h'$ and $g' = \Re(\frac{\alpha}{\alpha'} h) = \Re((\mu + \lambda\alpha)h)$ for some real numbers λ and μ, what gives the result. Furthermore, $\forall u, v$, $h(u, J.v) = ih(u, v) = i\frac{\alpha'}{\alpha} h'(u, v) = \frac{\alpha'}{\alpha} h'(u, J'.v) = h(u, J'.v)$, so $J = J'$. If M admits another J'', it induces other metrics of type 2, which are then in the family below, and $J'' = J$. □

Remark: *The assumption "*Ric *is non-degenerate" shall not be omitted. Indeed, there exist indecomposable Ricci-parallel manifolds M that admit any number of linear independant metrics associated to the same connection. For those cases,* Ric *is nilpotent. Such M may even be choosen symmetric.*

6. d Some examples

A family of symmetric spaces due to Cahen and Wallach (*cf.* [CW70]) may be used here as example.

As explained in [CP70], a simply connected pseudo-Riemannian symmetric space is associated to each pseudo-Riemannian symmetric triple $(\mathcal{G}, \sigma, <\,,\,>)$. Such a triple consists of a finite dimensional Lie algebra $\mathcal{G}$, a non-degenerate symmetric bilinear form $<\,,\,>$ on $\mathcal{G}$, invariant by $\mathcal{G}$, and an involutive automorphism σ of $\mathcal{G}$, orthogonal for $<\,,\,>$, satisfying the following property: $[\mathcal{Q}, \mathcal{Q}] = \mathcal{H}$, where $\mathcal{H} = \ker(\sigma - \mathrm{Id})$ and $\mathcal{Q} = \ker(\sigma + \mathrm{Id})$. The associated symmetric space M is a submanifold of the simply connected Lie group G associated to $\mathcal{G}$. Its tangent space at the point 0 is $T_0M = \mathcal{Q}$ and its holonomy algebra is $\mathcal{H}$, acting by Ad on $\mathcal{Q}$.

The announced examples are provided by a family of pseudo-Riemannian symmetric triples characterized as follows: $\dim \mathcal{G} = 2n+2$, $\dim \mathcal{H} = n$ and $\dim \mathcal{Q} = n+2 =$ dimension of the obtained symmetric space M. There is a basis $(U_i)_{i=1}^n$ of H, and a basis $(Y^*, (X_i)_{i=1}^n, Y)$ of $\mathcal{Q}$ such that, if the basis $(Y^*, (U_i)_{i=1}^n, (X_i)_{i=1}^n, Y)$ is denoted by β:

- Y^* is central,

- $\mathrm{Mat}_\beta(\mathrm{Ad}\, Y) = \left(\begin{array}{c|ccc|ccc|c} 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \hline 0 & & & & \lambda_1 & & & 0 \\ \vdots & & 0 & & & \ddots & & \vdots \\ 0 & & & & & & \lambda_n & 0 \\ \hline 0 & -\lambda_1 & & & & & & 0 \\ \vdots & & \ddots & & & 0 & & \vdots \\ 0 & & & -\lambda_n & & & & 0 \\ \hline 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{array}\right), \quad (\lambda_i)_{i=1}^n \in \mathbb{R}^{*n}$

  (remark: turning some X_i into $-X_i$, one may then require: $(\lambda_i)_{i=1}^n \in \mathbb{R}_+^{*n}$),

- $\forall (i,j) \in \{1, \dots, n\}$, $[U_i, X_j] =< [Y, U_i], X > Y^*$,

- $\mathrm{vect}\{U_i\}$ and $\mathrm{vect}\{X_j\}$ are abelian subalgebras,

- $\mathcal{H} \perp \mathcal{Q}$ for $<\,,\,>$,

- $< Y, Y^* >= 1$, $< Y, Y >=< Y^*, Y^* >= 0$,

- $\forall (i,j) \in \{1, \dots, n\}$, $< U_i, U_j >=< X_i, X_j >= \delta_{i,j}$ and $< X_i, Y >=< X_i, Y^* >= 0$.

It is easy to check that:

- $\forall V, W \in \mathcal{G}$, $\sigma([V, W]) = [\sigma(V), \sigma(W)]$

- σ is $<\,,\,>$-orthogonal and: $\forall V \in \mathcal{G}$, $\mathrm{Ad}\, V$ is $<\,,\,>$-skew symmetric,

- $[\mathcal{Q}, \mathcal{Q}] = \mathcal{H}$.

Therefore, $(\mathcal{G}, \sigma, <\,,\,>)$ is a pseudo-Riemannian (here Lorentzian) symmetric triple. Let $(M, <\ ,\ >)$ be its associated pseudo-Riemannian symmetric space. $\mathcal{H}$ is the holonomy algebra of M, acting on $\mathcal{Q} \simeq T_0 M$ by Ad. Therefore, by this representation, H is identified with the (abelian) subalgebra of $\mathfrak{so}(\mathcal{Q}, <\,,\,>)$ consisting of the matrices of the form:

$$\begin{pmatrix} 0 & X & 0 \\ 0 & 0 & -{}^t X \\ 0 & 0 & 0 \end{pmatrix} \quad \text{where } X = (x_1, \ldots, x_n) \text{ is some element of } \mathbb{R}^n,$$

in the basis $(Y^*, (X_i)_{i=1}^n, Y)$. Here M is indecomposable, non-irreducible, the only eigenvalue of Ric is 0, and Ric is nilpotent of order 2. More precisely, $\forall i \in \{1, \ldots, n\}$, we have $\mathrm{Ric}(X_i) = \mathrm{Ric}(Y^*) = 0$ and $\mathrm{Ric}(Y) = (\sum \lambda_i^2) Y^*$.

Now, $(\mathrm{Ad}\, Y)^2$ is diagonalizable in a $<\,,\,>$-pseudo-orthonormal basis, with the eigenvalues $\{0, -\lambda_1^2, \ldots, -\lambda_n^2\}$. Each eigenspace E_λ associated to one of the λ_i^2 is of even dimension $2d_\lambda$ where: $d_\lambda = \sharp\{i/\lambda_i = \lambda\}$). $E_\lambda = (E_\lambda \cap \mathcal{H}) \oplus (E_\lambda \cap \mathcal{Q})$, each term being of dimension d_λ. Then, on each E_λ, $<\,,\,>$ may be replaced by every scalar product $<\,,\,>'$:

- preserving the relation $\mathcal{H} \perp \mathcal{Q}$,
- such that $\mathrm{Ad}\, Y$ remains skew-symmetric, *i.e.* such that: $\forall i, j \in \{1, \ldots, n\}$, $< U_i, U_j >' = < X_i, X_j >'$;

and $(\mathcal{G}, \sigma, <\,,\,>')$ remains a pseudo-Riemannian symmetric triple, with the same brackets. Actually, one reaches here all the scalar products satisfying this property. In the basis of E_λ built with the U_i and the X_i, the matrix of $<\,,\,>'$ is:

$$\begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}, \quad \text{where } S \text{ is some symmetric matric of } M_{d_\lambda}(\mathbb{R}).$$

So, each E_λ may be equipped of $\frac{d_\lambda(d_\lambda+1)}{2}$ linearly independant metrics, letting the brakets unchanged, and thus the covariant derivative of the associated manifold M.

Notes: 1) These examples may be adapted to provide similar pseudo-Riemannian manifolds of every signature $(p+1, q+1)$. One take $n = p + q$, $< X_i, X_j >= \delta_{i,j}$ if $i \leq p$, else $< X_i, X_j >= -\delta_{i,j}$; $[Y, U_i] = -\lambda_i X_i$ if $i \leq p$, else $[Y, U_i] = \lambda_i X_i$. The other data are the same. $(\mathrm{Ad}\, Y)^2$ is diagonalizable, with the eigenvalues $\{0, -\lambda_1^2, \ldots, -\lambda_p^2, \lambda_{p+1}^2, \ldots, \lambda_n^2\}$. On the $E_\lambda^+ = \ker[(\mathrm{Ad}\, Y)^2 + \lambda^2\, \mathrm{Id}]$ (for one of the λ_i, $i \leq p$), the phenomenon is the same; whereas on the $E_\lambda^- = \ker[(\mathrm{Ad}\, Y)^2 - \lambda^2\, \mathrm{Id}]$ (for one of the other λ_i), the matrix of the other possible $<\,,\,>'$ are:

$$\begin{pmatrix} S & 0 \\ 0 & -S \end{pmatrix}, \quad \text{where } S \text{ is some symmetric matric of } M_{d_\lambda}(\mathbb{R}).$$

2) This provides a family of symmetric spaces such that $\mathrm{Ric} \neq 0$ and $\mathrm{Ric}^2 = 0$. Another such family is constructed in [CP70], pp.40 *sq.* See [CP80] too.

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