On the approximation to algebraic numbers by algebraic numbers

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Abstract. *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. It follows from a deep result of W. M. Schmidt that, for every positive real number ε, there are infinitely many algebraic numbers α of degree at most n such that $|\xi - \alpha| < H(\alpha)^{-n-1+\varepsilon}$, where $H(\alpha)$ denotes the naïve height of α. We sharpen this result by replacing ε by a function $H \mapsto \varepsilon(H)$ that tends to zero when H tends to infinity. We make a similar improvement for the approximation to algebraic numbers by algebraic integers, as well as for an inhomogeneous approximation problem.*

1. Introduction

In 1955, Roth [14] established that, like almost all real numbers (throughout the present paper, 'almost all' refers to the Lebesgue measure), an algebraic irrational number cannot be approximated by rationals at an order greater than two. This was subsequently generalized by W. M. Schmidt [15] to the approximation by algebraic numbers of bounded degree. Throughout the present note, the height of an integer polynomial $P(X)$, denoted by $H(P)$, is the maximum of the absolute values of its coefficients, and the height of an algebraic number α, denoted by $H(\alpha)$, is the height of its minimal defining polynomial over the integers.

Theorem (Schmidt, 1970). *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. Then, for every positive real number ε, there are only finitely many algebraic numbers α of degree at most n such that*

$$|\xi - \alpha| < H(\alpha)^{-n-1-\varepsilon}.$$

The above statement is one of the many consequences of the Schmidt Subspace Theorem [17]. Combined with a result of Wirsing [20], it implies that real algebraic numbers can be quite well approximated by real algebraic numbers of smaller degree, as stated in Theorem 7I from [16], reproduced below (see also Theorem 2.8 of [2]).

Theorem A. *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. Then, for every positive real number ε, there are infinitely many algebraic*

2000 *Mathematics Subject Classification : 11J68* *numbers α of degree at most n such that*

$$|\xi - \alpha| < H(\alpha)^{-n-1+\varepsilon}. \tag{1.1}$$

An improvement of Theorem A for rational approximation follows immediately from the theory of continued fractions (or from the Dirichlet *Schubfachprinzip*); namely, for any irrational number ξ, there are infinitely many rational numbers p/q such that $|\xi - p/q| < q^{-2}$. As for quadratic approximation, Davenport and Schmidt [8] established a considerably more general result, namely, that for every real number ξ that is not algebraic of degree at most two, there are infinitely many algebraic numbers of degree at most two such that

$$|\xi - \alpha| < 18 \max\{1, |\xi|^2\} H(\alpha)^{-3}.$$

Furthermore, it is well-known (see e.g., [20] or Theorem 2.9 of [2]) that, for every positive integer n and any real algebraic number ξ of degree $n+1$, there are effectively computable positive constants $c_1(\xi)$ and $c_2(\xi)$, depending only on ξ, such that

$$|\xi - \alpha| \geq c_1(\xi)\, H(\alpha)^{-n-1}, \quad \text{for any algebraic number } \alpha \text{ of degree at most } n, \tag{1.2}$$

and

$$|\xi - \alpha| \leq c_2(\xi)\, H(\alpha)^{-n-1}, \quad \text{for infinitely many real algebraic numbers } \alpha \text{ of degree } n.$$

Besides these results, no improvement of Theorem A is known.

The main purpose of the present note is to slightly sharpen Theorem A for all values of n greater than two. We are able to replace ε in (1.1) by a function $H \mapsto \varepsilon(H)$ that tends to zero when H tends to infinity. Furthermore, we display a similar statement regarding approximation to algebraic numbers by algebraic integers, a problem first studied by Mahler [13] and by Davenport and Schmidt [9]. We stress that while (1.1) is a consequence of the Schmidt Subspace Theorem, our improvement rests ultimately on a quantitative version of this powerful tool [18, 10]. Our main results are stated in Section 2 and proved in Section 4. Section 3 is devoted to another inhomogeneous approximation problem, which was investigated in [13, 16].

2. Main results

Our first result sharpens Theorem A.

Theorem 1. *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. Then, there are a positive constant c, depending only on ξ, and infinitely many algebraic numbers α of degree n such that*

$$|\xi - \alpha| < H(\alpha)^{-n-1+c(\log\log 3H(\alpha))^{-1/(2n+6)}}.$$

The key ingredient in the proof of Theorem 1 is a multidimensional extension of the Cugiani–Mahler Theorem [7, 12] obtained in [3] and whose proof rests on the quantitative version of the Subspace Theorem worked out by Evertse and Schlickewei [10]. Apart from this, our argument is classical and originates in [9] (see also [5, 19, 1, 2]). It illustrates the relationship between uniform homogeneous approximation to a system of linear forms and inhomogeneous approximation to the system of dual linear forms [6, 4].

Besides approximation by algebraic numbers, one may also study approximation by algebraic *integers*, as Mahler [13] and Davenport and Schmidt [9] did. At the beginning of the latter paper, the authors established that, for every real irrational number ξ, there are a constant $c_3(\xi)$, depending only on ξ, and infinitely many algebraic integers of degree two such that

$$|\xi - \alpha| < c_3(\xi)\, H(\alpha)^{-2}.$$

We reproduce below Theorem 7K from [16], whose proof combines ideas from [9] with an application of the Subspace Theorem.

Theorem B. *Let $n \geq 3$ be an integer. Let ξ be an algebraic real number of degree greater than or equal to n, which is not an algebraic integer of degree n. Then, for every positive real number ε, there are infinitely many algebraic integers α of degree at most n such that*

$$|\xi - \alpha| < H(\alpha)^{-n+\varepsilon}.$$

Using the same approach as for the proof of Theorem 1, we are able to sharpen Theorem B as follows.

Theorem 2. *Let $n \geq 3$ be an integer. Let ξ be an algebraic real number of degree greater than n. Then there are a positive constant c, depending only on ξ, and infinitely many algebraic integers α of degree n such that*

$$|\xi - \alpha| < H(\alpha)^{-n+c(\log\log 3H(\alpha))^{-1/(2n+4)}}. \tag{2.1}$$

A much better result is known for approximation to an algebraic number by algebraic integers of the same degree or of larger degree. Namely, Theorem 2.11 of [2] asserts that if ξ is an algebraic number of degree d, then, for any integer $n \geq d$, there exist a constant $c_4(n, \xi)$, depending only on n and on ξ, and infinitely many algebraic integers α of degree n such that

$$|\xi - \alpha| < c_4(n, \xi)\, H(\alpha)^{-d}.$$

Furthermore, under the same assumption, there exists a constant $c_5(n, \xi)$, depending only on n and on ξ, such that

$$|\xi - \alpha| > c_5(n, \xi)\, H(\alpha)^{-d}$$

holds for every algebraic number α of degree at most n and different from ξ. The latter statement is simply a version of the so-called Liouville inequality, as it is the case for (1.2) above.

We stress that in Theorems 1 and 2 we impose the exact degree of the approximants, unlike in Theorems A and B and unlike in the result of Davenport and Schmidt [8] mentioned in the Introduction. This is achieved by means of an argument from [5].

Probably, more than Theorem 1 should be true and we believe that the answer to Problem 1 below is positive.

Problem 1. *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. To prove that there are a positive constant c, depending only on ξ, and infinitely many algebraic numbers α of degree n such that*

$$|\xi - \alpha| < cH(\alpha)^{-n-1}.$$

A positive answer to Problem 1 would follow from a positive answer to Problem 2 below.

Problem 2. *Let n be a positive integer. Let ξ be an algebraic real number of degree greater than n. To prove that there are a positive constant c, depending only on ξ, and arbitrarily large real numbers H such that the system of inequalities*

$$|P(\xi)| < cH^{-n}, \quad H(P) \leq H,$$

has no solution in non-zero integer polynomial $P(X)$ of degree at most n.

Problem 2 amounts to prove that the n-tuple $(\xi, \ldots, \xi^n)$, with ξ as above, is regular, in the sense of Cassels [6], page 92. Note that Theorem 7 of [1] asserts that, for almost all real numbers ξ, the n-tuple $(\xi, \ldots, \xi^n)$ is regular for every positive integer n (see [11] for an alternative proof).

We do not mention further related results and problems on approximation by algebraic numbers. The interested reader is directed to the monograph [2].

3. An inhomogeneous approximation problem

By means of a suitable extension of the proof of Roth's Theorem, Mahler [13] established that, for every quadratic irrationality ξ, every real algebraic number β and every positive real number ε, there exist at most finitely many pairs of rational integers $x \neq 0$, y such that

$$|x\xi + y + \beta| \leq |x|^{-1-\varepsilon}.$$

This result was generalized by Schmidt [16], Corollary 7F, who applied his Subspace Theorem to get the following statement.

Theorem (Schmidt, 1971). *Suppose $\xi_1, \ldots, \xi_\ell, \beta$ are real algebraic numbers such that β is not a linear combination of $\xi_1, \ldots, \xi_\ell$ with rational integer coefficients. Then, for every positive real number ε, there are only finitely many integer ℓ-tuples $(q_1, \ldots, q_\ell)$ with $q = \max\{|q_1|, \ldots, |q_\ell|\} > 0$ and*

$$|q_1\xi_1 + \ldots + q_\ell\xi_\ell + \beta| < q^{-\ell+1-\varepsilon}.$$

In the opposite direction, Schmidt observed that combining Corollary 7D from [16] with certain transference theorems (e.g., from Chapter V of Cassels' monograph [6]) yields

Corollary (Schmidt, 1971). *Suppose $\xi_1, \ldots, \xi_\ell$ are real algebraic numbers that are linearly independent over the field of rational numbers. Then, for every real number β and every positive real number ε, there are infinitely many integer ℓ-tuples $(q_1, \ldots, q_\ell)$ with $q = \max\{|q_1|, \ldots, |q_\ell|\} > 0$ and*

$$|q_1\xi_1 + \ldots + q_\ell\xi_\ell + \beta| < q^{-\ell+1+\varepsilon}.$$

Like in the previous section, we are able to improve the above corollary by means of the results from [3].

Theorem 3. *Suppose $\xi_1, \ldots, \xi_\ell$ are real algebraic numbers that are linearly independent over the field of rational numbers. Then, for every real number β, there are a constant c, depending only on $\xi_1, \ldots, \xi_\ell, \beta$, and infinitely many integer ℓ-tuples $(q_1, \ldots, q_\ell)$ with $q = \max\{|q_1|, \ldots, |q_\ell|\} \geq 3$ and*

$$|q_1\xi_1 + \ldots + q_\ell\xi_\ell + \beta| < q^{-\ell+1+c(\log\log q)^{-1/(2\ell+4)}}. \tag{3.1}$$

The proof of Theorem 3 combines Theorem 3.1 of [3] with a transference lemma relating problems of uniform homogeneous approximation to problems of inhomogeneous approximation, see [6] or [4] (in particular Lemma 3). We omit the details. We content ourselves to point out that the quantity $1/(2\ell+4)$ occurs in (3.1) since the proof of Theorem 3 begins with an application of Theorem 3.1 of [3] to the $\ell-1$ real numbers $\xi_1/\xi_\ell, \ldots, \xi_{\ell-1}/\xi_\ell$. In particular, for $\ell = 2$, the version of the Cugiani–Mahler Theorem stated as Corollary 2.3 in [3] allows us to replace the exponent $1/(2\ell+4)$ in (3.1) by any real number smaller than $1/3$.

4. Proofs

The proofs of Theorems 1 and 2 heavily depend on an immediate consequence of a multidimensional extension of the Cugiani–Mahler theorem. We reproduce below Corollary 2 from [3].

Theorem (Bugeaud, 2007). *Let n be a positive integer and ξ be a real algebraic number of degree greater than n. Let $\varepsilon : \mathbf{Z}_{\geq 1} \to \mathbf{R}_{>0}$ be a non-increasing function satisfying*

$$\lim_{H\to+\infty} \frac{\varepsilon(H)}{(\log\log H)^{-1/(2n+6)}} = +\infty.$$

Let $(P_j(X))_{j\geq 1}$ be the sequence of distinct primitive, integer polynomials of degree at most n that satisfy

$$|P(\xi)| < H(P)^{-n-\varepsilon(H(P))},$$

ordered such that $1 \leq H(P_1) \leq H(P_2) \leq \ldots$ If this sequence is infinite, then

$$\limsup_{j\to+\infty} \frac{\log H(P_{j+1})}{\log H(P_j)} = +\infty.$$

From the above theorem we derive the following auxiliary statement.

Corollary 1. *Let n be a positive integer and ξ be a real algebraic number of degree greater than n. Then there are a positive constant c, depending only on ξ, and arbitrarily large real numbers H such that the system of inequalities*

$$|P(\xi)| < H^{-n-c(\log\log H)^{-1/(2n+6)}}, \quad H(P) \le H,$$

has no solution in non-zero integer polynomial $P(X)$ of degree at most n.

Proof. We argue by contradiction. If Corollary 1 does not hold, then there exists a function ε satisfying the hypotheses of the preceding theorem with the property that for each positive real number H there exists a non-zero integer polynomial $P(X)$ of degree at most n and height at most H with $|P(\xi)| \le H^{-n-\varepsilon(H)}$. From this one infers the existence of an infinite sequence of non-zero primitive integer polynomials $P_1(X), P_2(X), \dots$ of degree at most n and with $H(P_1) < H(P_2) < \dots$ such that

$$|P_j(\xi)| \le H(P_{j+1})^{-n-\varepsilon(H(P_{j+1}))}, \quad j \ge 1.$$

Applying the preceding theorem then gives $|P_j(\xi)| \le H(P_j)^{-d}$ for each j sufficiently large, where d is the degree of ξ. This contradicts Liouville's inequality (see [2], Theorem A.1) since $H(P_j)$ tends to infinity with j, while none of the algebraic numbers $P_j(\xi)$ vanishes. □

Proofs of Theorems 1 and 2. We proceed now following Davenport and Schmidt [9] and Bugeaud and Teulié [5] (see also [19, 1, 2]). Although this method is now well-known and has appeared in a few papers, we decide to include all the details.

Let n and ξ be as in the statement of Theorem 1. It follows from Corollary 1 that there are a positive constant c, depending only on ξ, and arbitrarily large real numbers H for which the first minimum $\lambda_1(H)$ of the convex body $\mathcal{C}(H)$ defined by

$$|x_n\xi^n + \dots + x_1\xi + x_0| \le H^{-n}, \quad |x_1|, \dots, |x_n| \le H,$$

satisfies

$$\lambda_1(H) \ge H^{-c(\log\log H)^{-1/(2n+6)}}.$$

Consequently, by Minkowski's second theorem (see [2], Theorem B.3), the last minimum of $\mathcal{C}(H)$ satisfies

$$\lambda_{n+1}(H) \le 2^{n+1}\, H^{nc(\log\log H)^{-1/(2n+6)}}. \tag{4.1}$$

Consider a real number H for which (4.1) holds and set $\lambda := \lambda_{n+1}(H)$ for shortness. Thus, there exist $n+1$ linearly independent integer polynomials $P_j(X) := x_n^{(j)}X^n + \dots + x_1^{(j)}X + x_0^{(j)}$, for $j = 1, \dots, n+1$, of degree at most n, satisfying

$$|P_j(\xi)| \le \lambda H^{-n} \quad \text{and} \quad H(P_j) \le \lambda H, \tag{4.2}$$

for $j = 1, \dots, n+1$, and $\delta := |\det(x_i^{(j)})| \le (n+1)!$ (see [19] or use again [2], Theorem B.3). Let p be a prime number which does not divide δ. Then, there exists a superscript j such that p does not divide $x_0^{(j)}$; without loss of generality, we assume that $j = 1$.

Now, we construct integer, irreducible polynomials of prescribed degree, that are small when evaluated at ξ. To deal simultaneously with Theorems 1 and 2, we introduce an integer m which is greater than or equal to n. We consider the following linear system of $n+1$ equations in the $n+1$ real unknowns $\theta_1, \ldots, \theta_{n+1}$:

$$\begin{gathered}
\xi^m + p(\theta_1 P_1(\xi) + \ldots + \theta_{n+1} P_{n+1}(\xi)) = p(n+2)\lambda H^{-n} \\
m\xi^{m-1} + p(\theta_1 P_1'(\xi) + \ldots + \theta_{n+1} P_{n+1}'(\xi)) = p\lambda H + p \sum_{1 \le j \le n+1} |P_j'(\xi)| \\
\theta_1 x_h^{(1)} + \ldots + \theta_{n+1} x_h^{(n+1)} = 0 \ (h = 2, \ldots, n).
\end{gathered} \tag{4.3}$$

Since the polynomials $P_j(X)$, for $j = 1, \ldots, n+1$, are linearly independent, this system has one and only one solution $(\theta_1, \ldots, \theta_{n+1})$. We take a $(n+1)$-tuple $(t_1, \ldots, t_{n+1})$ of integers such that $|\theta_i - t_i| \le 1$ for $i = 1, \ldots, n+1$, and we set

$$x_h = t_1 x_h^{(1)} + \ldots + t_{n+1} x_h^{(n+1)}, \text{ for } h = 0, \ldots, n.$$

We consider the polynomial

$$\begin{aligned}
P(X) &= X^m + p(x_n X^n + \ldots + x_1 X + x_0) \\
&= X^m + p(t_1 P_1(X) + \ldots + t_{n+1} P_{n+1}(X)),
\end{aligned}$$

which, by a suitable choice of $(t_1, \ldots, t_{n+1})$, is irreducible. Indeed, by using Eisenstein's Criterion, it is sufficient to check that its constant coefficient, namely $p(t_1 x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)})$, is not divisible by p^2, since its leading coefficient is congruent to 1 modulo p (recall that $m \ge n$). We fix an n-tuple $(t_2, \ldots, t_{n+1})$ and there remain two possible choices for t_1, which we denote by $t_{1,0}$ and $t_{1,1} = t_{1,0} + 1$. Since p does not divide $x_0^{(1)}$, at least one of the integers $t_{1,0} x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)}$ or $t_{1,1} x_0^{(1)} + \ldots + t_{n+1} x_0^{(n+1)}$ is not divisible by p. This enables us to choose t_1 such that the polynomial $P(X)$ is Eisensteinian with respect to the prime number p, hence, irreducible.

By (4.2) and the first two equations of (4.3), the polynomial $P(X)$ satisfies

$$0 < p\lambda H^{-n} \le P(\xi) \le p(2n+3)\lambda H^{-n} \tag{4.4}$$

and

$$p\lambda H \le P'(\xi) \le p\lambda H + 2p \sum_{j=1}^{n+1} |P_j'(\xi)| \ll p\lambda H. \tag{4.5}$$

Here and below, the numerical constant implied by $\ll$ depends at most on ξ and on m. Finally, by (4.1), (4.2) and the last equations of (4.3), we get

$$H(P) \ll \lambda H \ll H^{1+nc(\log\log H)^{-1/(2n+6)}}. \tag{4.6}$$

By (4.4), (4.5), and the first assertion of Lemma A.5 of [2], the polynomial $P(X)$ has a root α satisfying

$$|\xi - \alpha| \le m \frac{P(\xi)}{P'(\xi)} \le m(2n+3)H^{-n-1}. \tag{4.7}$$

It then follows from (4.6) and (4.7) that there exists a positive constant κ, depending only on ξ and on m, such that

$$|\xi - \alpha| < H(\alpha)^{-n-1+\kappa(\log\log 3H(\alpha))^{-1/(2n+6)}}. \tag{4.8}$$

Since there are arbitrarily large values of H such that (4.1) holds, we get infinitely many algebraic numbers α of degree m satisfying (4.8).

For a proof of Theorem 1, we simply choose $m = n$. To get Theorem 2, we take $m = n+1$ and notice that the resulting statement deals then with approximation by algebraic integers of degree $n+1$. This explains why the exponent of $\log\log 3H(\alpha)$ in (2.1) is $1/(2n+4)$ and not $1/(2n+6)$.

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