Isometric actions of Lie subgroups of the Moebius group.

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Abstract. We prove here, by geometric, or rather dynamical, methods, the following theorem. Let G be a non-compact connected Lie subgroup of the isometry group $\mathrm{Isom}(\mathbb{H}^n)$ of the real hyperbolic space $\mathbb{H}^n$, which does not fix any point at infinity, i.e. on $\partial\mathbb{H}^n \simeq \mathbb{S}^{n-1}$. Then G preserves a certain hyperbolic subspace $\mathbb{H}^d \subset \mathbb{H}^n$ and "contains" all the identity component $\mathrm{Isom}^0(\mathbb{H}^d)$ of its isometry group. We provide an "algebra-free" proof and present the dynamical tools used so that the exposition is "self-contained".

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1 Introduction

We provide here a geometric —essentially dynamic— proof of the following result:

Theorem 1.1 *Let G be a non-compact connected Lie subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*, which does not fix any point at infinity (i.e. on $\partial\mathbb{H}^n \simeq \mathbb{S}^{n-1}$). Then, up to conjugacy, G preserves a hyperbolic subspace $\mathbb{H}^d$ with $1 \leq d \leq n$ and contains* $\mathrm{O}^0(d, 1)$.

In [DO], A. J. Di Scala and C. Olmos proved some equivalent results, describing additionally the case where G admits a fixed point. Their motivation is, like ours, to give a geometrical proof of a result classically proved by algebraic means. This takes place in a more general background we recall in 1.1 just below. However, some nontrivial algebraical tools remain used in their proof. We propose here an "algebra-free" approach (actually, we allow the use of some rudimentary notions such as the radical of a Lie group, to show that Theorems 1.1 and 1.2 are essentially equivalent). In a pedagogical view, the proof is fully self-contained (see 1.2) and one of its steps is linked with a more general problem in dynamics of algebraic groups (see 1.3). Finally, we present an application of Theorem 1.1 to the holonomy of Lorentzian manifolds, see 1.4.

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1.1 Isometric dynamics on Riemannian symmetric spaces

Let X be a Riemannian symmetric space of non positive curvature. The following two properties of subgroups of $\mathrm{Isom}(X)$ are well known. Firstly, if G is compact, then it has a fixed point in X. Secondly, if G is a semi-simple (e.g. diagonalizable) one parameter group, then it keeps invariant a geodesic, which is moreover unique in the negatively curved case and called the axis of G). The original proofs were algebraic in nature, but now there are also geometrical ones, applying to any Hadamard space [BGS].

However, both these facts are corollaries of the following less known statement:

Theorem 1.2 *Let G be a semi-simple Lie group acting isometrically on a Riemannian symmetric space X of nonpositive curvature. Then G admits a totally geodesic orbit Y (which is obviously a symmetric space). Moreover G contains the identity component of the isometry group of Y; more precisely, the group of the restrictions of elements of G to Y contains the "intrinsic" group* $\mathrm{Isom}^0(Y)$.

Though this statement is purely geometric, only algebraic approaches to this result are in the literature, with the exception of [DO]. One can find in [K] an algebraic proof of the first part of Theorem 1.2 (existence of a totally geodesic orbit); this reference is recalled in [AVS], theorem 3.7. p.103, where Theorem 1.2 is stated in the case of constant curvature. We thank P. de la Harpe for this remark. Also, an algebraic proof of a proposition implying the second part of Theorem 1.2, in the case $G = \mathrm{O}(n,1)$, appears in [BH], §3, prop. 1.

The available formulations of Th. 1.2 are mostly algebraic. For example, for the "universal" symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$, up to conjugacy, G is adjoint, *i.e.* invariant under the canonical automorphism $A \mapsto (A^*)^{-1}$ of $\mathrm{SL}(n,R)$. In this case, the orbit of the base point 1 is totally geodesic. Even this geometrical interpretation is not well known; see for instance [GP] where the algebraic formulation is attributed to [Mo].

Now, all proofs of the different algebraic formulations of Theorem 1.2 use nontrivial results from the classification of semi-simple Lie groups, or at least significant steps towards it.

We would like to give a geometric proof of it. This would emphasize exactly which properties are behind it; also, as by-product, one should get geometric proofs of other well known algebraic facts on semi-simple Lie groups.

1.2 Philosophy of the article

The essential mathematical aim of the present paper is to realize the wish above, in the particular case of the hyperbolic space. More precisely, we prove the slightly different version "theorem 1.1" given at the beginning which implies it (see Lemma 2.1, section 2). We recognize this is modest with respect to the global goal. However:

• Paying attention to this question is by itself interesting. Beyond the technical content, it forces one to think about the "geometrization of Lie groups".

• In our exposition, we have tried to present the proof, based on geometric (essentially dynamic) arguments, on a completely self-contained way. For ease of reading, the background and proofs of the few standard lemmas used here are also recalled. The geometric (this time in a more analytical sense) motivation of [DO] was similar, yet this article still uses (implicitly) non-trivial algebraic tools in the proofs, *e.g.* the Iwasawa decomposition.

• Step two of the proof is an elementary result on conformal dynamics on the sphere, which suggests more serious questions in this special case as well as in the general case of boundaries of symmetric spaces (see 1.3).

• We are confident that the hyperbolic part of the proof can be adapted to the general rank one case, and this would naturally involve beautiful geometric properties of the boundaries.

1.3 Dynamics on boundaries, one step of the proof

Step 2 of the proof rests on a rigidity property (Lemma 4.1) of invariant sets under conformal dynamics on the sphere $\mathbb{S}^{n-1}$. Let us state it here —in a slightly different and weaker form— in order to comment it. Denote by $\mathcal{M}_d^k$ the space of C^k embedded compact submanifolds of dimension d in $\mathbb{S}^{n-1}$, and $\mathcal{S}_d$ the space of the d-dimensional (round) spheres.

Fact 1.3 (Quick and weaker version of Lemma 4.1) *If $V \in \mathcal{M}_d^1$ is invariant under a noncompact subgroup of* $\mathrm{Conf}(\mathbb{S}^{n-1})$*, then $V \in \mathcal{S}_d$.*

Let us make the following observations:

1) Other symmetric spaces. It is very useful to consider how this fact comes into play on other symmetric spaces.

Lemma 4.1, in step 2 of the proof, consists in finding an orbit of G in $\mathbb{S}^{n-1}$ which is a (round) sphere. One then fills it (taking its convex hull) and gets a hyperbolic subspace. In the most general case (say for the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$), algebraic geometric dynamics yields closed orbits of G on the boundary. The filling process is then sufficiently regular in the general rank one case, but delicate in higher rank. In any case, it seems not to be easy to describe the geometry of submanifolds of the boundary which are the boundary of geodesic submanifolds of the symmetric space. Even in the case of the complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, these boundary submanifolds may be quite complicated (see for instance [G]).

2) Stability. One may ask if a stable version of Fact 1.3 holds. Indeed, it turns out that the following is true; we shall provide the details of its proof elsewhere:

Theorem 1.4 *Endow $\mathcal{M}_d^1$ with the C^0 topology and consider the action of* $\mathrm{Conf}(\mathbb{S}^{n-1})$ *on it. This is proper on $\mathcal{M}_d^1 \setminus \mathcal{S}_d$ (and obviously transitive and nonproper on $\mathcal{S}_d$).*

This means, roughly speaking, that there is a good complete system of conformal invariants for elements of $\mathcal{M}_d^1 \setminus \mathcal{S}_d$.

3) Topological submanifolds, fractals. The example of limit sets of quasi-Fuschian groups shows that no rigidity condition extends to $\mathcal{M}_d^0$. It remains nevertheless interesting to understand the conformal dynamics on $\mathcal{M}_d^0$.

1.4 An application: the holonomy of the Lorentz symmetric spaces

Finally, let us mention an application of Theorem 1.1, already indicated in [DO]. We state the result in terms of linear algebra.

Theorem 1.5 *Let G be a non-compact connected Lie subgroup of $\mathrm{O}(n,1) = \mathrm{O}(\mathbb{R}^{n+1}, g)$, $g = -\,\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_n^2$, which does not preserve any isotropic direction. Then:*

- *G preserves a subspace $E \subset \mathbb{R}^{n+1}$, with $\mathrm{sgn}(g_{|E}) = (d,1)$ and $\dim E = d+1 \geq 2$,*
- *E and G are such that $\{h_{|E} \,/\, h \in G\} \supset \mathrm{SO}^0(g_{|E}) \simeq \mathrm{SO}^0(d,1)$.*

Corollary 1.6 *Let* Hol^0 *be the restricted holonomy group of an indecomposable (*i.e. *not locally decomposable into a Riemannian product) Lorentzian manifold of dimension n. Then either* $\mathrm{Hol}^0 = \mathrm{SO}^0(n-1,1)$ *or* Hol^0 *stabilizes an isotropic direction.*

This fact was known since 1957, as a consequence of Berger's list [B2] of the possible holonomy groups for symmetric spaces. However, L. Bérard Bergery and A. Ikemakhen asked for a direct proof in [BBI], remark p.31. As indicated in [BBI], Berger's theorem [B1] implies the above corollary for non locally symmetric Lorentzian spaces. Now, if the manifold is locally symmetric and indecomposable, its isotropy subgroup Iso^0 is indecomposable, so it can neither fix a non-isotropic vector (so cannot be compact), nor stabilize a subspace E as in the statement of theorem 1.5. So it has to fix an isotropic direction. Now for locally symmetric spaces, $\mathrm{Hol}^0 \subset \mathrm{Iso}^0$. The result follows. (Note also [Z] for another proof.) □

2 Background and structure of the article

Some brief recalls. Throughout, g is the lorentzian metric $-\mathrm{d}x_0^2 + \mathrm{d}x_1^2 + \ldots + \mathrm{d}x_n^2$ on $\mathbb{R}^{n+1}$ (d replacing n in section 5). The n-dimensional hyperbolic space is $\mathbb{H}^n = \{m \in \mathbb{R}^+ \times \mathbb{R}^n \,/\, g(m,m) = -1\}$, endowed with its riemannian metric $g_{\mathrm{hyp}} = g_{|\mathbb{H}^n}$, of constant curvature -1. So $\mathrm{PO}(n,1)$ is its isometry group $\mathrm{Isom}(\mathbb{H}^n, g_{\mathrm{hyp}})$. Hence, $\mathrm{O}(n,1)$ acts on $\mathbb{S}^{n-1}$, which bounds $\mathbb{H}^n$ at infinity. This action is conformal for the canonical conformal structure of $\mathbb{S}^{n-1}$; more precisely, $\mathrm{PO}(n,1)$ is the conformal group $\mathrm{Conf}(\mathbb{S}^{n-1})$ of $\mathbb{S}^{n-1}$. Indeed, any sphere $\mathbb{S}^{n-1}$ imbedded into the g-isotropic cone of $\mathbb{R}^{n+1}$, transversally to the isotropic lines, inherits from g a riemannian metric. These inherited metrics are conformally equivalent to the canonical metric of $\mathbb{S}^{n-1}$. Finally, the topology on $\overline{\mathbb{H}^n} = \mathbb{H}^n \cup \mathbb{S}^{n-1}$ is the topology induced by that of $\mathbb{R}^n$ on the closed unit ball $\overline{\mathbb{D}^n}$, with which $\overline{\mathbb{H}^n}$ is identified by the stereographic projection of $\mathbb{R}^{n+1}$ on $\{0\} \times \mathbb{R}^n$, of pole $(-1, 0, \ldots, 0)$. This topology gives naturally back, on $\mathbb{H}^n$ and $\mathbb{S}^{n-1}$, their canonical topology. Note that other topologies on $\overline{\mathbb{H}^n}$ with this properties exist and are used. An inversion centered at a point of the boundary of $\overline{\mathbb{D}^n}$ maps $\overline{\mathbb{D}^n}$ on the Poincaré half space ($\simeq \mathbb{R}^+ \times \mathbb{R}^{n-1}$), another conformal model for $\mathbb{H}^n$.

The space of the oriented geodesics of $\mathbb{H}^n$ is parametrized by the couples of their limit points in $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$, so this space is identified with $(\mathbb{S}^{n-1})^2 \setminus \Delta$, where Δ is the diagonal, *i.e.* the graph of $\mathrm{Id}_{\mathbb{S}^{n-1}}$. The application $\Phi : \mathrm{T}^1\mathbb{H}^n \to (\mathbb{S}^{n-1})^2 \setminus \Delta$ associating, to each vector $v \in \mathrm{T}^1\mathbb{H}^n$, the geodesic with initial condition v, is a $\mathcal{C}^\infty$ submersion; its fibres are naturally the orbits of the geodesic flow φ^t of $\mathrm{T}^1\mathbb{H}^n$.

An element $h \in \mathrm{O}(n,1)$ is said to be *elliptic* if it fixes a point of $\mathbb{H}^n$, *i.e.* if it admits an eigenvector $v \in \mathbb{R}^{n+1}$ such that $g(v,v) < 0$. Else, it must fix a point of $\mathbb{S}^{n-1}$ *i.e.* admit a g-isotropic eigenvector v. Two cases are then possible: either this fixed point is unique (then $h(v) = v$ and $v \in \mathrm{Im}(h - \mathrm{Id})$) and h is said to be *parabolic*, or h admits exactly two fixed points, corresponding to two isotropic eigenvectors v and v' in $\mathbb{R}^{n+1}$ with $h(v) = \lambda v$ and $h(v') = \lambda^{-1}v'$ with $\lambda < 1$; then h is said to be *loxodromic*.

As announced in 1.2, Theorem 1.1 implies Theorem 1.2. Actually, they are essentially equivalent. Let us already recall briefly why (the following Lemma holds the general rank one case but not in higher rank).

Lemma 2.1 *If G is a semi-simple non-compact connected subgroup of* $\mathrm{Isom}(\mathbb{H}^n)$*, it has no fixed point (neither at infinity nor in the interior).*

If G has no fixed point, then it fixes some hyperbolic subspace $\mathbb{H}^d$, and the so obtained homomorphism $G \to$ $\mathrm{Isom}(\mathbb{H}^d)$ *has compact kernel and a semi-simple image.*

Proof. Assume G fixes a point x. If $x \in \mathbb{H}^n$, G is compact as it is contained in the stabilizer of x which is isomorphic to $\mathrm{O}(n)$. If $x \in \partial\mathbb{H}^n$, its stabilizer is naturally isomorphic to the similarity group of a Euclidean space (see the model of the Poincaré half-space, with $x = \infty$) which cannot contain a noncompact semi-simple Lie group.

Conversely, let R be the solvable radical of G. It will have a fixed point in $\overline{\mathbb{H}} = \mathbb{H} \cup \partial\mathbb{H}$. This is true in general and essentially equivalent to the fact that representations of solvable Lie groups preserve flags; see for instance [BGS] for an overview of the subject).

If R is non-compact, then it contains some hyperbolic or parabolic element (see Lemma 3.1 in section 3 below). The set of fixed points of such a (parabolic or hyperbolic) element consists of one or two points. Therefore, this property holds for the set of fixed points of R. The group G acts on it and thus, by connectedness, fixes some point at infinity. This is a contradiction, so R must be compact. In this case, its fixed point set is a geodesic hyperbolic subspace $\mathbb{H}^d$. This determines a homomorphism $G \to \mathrm{Isom}(\mathbb{H}^d)$. Its kernel is compact, since all its elements fix some interior point. Its image is semi-simple, since this procedure kills the radical. By definition, the quotient of a Lie group by its radical has no radical, and is therefore semi-simple. □

Now the proof of Theorem 1.5 is divided into one (classical) preliminary step and two additional steps, each of them corresponding to one section of the article.

Scheme of the proof. *Step 1: Preliminary.* We recall, with its sketch of proof, a standard sufficient condition for the existence of loxodromic elements in a subgroup of $\mathrm{O}(n, 1)$.

Step 2: "rigidity" of some G-orbits on $\mathbb{S}^{n-1}$. We exhibit here a subspace E as announced in the theorem. More exactly, we will prove that G stabilizes a subsphere of $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$, and hence its convex hull in $\mathbb{H}^n$, which is a hyperbolic subspace of $\mathbb{H}^n$, that is $H = \mathbb{H}^n \cap E$, where E is a sub-vectorspace of $\mathbb{R}^{n+1}$. Moreover, G acts transitively on this subsphere.

Step 3: "maximality" of G in the stabilizer of E. We show how this leads to the result.

3 Step 1 (standard): existence of loxodromic elements in G

We prove the following lemma —the notion of limit set is defined just below.

Lemma 3.1 *If G is a non precompact subgroup of $\mathrm{O}(n, 1)$, G has a loxodromic or a parabolic element. If its limit set $L(G)$ contains two distinct points x, y (so in particular if G stabilizes no isotropic direction), G contains a loxodromic transformation the two fixed points of which are arbitrarily near x and y.*

This result is a consequence of a few classical (and basic) notions and results in hyperbolic dynamics. You can find a good introduction to them for example in [Ma], in the framework of flat conformal geometry, or [EN], in that of Hadamard manifolds. The first notion is that of *limit set* of G, which is most usually used for discrete groups, but works on the same way for any group with the following definition.

Definition 3.2 *The* limit set *$L(G)$ of a group G acting isometrically on $\mathbb{H}^n$ is $L(G) = \mathbb{S}^{n-1} \cap \overline{G.a}$, where $\mathbb{S}^{n-1}$ is the boundary of $\mathbb{H}^n$ at infinity and where $\overline{G.a}$ is the closure, in $\overline{\mathbb{H}^n}$, of the G-orbit of a point $a \in \mathbb{H}^n$. This definition does not depend on the choice of a.*

Definition 3.3 *Two points x and y (not necessarily distinct) of $L(G)$ are called* dual *if there exists a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of elements of G such that, for one point $a \in \mathbb{H}^n$ (and then for any such point), $\varphi_n(a) \underset{n\to\infty}{\longrightarrow} x$ and $\varphi_n^{-1}(a) \underset{n\to\infty}{\longrightarrow} y$. We denote by $D(x)$ the set of the points of $\mathbb{S}^{n-1}$ dual to x.*

We summarize the keyfacts needed in our proof in the following lemma. Its first point gives the "independance of a" in the definitions.

Lemma 3.4
(i) *If $a, b \in \mathbb{H}^n$, $x \in \mathbb{S}^{n-1} = \partial\mathbb{H}^n$, $(\varphi_n)_{n\in\mathbb{N}} \in \mathrm{O}(n,1)^{\mathbb{N}}$ and $\varphi_n(a) \underset{n\to\infty}{\longrightarrow} x$, then $\varphi_n(b) \underset{n\to\infty}{\longrightarrow} x$.*
(ii) *For all x in $\mathbb{S}^{n-1}$, $D(x)$ is G-invariant.*
(iii) *If G fixes no point on $\mathbb{S}^{n-1}$, any two points of $L(G)$ are dual.*
(iv) *If x, y in $\mathbb{S}^{n-1}$ are dual, G has a parabolic or a loxodromic element. If $x \neq y$, G contains a loxodromic element φ the two fixed points of which are arbitrarily near x and y.*

Proof. **(i)** This follows from the comparison between the hyperbolic and the euclidean metrics on $\mathbb{D}^n$: if $x \in \mathbb{S}^{n-1}$, $\mathbb{D}^n \ni a_n \to x$, $\mathbb{D}^n \ni b_n \to x$ and $d_{\mathrm{hyp}}(a_n, b_n)$ is bounded, $d_{\mathrm{eucl}}(a_n, b_n) \to 0$.

(ii) Let us take y dual to x, $a \in \mathbb{H}^n$ and $\varphi \in G$. By definition of duality, there is a sequence $(\varphi_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that $\varphi_n(a) \underset{n\to\infty}{\longrightarrow} x$ and $\varphi_n^{-1}(a) \underset{n\to\infty}{\longrightarrow} y$. Then naturally $\varphi(\varphi_n^{-1}(a)) \underset{n\to\infty}{\longrightarrow} \varphi(y)$ and by **(i)**, $\varphi_n(\varphi^{-1}(a)) \underset{n\to\infty}{\longrightarrow} x$, that is to say, $\varphi(y)$ is dual to x.

(iii) Notice that if $E \subset \mathbb{S}^{n-1}$ is G-invariant, so is its convex hull $C(E)$ in $\mathbb{H}^n$, *i.e.* the convex hull of the union of the geodesics joining the pairs of points of E. Therefore, for any $a \in C(E)$, $G.a \subset C(E)$, so $\overline{G.a} \cap \mathbb{S}^{n-1} \subset E$. By **(i)**, it implies that $L(G) \subset E$ as soon as $C(E) \neq \emptyset$, *i.e.* as $\sharp E \geq 2$. Now if $x \in L(G)$, by compactness of $\overline{\mathbb{H}^n}$, $D(x) \neq \emptyset$; if moreover G has no fixed point in $\mathbb{S}^{n-1}$, $D(x)$, which is G-invariant, contains at least two points. Applying the remark, we get $D(x) \supset L(G)$. This shows **(iii)**.

(iv) Let us take x and y dual points in $\mathbb{S}^{n-1}$, represented by two isotropic vectors e_x and e_y of $\mathbb{R}^{n+1}$, and $(\varphi_n)_{n\in\mathbb{N}}$ a sequence in G such that $\varphi_n(a) \underset{n\to\infty}{\longrightarrow} x$ and $\varphi_n^{-1}(a) \underset{n\to\infty}{\longrightarrow} y$. Then let us take e a point in $\mathbb{H}^n$, *i.e.* a vector in $\mathbb{R}^{n+1}$ with $g(e,e) = -1$, and $K = G_e \simeq \mathrm{O}(n)$ is stabilizer; K is a maximal compact subgroup of $\mathrm{O}(n,1)$. Let $\varphi_n = K_n A_n K'_n$ be the decomposition of each φ_n with K_n and K'_n in K and $A_n \in \mathrm{O}(n,1)$ diagonal; note that A has then at most two eigenvalues in $\mathbb{R} \setminus \{-1, 1\}$, λ_n and λ_n^{-1}, of multiplicity one.By compactness of K, taking possibly a subsequence of $(\varphi_n)_{n\in\mathbb{N}}$, K_n and K'_n tend respectively to some K_∞ and K'_∞ in K. Now as, for $a \in \mathbb{H}^n$, $\varphi_n(a) \underset{n\to\infty}{\longrightarrow} x$, A_n leaves all compact set of $\mathrm{O}(n,1)$, *i.e.* its eigenvalues $\lambda_n > 1$ and λ_n^{-1}, associated to (isotropic) eigenvectors e_n and e'_n, leave all compact of $\mathbb{R}^*$. Let us denote by x_n and y_n the points of $\partial\mathbb{H}^n$ associated to e_n and e'_n. Then, for n big enough, $\varphi_n(K'^{-1}_\infty(a))$ must be close to $K_\infty x_n$. As it is close to x, necessarily $x_n \underset{n\to\infty}{\longrightarrow} K_\infty^{-1} x$. Similarly $y_n \underset{n\to\infty}{\longrightarrow} K'_\infty y$. Besides, if $\varepsilon > 0$, if B'_ε and B_ε are the euclidean ε-balls of respective centers $\mathbb{K}'^{-1}_n y_n$ and $\mathbb{K}_n x_n$ (in the model of the Poincaré ball $\mathbb{D}^n$), and if $|\lambda_n|$ is big enough, then $K_n A_n K'_n$ sends $\overline{\mathbb{D}^n} \setminus (B'_\varepsilon \cap \overline{\mathbb{D}^n})$ inside $\overline{\mathbb{D}^n} \cap \overline{B_\varepsilon}$ ("North-South dynamics"). One checks then that φ_n has to be parabolic or loxodromic with fixed point(s) ε-close to $K'^{-1}_n y_n$ and $K_n x_n$. If $x \neq y$, for n big enough, these fixed point(s) are distinct, so φ_n is loxodromic. □

Notes. The "KAK decomposition" —given by the existence of a square root of any symmetric matrix—, shortens the proof of **(iv)**. Alternative tools may be used, *e.g.* the notion of "vision angle" in a Hadamard manifold ([EN] propositions 8.4–8.6, 4.7 and 6.4) or of isometric sphere of a conformal map ([Ma] section 5, notably Lemma 5.8).

Note that in general, G does *not* contain a parabolic or loxodromic element the set of fixed points of which *is* $\{x, y\}$, even if G is closed in $\mathrm{O}(n,1)$. For example, if G is a cocompact lattice, all pairs $\{x,y\} \subset \partial\mathbb{H}$ are dual, though only a countable number are the set of fixed points of elements of G.

Proof of Lemma 3.1. If $L(G) \neq \emptyset$, the result follows from **(iii)** and **(iv)** above. Now $L(G) = \emptyset$ if and only if G is precompact. The "if" part is immediate, let us check the "only if". If $L(G) = \emptyset$, by definition of $L(G)$, the orbit $G.a$ of any $a \in \mathbb{H}^n$ is bounded. So is $\overline{G}.a$, where $\overline{G}$ is the closure of G in $\mathrm{O}(n,1)$. Now, denoting by $\overline{G}_a$ the stabilizer of a in $\overline{G}$, $\pi : \overline{G} \to \overline{G}/\overline{G}_a$ is a fibration over $\overline{G}/\overline{G}_a$, homeomorphic to $\overline{G}.a$, hence compact, with fibre $\overline{G}_a$ which is also compact, as included in $\mathrm{O}(\mathrm{T}_a\mathbb{H}^n, (g_{\mathrm{hyp}})_{|a}) \simeq \mathrm{O}(n)$. So $\overline{G}$ is compact. □

4 Step 2: A rigidity result about differentiable submanifolds of $\mathbb{S}^{n-1}$

The fundamental lemma on which the second step is based is the following.

Lemma 4.1 *Let $\mathcal{M}$ be a connected differentiable manifold of dimension k and $F : \mathcal{M} \to \mathbb{S}^m$ an injective continuous map of $\mathcal{M}$ into $\mathbb{S}^m$. Let us suppose that $\mathcal{F} = F(\mathcal{M})$ is h-invariant, where h is a loxodromic transformation of $\mathbb{S}^m$ with attractive point $q_+ \in \mathcal{F}$, and that F is differentiable at $F^{-1}(q_+)$. Then $\mathcal{F}$ is a k-sphere of $\mathbb{S}^m$ passing by the poles q_+ and q_- of h, or such a sphere, without the point q_-.*

Proof of the lemma. A stereographic map from $\mathbb{S}^m \setminus \{q_-\}$ onto $\mathbb{R}^m$ sending q_+ on 0 conjugates h with a linear transformation of $\mathbb{R}^n$ of the type λA, where $\lambda \in]0,1[$ and $A \in \mathrm{O}(m)$. We go further in this framework.

Let us denote by p the inverse image of q_+ by F. Using the differentiability hypothesis about F, we may choose an open set $\mathcal{U}$ of $\mathcal{M}$ containing p and such that, in a well-chosen coordinate system of $\mathbb{R}^m$, centered at 0, $F(\mathcal{U})$ is the graph of a certain function $f : \mathbb{R}^k \to \mathbb{R}^{n-k}$, differentiable at 0 (and, for convenience, satisfying $f(0) = 0$):

$$F(\mathcal{U}) = \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \,/\, x \in B(0,r) \text{ and } y = f(x)\}.$$

Now necessarily, f is linear. Let us prove it.

Remark: the particular case $A = \mathrm{Id}$. In this case, the h-invariance of $\mathcal{F}$ implies that f commutes with $x \mapsto \lambda x$. Now, lots of such functions exist, but only the *linear* ones may be differentiable at 0. Indeed, by definition of differentiability:

$$\lambda^{-n} f(\lambda^n x) \underset{n\to\infty}{\longrightarrow} \mathrm{d}f(0).x,$$

so, if $\lambda^{-n} f(\lambda^n x) = f(x)$, f has to be equal to its differential $\mathrm{d}f(0)$.

The proof of the general case, modulo technical modifications, is the same, as $\mathrm{O}(m)$ is compact. For convenience, let us choose more precise coordinates. The first factor $\mathbb{R}^k \times \{0\}$ may be chosen equal to $\mathrm{T}_{q_+}\mathcal{F}(\mathcal{U})$, so that $\mathrm{d}f(q_+) = 0$. Besides, as $\mathcal{F} = F(\mathcal{M})$ is h-stable, $\mathrm{T}_{q_+}\mathcal{F}(\mathcal{U})$ is A-stable. Choosing $(\mathrm{T}_{q_+}\mathcal{F}(\mathcal{U}))^\perp$ as second factor $\{0\} \times \mathbb{R}^{m-k}$, that factor is also A-stable, A being orthogonal. Then:

$$\begin{aligned} h(F(\mathcal{U})) &= \{A.(x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \,/\, x \in B(0,\lambda r) \text{ and } y = \lambda.f(\lambda^{-1}.x)\}. \\ &= \{(x,y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \,/\, x \in B(0,\lambda r) \text{ and } y = A''\lambda.f(\lambda^{-1}.A'^{-1}x)\}, \\ &\qquad \text{where } A' = A_{|\mathbb{R}^k \times \{0\}} \text{ and } A'' = A_{|\{0\}\times\mathbb{R}^{m-k}} \end{aligned}$$

Now, as $\mathcal{F} = F(\mathcal{M})$ is preserved by h, $h(F(\mathcal{U})) \subset \mathcal{F}$. As the projection from $\mathbb{R}^m$ on its first factor $\mathbb{R}^k$ maps $h(F(\mathcal{U}))$ onto $B(0, \lambda r)$, necessarily:

$$h(F(\mathcal{U})) = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} / \, x \in B(0, \lambda r) \text{ and } y = f(x)\}.$$

Therefore, on $B(0, r)$, $f \circ \lambda A' = \lambda A'' \circ f$. Fixing an x near 0, we get, for any $n \in \mathbb{N}$:

$$f(\lambda^n A'^n.x) = \lambda^n A''^n.f(x), \text{ or equivalently } \lambda^{-n} A''^{-n}.f(\lambda^n A'^n.x) = f(x).$$

Now $f(0) = 0$, f is differentiable at 0, the coordinates are such that $\mathrm{d}f(0) = 0$, and A' and A'' preserve the euclidean norm, so: $\lambda^{-n} A''^{-n}.f(\lambda^n A'^n.x) \underset{n\to\infty}{\longrightarrow} 0$. Thus $f = 0$ on $B(0, r)$, that is to say, f is *linear* on $B(0, r)$, hence moreover $f \circ A' = A'' \circ f$.

Finally, as $\mathcal{F}$ is h-invariant, thus also h^{-1} invariant, it contains $\mathcal{S}' = \cup_{n\in\mathbb{N}} h^{-n}(F(\mathcal{U}))$, which is $f(\mathbb{R}^k)$, a vectorspace of dimension k. Going back to $\mathbb{S}^m$ by the inverse of the stereographic map, $\mathcal{S}' = \mathcal{S} \setminus \{q_-\}$, where $\mathcal{S}$ is a k-sphere passing by q_+ and q_-. As the only proper extension, as an immersed manifold, of $\mathcal{S}'$ in $\mathbb{S}^m$, is $\mathcal{S}$ and as $\mathcal{M}$ is connected, $\mathcal{F} = \mathcal{S}$ or $\mathcal{F} = \mathcal{S}' = \mathcal{S} \setminus \{q_-\}$. □

Proof of step 2. The group G is supposed to preserve no isotropic direction, that is to say it has no fixed point on $\mathbb{S}^{n-1} = \partial\mathbb{H}^n$. So by Lemma 3.1, there is a loxodromic element γ in G. Let $q_+ \in \mathbb{S}^{n-1}$ be the attractive point of γ and $G_{q_+} \subset G$ the stabilizer of q_+ in G. We apply Lemma 4.1 to the G-orbit of q_+, *i.e.* with $\mathcal{M} = G/G_{q_+}$ and $F : h \mapsto h(q_+)$, so $\mathcal{F} = F(\mathcal{M})$ is the G-orbit $G.q_+$ of q_+. So $G.q_+$ is a certain sphere $\mathcal{S}$ or $\mathcal{S} \setminus \{q_-\}$. The second case is impossible, for $\{q_-\} = \mathrm{bound}(\mathcal{S} \setminus \{q_-\})$ would be G-invariant. So $G.q_+$ is a G-invariant sphere of $\mathbb{S}^{n-1}$, step 2 is achieved. □

5 Step 3: "Maximality" of G

We prove now the end of the statement of theorem 1.5 that is:

Proposition 5.1 *Let G be a non compact (closed) connected Lie subgroup of* $\mathrm{SO}^0(d, 1)$ *which acts transitively on $\mathbb{S}^{d-1}$. Then $G = \mathrm{SO}^0(d, 1)$.*

By Lemma 3.1, section 3, we can take a loxodromic element γ in G. The space of the geodesics of $\mathbb{H}^d$ is diffeomorphic to $(\mathbb{S}^{d-1})^2 \setminus \Delta$; with this identification, the axis θ of this loxodromic element γ corresponds to the couple (q_-, q_+) of its endpoints. Step 3 rests on two lemmas.

Lemma 5.2 *The G-orbit of θ is open in the space of the geodesics of $\mathbb{H}^d$.*

Proof. Let π_1 and π_2 be the restrictions to $(\mathbb{S}^{d-1})^2 \setminus \Delta$ of the projections on the first, respectively second, factor $\mathbb{S}^{d-1}$. Let us also set $q = (q_-, q_+)$. After Step 2, $G.q_+ = \mathbb{S}^{d-1}$, so $\pi_2(G.q) = G.q_+ = \mathbb{S}^{d-1}$ and symmetrically $\pi_1(G.q) = \mathbb{S}^{d-1}$.

Now $G.q$ is the image of an immersion of G/G_q in $(\mathbb{S}^{d-1})^2 \setminus \Delta$, so, differentiating this immersion at the identity, we get a subspace $F = \mathrm{T}_q(G.q) \subset \mathrm{T}_q((\mathbb{S}^{d-1})^2 \setminus \Delta)$ which is stable under the action of the stabilizer G_q. In particular, it is $\mathrm{d}\gamma(q) = (\mathrm{d}\gamma(q_-), \mathrm{d}\gamma(q_+))$-stable, where γ is seen, successively, as a diffeomorphism of $(\mathbb{S}^{d-1})^2$ and of $\mathbb{S}^{d-1}$.

In a chart $(s, i \circ s)$ of $(\mathbb{S}^{d-1})^2$, where s is a stereographic map from $\mathbb{S}^{d-1}$ onto $\mathbb{R}^{d-1} \cup \{\infty\}$ sending q_- on 0 and q_+ on ∞ and where i is the inversion of $\mathbb{R}^{d-1}$ of center 0 and radius 1 (the radius does not matter), $d\gamma(q)$ is represented by a linear isomorphism B of $(\mathbb{R}^{d+1})^2$ of the type :

$$B = \begin{pmatrix} \lambda^{-1}A & 0 \\ 0 & \lambda A \end{pmatrix}, \text{ where } A \in \mathrm{O}(d-1) \text{ and } \lambda \in]0,1[.$$

The image E of $\mathrm{T}_q(G.q)$ by this chart is B-invariant. Let us denote E_1 and E_2 the first and second factors of the product $(\mathbb{R}^{d-1})^2$ and $\mathrm{pr}_i : E_1 \oplus E_2 \to E_i$ the canonical projections. By assumption, $\pi_i(G.q) = \mathbb{S}^{d-1}$; differentiating, in the chart, these equalities in 0, we get $\mathrm{pr}_i(E) = E_i$ for each i.

Remark: the particular case $A = \mathrm{Id}$. As E is B-stable, it has immediately to be, in this case, $\{0\}$, E_1, E_2 or $E_1 \oplus E_2$. As $\mathrm{pr}_i(E) = E_i$ for $i = 1, 2$, $E = E_1 \oplus E_2 = (\mathbb{R}^{d-1})^2$, *i.e.* $\dim G.q = 2(d-1)$, which is the wanted result.

The general case works similarly. Since each E_i is a sum of spectral subspaces of B (associated with complex eigenvalues of the respective type $\lambda^{-1}e^{i\theta}$ and $\lambda e^{i\theta}$, $\theta \in \mathbb{R}$), the projections pr_i are polynomials in B. So, as E is B-stable, $\mathrm{pr}_i(E) \subset E$. As $\mathrm{pr}_i(E) = E_i$, $E \supset E_1 \oplus E_2$ and the result comes like above. $\square$

Lemma 5.3 *Take G as in Proposition 5.1, with $d \geq 2$. Then G acts transitively on the unit tangent bundle $\mathrm{T}^1\mathbb{H}^d$ of $\mathbb{H}^d$.*

Remark. In the case $d = 1$, which we shall not need, the following proves the transitivity of G on $\mathbb{H}^d$.

Proof. Throughout, we take γ a loxodromic element of G, θ its axis and v a unit tangent vector to θ, at some point x of θ.

As G acts isometrically on $\mathbb{H}^d$, its action on $\mathrm{T}^1\mathbb{H}^d$ commutes with the geodesic flow φ^t : $\forall \gamma \in G$, $\forall t \in \mathbb{R}$, $\forall v \in \mathrm{T}^1\mathbb{H}^d$ $\varphi^t(\gamma(v)) = \gamma(\varphi^t(v))$. In particular, Φ is G-equivariant, with Φ the canonical submersion from $\mathrm{T}^1\mathbb{H}^d$ onto $(\mathbb{S}^{d-1})^2 \setminus \Delta$, as set in the introduction. By Lemma 5.2, $\Phi(G.v) = G.\theta$ is open, so

- either $G.v$ is open in $\mathrm{T}^1\mathbb{H}^d$,
- or $G.v$ is transverse to the fibres of Φ, that is to the orbits of the geodesic flow φ^t.

Now, as $\mathrm{Isom}(\mathbb{H}^d, g_{\mathrm{hyp}})$ acts properly on $(\mathbb{H}^d, g_{\mathrm{hyp}})$, it acts properly on its unit tangent bundle $\mathrm{T}^1\mathbb{H}^d$, hence so does G. Consequently, the orbits of G in $\mathrm{T}^1\mathbb{H}^d$ are closed.

So in the first case, $G.v$ is open and closed in $\mathrm{T}^1\mathbb{H}^d$ which is connected, so $G.v = \mathrm{T}^1\mathbb{H}^d$ and we are done.

We complete the proof by supposing that the second case holds and obtaining a contradiction. Before doing this, let us recall a classical fact about the loxodromic elements of $\mathrm{Isom}(\mathbb{H}^d, g_{\mathrm{hyp}})$.

Recall. If h is a loxodromic element of $\mathrm{Isom}(\mathbb{H}^d, g_{\mathrm{hyp}})$ and θ_h its axis, then $h_{|\theta_h}$ is a translation. The (constant) distance $\tau(h) = d(h(m), m)$, for $m \in \theta_h$, is equal to $2\ln(\rho(h))$, where $\rho(h) > 1$ is the spectral radius of $h \in \mathrm{O}(d, 1)$.

As G and φ^t commute, we may associate to each vector w of $\mathrm{T}^1\mathbb{H}^d$, the group $T(w) = \{t \in \mathbb{R}\,;\, \varphi^t(G.w) = G.w\}$. Let us consider $T(v)$. It is a subgroup of $(\mathbb{R}, +)$, so $T(v) = \{0\}$, $T(v) = a\mathbb{Z}$ for some $a \in \mathbb{R}$ or $T(v)$ is dense. The last case is impossible, as $G.v$ is transverse to φ^t and *closed.* So does the first one: $\gamma(v)$ is still tangent, with the same orientation as v, to the axis θ of γ and $\gamma(v) \neq v$, so there is a $t \in \mathbb{R}^*$ such that $\gamma(v) = \varphi^t(v)$ (actually $t = \tau(\gamma)$, see recall above). So $\varphi^{\tau(\gamma)}(G.v) = G.v$ and $T(v) \supset \tau(\gamma)\mathbb{Z} \supsetneq \{0\}$.

Let us finally rule out the second case. As G and φ^t commute, T is invariant along the orbits of φ^t (the first author thanks Thierry Barbot for this remark), so T may be viewed as a function of the (oriented or not, we choose "oriented") geodesics of $\mathbb{H}^d$. For θ such a geodesic, let us set $a(\theta) = \inf(T(\theta) \cap \mathbb{R}_+^*)$. Trivially, T, hence a, is invariant along the orbit $G.\theta$ of θ. If h is a loxodromic element of G, of axis θ_h, the same argument as used above for v gives: $a(\theta_h) \leq \tau(h)$.

By Lemma 5.2, the G-orbits of such geodesics θ_h are open in $(\mathbb{S}^{d-1})^2 \setminus \Delta$, so T and hence a are locally constant on the union $\mathcal{U}$ of these orbits — note that $\mathcal{U}$ is simply the set of the (couples of endpoints of the) axis of loxodromic elements of G. Besides, an element h of $\mathrm{O}(d,1)$ is loxodromic if and only if $\rho(h) > 1$, so $\mathcal{U} = \rho^{-1}(]1, +\infty[)$. As G is connected and as $\mathrm{Id} \in \rho^{-1}(1) \neq \emptyset$, the closure of each connected component of $\mathcal{U}$ contains $\rho^{-1}(1)$; in particular, if $\mathcal{U}_1$ is such a component, $\inf_{\mathcal{U}_1} \rho = 1$. Let a_1 be the constant value of a on $\mathcal{U}_1$; if $h \in \mathcal{U}_1$ and if θ_h is its axis, $a_1 = a(\theta_h) < \tau(h) = 2\ln(\rho(h))$, so $a_1 \leq \inf_{\mathcal{U}_1}(2\ln\circ\rho) = 0$, therefore $a = 0$ on $\mathcal{U}$, which is a contradiction. □

We will also need a little standard lemma, of which we give a proof.

Lemma 5.4 *Let G be a Lie group acting differentiably and transitively on a connected manifold $\mathcal{M}$. Then the identity component G^0 of G acts also transitively on $\mathcal{M}$.*

Proof. Let a be a point in $\mathcal{M}$. As G is a Lie group, the G^0-orbit of a is open in $G.a = \mathcal{M}$, as well as the orbit of a under the action of any connected component of G. As those orbits are pairwise disjoint or equal, they are also closed; in particular, $G^0.a$ is closed. As $\mathcal{M}$ is connected, $G^0.a = \mathcal{M}$. □

Proof of Proposition 5.1. By Lemma 5.3, G acts transitively on $T^1\mathbb{H}^d$. By induction, we will prove that $G \cap \mathrm{Isom}_+(\mathbb{H}^d, g_{\mathrm{hyp}})$ acts transitively on the fibre bundle $\mathcal{B}^+\mathbb{H}^d$ of the positive orthonormal frames on $\mathbb{H}^d$. As an isometry of $(\mathbb{H}^d, g_{\mathrm{hyp}})$ is determined by its differential at any point, it means that $G \supset \mathrm{Isom}_+(\mathbb{H}^d, g_{\mathrm{hyp}}) = \mathrm{SO}^0(d,1)$, which is the wanted result.

Let us take v any element of $\mathrm{T}^1\mathbb{H}^d$ and $H_1 = \exp(v^\perp)$ the hyperbolic hyperplane of $\mathbb{H}^d$, passing through the base point x of v and orthogonal to v. Let $G_1 \subset G$ be the stabilizer of H_1 in G and G_1^0 its identity component. Then G_1^0, which is a closed subgroup of $\mathrm{Isom}(H_1)$, as the condition $g(H_1) \subset H_1$ is closed, acts transitively on H_1. Let us check it.

Let us take $y \in H_1$ and w one of the two unit normal vectors to H_1 at y. As G acts transitively on $\mathrm{T}^1\mathbb{H}^d$, there is an h in G such that $h(v) = w$. The differential $\mathrm{d}h$ of h maps $v^\perp$ on $w^\perp$, so h maps $H_1 = \exp(v^\perp)$ on $H_1 = \exp(w^\perp)$ —both are H_1 as H_1 is totally geodesic—, *i.e.* $h \in G_1$; h maps the base point x of v on y, so G_1 acts transitively on H_1. So does G_1^0, by Lemma 5.4. Notice that G_1^0 cannot then be compact, as H_1 is not compact; so it has loxodromic elements. We will moreover prove the

Claim. G_1^0 has no fixed point at infinity on $\partial H_1 \simeq \mathbb{S}^{d-2}$.

Let us suppose the contrary, then this fixed point p_1 is unique; indeed else G_1^0 would stabilize any geodesic linking two such fixed points, which excluded as G_1^0 is transitive on H_1. Thus, there is an application ψ associating, to each "hyperbolic hyperplane" H_1' of $\mathbb{H}^d$, the fixed point of the identity component $G_{H_1'}^0$ of its stabilizer.

Recall. The set of the hyperbolic hyperplanes of $\mathbb{H}^d$ is parametrized by the set $\mathcal{E} \subset \mathbb{RP}^d$ of the vectorial lines of $\mathbb{R}^{d+1}$ on which g is positive definite. Indeed, H_1' is such an hyperplane of if and only if it is of the form $\mathbb{H}^d \cap E'$, where E' is a hyperplane of $\mathbb{R}^{d+1}$ such that $\mathrm{sgn}(g_{|E'}) = (d-1,1)$, *i.e.* such that $E'^\perp \in \mathcal{E}$. [Hence, $\mathcal{E}$ is canonically diffeomorphic to the de Sitter space $\mathrm{dS}^d = \{m \in \mathbb{R}^{d+1}; g(m,m) = 1\}$, quotiented by antipody $m \sim -m$: a $\delta \in \mathcal{E}$ cuts dS^d in two antipodal points.]

Now G is transitive on $\mathrm{T}^1\mathbb{H}^d$, and consequently also on the set $\{\exp(w^\perp); w \in \mathrm{T}^1\mathbb{H}^d\}$ of the hyperbolic hyperplanes of $\mathbb{H}^d$. So ψ, which is G-equivariant, is defined by $\psi(g.H_1) = g.p_1$, so ψ is differentiable, thus continuous.

Let us fix a geodesic θ of H_1, with endpoints $\{q_-, q_+\} \not\ni p_1$. If θ is the axis of some loxodromic element γ of G_1, we are done. Indeed if so, let us take a loxodromic element h in G_1^0, we denote its axis θ_h; p_1 is necessarily one of its endpoints. Then $\gamma h \gamma^{-1}$ is loxodromic with axis $\gamma(\theta_h)$: as $p_1 \notin \{q_-, q_+\}$, $\gamma(p_1) \neq p_1$, so p_1 is not an endpoint of $\gamma(\theta_h)$. So, $\gamma h \gamma^{-1}(p_1) \neq p_1$, which cannot hold: as G_1^0 is normal in G^1, $\gamma h \gamma^{-1} \in G_1^0$. So we are left with finding a loxodromic element in G_1 with axis θ.

As G acts transitively on $\mathrm{T}^1\mathbb{H}^d$, the stabilizer G_θ of θ in G acts transitively on θ; so does its identity component G_θ^0, by Lemma 5.4. Now, $\mathrm{O}(d,1)_\theta^0$, the full connected stabilizer of θ in $\mathrm{O}(d,1)$, is isomorphic to $\mathbb{R} \times \mathrm{SO}^0(d-1)$, with the first factor $\mathbb{R}$ standing for the translation along θ (see, in the model of the Poincaré half-space, the stabilizer of the half line $\mathrm{i}\mathbb{R}_+^*$; we denote it additively, so $(\mathbb{R},+)$). Note that an $h \in \mathrm{O}(d,1)_\theta^0 \simeq \mathbb{R} \times \mathrm{SO}^0(d-1)$ is loxodromic if and only if its component on $\mathbb{R}$ is not 0. Now G_θ^0 is transitive on θ, which exactly means that its projection on this first factor $\mathbb{R}$ is surjective. So one can find a one-parameter subgroup $\Gamma = \{\gamma_t; t \in \mathbb{R}\}$ of G_θ^0, consisting, except Id, of loxodromic elements, and transitive on θ.

Now let us choose an x in θ and denote by $\mathrm{N}\theta$ the normal bundle of θ in $\mathbb{H}^d$. Each $\gamma^t.H_1$ is a hyperbolic hyperplane containing θ, so it may be parametrized by its unit normal vector $v^t \in \mathbb{S}_x^{d-2} \subset \mathrm{N}_x\theta$ at x. The group Γ acts isometrically on $\mathbb{S}_x^{d-2}$: the image of $w \in \mathbb{S}^{d-2}$ is obtained by pushing back $\gamma^t.w$, by parallel transport along θ, from $\mathrm{T}_{\gamma^t.x}\mathbb{H}^d$ to $\mathrm{T}_x\mathbb{H}^d$; so $v^t = \gamma^t.v^0$. Examining the matricial form of such an isometry, one sees that:

- either $\Gamma.v^0 = \{v^t; t \in \mathbb{R}\}$ is periodic

- or it is not, then its closure $\overline{\Gamma.v^0}$, equal to its ω-limit set $\cap_{n\in\mathbb{N}}\overline{\{v^t; t \geq n\}}$, is diffeomorphic to a torus of dimension 2 or more.

In the first case, taking t such that $v^t = v^0$, γ^t is what we need: a loxodromic element stabilizing H_1. The second case is impossible, for $\psi(\gamma^t.H_1) = \gamma^t.p_1$ would tend, when $t \to \infty$, to q_+: so ψ, which is continuous, would be constant, equal to q, on the ω-limit set of $\Gamma.v^0$, hence equal to q on $\overline{\Gamma.v^0} \ni x$. This contradicts $\psi(H_1) = p_1$. This proves the claim.

By induction, we now finish the proof of Proposition 5.1. As G_1^0 is a closed, connected and transitive subgroup of $\mathrm{Isom}(H_1)$, fixing no point at infinity on $\partial H_1 \simeq \mathbb{S}^{d-2}$, by Step 2, it stabilizes a subsphere of $\mathbb{S}^{d-2}$ (and the hyperbolic subspace of H_1 that it defines) and acts transitively on this sphere. As G_1^0 acts transitively on H_1, this sphere is the whole $\partial H_1 \simeq \mathbb{S}^{d-2}$. So we can apply Lemma 5.3: G_1^0 acts transitively on T^1H_1. In particular, the stabilizer G_v of v acts transitively on the unit vectors of $v^\perp \subset \mathrm{T}_xH_1$. Let $(v_1, \dots, v_d)$ be a positive orthonormal basis of $\mathrm{T}_x\mathbb{H}^d$, with $v_1 = v$. By induction, with $H_k = \exp(\mathrm{span}(v_1, \dots, v_k)^\perp)$, it comes that for each $k < d-1$, the stabilizer $G^0_{(v_1,\dots,v_k)}$ of $(v_1, \dots, v_k)$ acts transitively on the unit vectors of $(\mathrm{span}(v_1, \dots, v_k))^\perp$. Put together, steps $k = 1$ to $k = d-2$ mean exactly that the stabilizer G_x of x acts transitively on the direct orthonormal basis of $\mathrm{T}_x\mathbb{H}^d$. (Notice that the induction stops with step $k = d-2$. Indeed, once fixed the vectors $(v_1, \dots, v_{d-1})$, v_d is also fixed as $G^0_{(v_1,\dots,v_{d-1})} = \{\mathrm{Id}\}$.) As G acts transitively on $\mathbb{H}^d$ by Lemma 5.3, G acts then transitively on $\mathcal{B}^+\mathbb{H}^d$, which completes the proof. □

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