Automatic continued fractions are transcendental or quadratic

Les fractions continues automatiques sont transcendantes ou quadratiques

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Abstract. *We establish new combinatorial transcendence criteria for continued fraction expansions. Let $\alpha = [0; a_1, a_2, \ldots]$ be an algebraic number of degree at least three. One of our criteria implies that the sequence of partial quotients $(a_\ell)_{\ell \ge 1}$ of α is not 'too simple' (in a suitable sense) and cannot be generated by a finite automaton.*

Résumé. *Nous établissons de nouveaux critères combinatoires de transcendance pour des développements en fraction continue. Soit $\alpha = [0; a_1, a_2, \ldots]$ un nombre algébrique de degré au moins égal à trois. L'un de nos critères entraîne que la suite $(a_\ell)_{\ell \ge 1}$ des quotients partiels de α n'est pas trop simple (en un certain sens) et ne peut pas être engendrée par un automate fini.*

1. Introduction and results

A well-known open question in Diophantine approximation asks whether the continued fraction expansion of an irrational algebraic number α either is ultimately periodic (this is the case if, and only if, α is a quadratic irrational), or contains arbitrarily large partial quotients. As a preliminary step towards its resolution, several transcendence criteria for continued fraction expansions have been established recently [1, 4, 5, 9] (we refer the reader to these papers for references to earlier works, which include [23, 14, 27, 12]) by means of a deep tool from Diophantine approximation, namely the Schmidt Subspace Theorem (see Theorem 2.1 below). In the present note, we show how a slight modification of their proofs allows us to considerably improve two of these criteria. We begin by pointing out two important consequences of one of our new criteria. Thus, we solve two problems addressed

2000 *Mathematics Subject Classification :* 11J70, 11J81, 11J87. Keywords: continued fractions, transcendence. Mots clefs : fractions continues, transcendance.

and discussed in [1] and we establish for continued fraction expansions of algebraic numbers the analogues of the results of [3] on the expansions of algebraic numbers to an integer base.

Throughout this note, $\mathcal{A}$ denotes a finite or infinite set, called the alphabet. We identify a sequence $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ of elements from $\mathcal{A}$ with the infinite word $a_1 a_2 \ldots a_\ell \ldots$, as well denoted by $\mathbf{a}$. This should not cause any confusion.

For $n \geq 1$, we denote by $p(n, \mathbf{a})$ the number of distinct blocks of n consecutive letters occurring in the word $\mathbf{a}$, that is,

$$p(n, \mathbf{a}) := \mathrm{Card}\{a_{\ell+1} \ldots a_{\ell+n} : \ell \geq 0\}.$$

The function $n \mapsto p(n, \mathbf{a})$ is called the *complexity function* of $\mathbf{a}$. A well-known result of Morse and Hedlund [24, 25] asserts that $p(n, \mathbf{a}) \geq n + 1$ for $n \geq 1$, unless $\mathbf{a}$ is ultimately periodic (in which case there exists a constant C such that $p(n, \mathbf{a}) \leq C$ for $n \geq 1$).

Our first result asserts that the complexity function of the sequence of partial quotients $(a_\ell)_{\ell \geq 1}$ of an algebraic number

$$[0; a_1, a_2, \ldots, a_\ell, \ldots] = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

of degree at least three cannot increase too slowly.

Theorem 1.1. *Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of positive integers which is not ultimately periodic. If the real number*

$$[0; a_1, a_2, \ldots, a_\ell, \ldots]$$

is algebraic, then

$$\lim_{n \to +\infty} \frac{p(n, \mathbf{a})}{n} = +\infty. \tag{1.1}$$

Theorem 1.1 improves Theorem 7 from [12] and Theorem 4 from [1], where

$$\lim_{n \to +\infty} p(n, \mathbf{a}) - n = +\infty$$

was proved instead of (1.1). This gives a positive answer to Problem 3 of [1] (we have chosen here a different formulation).

An infinite sequence $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ is an automatic sequence if it can be generated by a finite automaton, that is, if there exists an integer $k \geq 2$ such that a_ℓ is a finite-state function of the representation of ℓ in base k, for every $\ell \geq 1$. We refer the reader to [13] for a more precise definition and examples of automatic sequences. Let $b \geq 2$ be an integer. In 1968, Cobham [19] asked whether a real number whose b-ary expansion can be generated by a finite automaton is always either rational or transcendental. A positive answer to Cobham's question was recently given in [3]. We addressed in [1] the analogous question for continued fraction expansions. Since the complexity function of every automatic sequence $\mathbf{a}$ satisfies $p(n, \mathbf{a}) = O(n)$ (this was proved by Cobham [20] in 1972), Theorem 1.1 implies straightforwardly a negative answer to Problem 1 of [1].

Theorem 1.2. *The continued fraction expansion of an algebraic number of degree at least three cannot be generated by a finite automaton.*

The proofs of Theorems 1.1 and 1.2 rest ultimately on a combinatorial transcendence criterion established by means of the Schmidt Subspace Theorem. This is also the case for the similar results about expansions of irrational algebraic numbers to an integer base, see [3, 10].

Before stating our criteria, we introduce some notation. The length of a word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$. We denote the mirror image of a finite word $W := a_1 \dots a_\ell$ by $\overline{W} := a_\ell \dots a_1$. In particular, W is a palindrome if, and only if, $W = \overline{W}$.

Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition $(*)$ if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, either the word $W_n U_n V_n U_n$ or the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;

(iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;

(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Equivalently, the word $\mathbf{a}$ satisfies Condition $(*)$ if there exists a positive real number ε such that, for arbitrarily large integers N, the prefix $a_1 a_2 \dots a_N$ of $\mathbf{a}$ contains two disjoint occurrences of a word of length $[\varepsilon N]$ or it contains a word U of length $[\varepsilon N]$ and its mirror image $\overline{U}$, provided that U and $\overline{U}$ do not overlap. Here and below, $[\cdot]$ denotes the integer part function.

We summarize our two new combinatorial transcendence criteria in the following theorem.

Theorem 1.3. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition $(*)$, then α is transcendental.

Theorem 1.3 is the disjoint union of two transcendence criteria. A first one applies to stammering continued fractions, where the terminology 'stammering' means that in (i) the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n; see Theorem 3.1. A second one is concerned with quasi-palindromic continued fractions, where the terminology 'quasi-palindromic' means that in (i) the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$ for infinitely many n; see Theorem 5.1. The condition that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ has to be bounded is not very restrictive, since it is satisfied by almost all real numbers (in the sense of the Lebesgue measure). Furthermore, it is clearly satisfied when $(a_\ell)_{\ell\geq 1}$ is bounded. Note that this condition can be removed if $\mathbf{a}$ begins with arbitrarily large squares $U_n U_n$ (Theorem 2.1 from [9]) or with arbitrarily large palindromes $U_n \overline{U}_n$ (Theorem 2.1 from [5]).

Theorem 1.3 encompasses all the combinatorial transcendence criteria for continued fraction expansions established in [1, 4, 5, 9] under the assumption that the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded.

Let $\mathbf{a}$ be a sequence of positive integers. If there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that $\limsup_{n\to+\infty} |W_n|/|U_n|$ is sufficiently small and $\mathbf{a}$ satisfies Condition $(*)$, then the transcendence of $[0; a_1, a_2, \ldots]$ was already proved in [1, 9, 5]. The novelty in Theorem 1.3 is that we allow $|W_n|$ to be large, provided however that the quotients $|W_n|/|U_n|$ remain bounded independently of n. This is crucial for the proofs of Theorems 1.1 and 1.2.

At present, we do not know any transcendence criterion involving palindromes for expansions to integer bases; however, see [2].

We end this section with an application of Theorem 3.1 to quasi-periodic continued fractions.

Theorem 1.4. *Consider the quasi-periodic continued fraction*

$$\alpha = [0; a_1, \ldots, a_{n_0-1}, \underbrace{a_{n_0}, \ldots, a_{n_0+r_0-1}}_{\lambda_0}, \underbrace{a_{n_1}, \ldots, a_{n_1+r_1-1}}_{\lambda_1}, \ldots],$$

where the notation means that $n_{k+1} = n_k + \lambda_k r_k$ and the λ's indicate the number of times a block of partial quotients is repeated. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to α. Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If the sequence $(a_\ell)_{\ell\geq 1}$ is not ultimately periodic and

$$\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1, \tag{1.2}$$

then the real number α is transcendental.

Theorem 1.4 improves Theorem 3.4 from [4], where, instead of the assumption (1.2), the stronger condition $\liminf_{k\to\infty} \lambda_{k+1}/\lambda_k > 2$ was required.

2. Auxiliary results

We gather below several classical results from the theory of continued fractions. Standard references include [26, 22, 29].

Let $\alpha := [0; a_1, a_2, \ldots]$ be a real irrational number. Set $p_{-1} = q_0 = 1$ and $q_{-1} = p_0 = 0$. For $\ell \geq 1$, the ℓ-th convergent to α is the rational number $p_\ell/q_\ell := [0; a_1, a_2, \ldots, a_\ell]$. Observe that

$$q_\ell = a_\ell q_{\ell-1} + q_{\ell-2}, \quad \ell \geq 1. \tag{2.1}$$

Furthermore, the sequence $(q_\ell)_{\ell\geq 1}$ is increasing and q_ℓ and $q_{\ell+1}$ are coprime for $\ell \geq 0$.

The theory of continued fraction implies that (see e.g. Theorem 164 of [22])

$$|q_\ell \alpha - p_\ell| < q_{\ell+1}^{-1}, \quad \text{for } \ell \geq 1, \tag{2.2}$$

and

$$q_{\ell+h} \geq q_\ell (\sqrt{2})^{h-1}, \quad \text{for } h, \ell \geq 1. \tag{2.3}$$

Indeed, an easy induction on h based on (2.1) proves (2.3) for every fixed value of $\ell \geq 1$.

Likewise, an induction based on (2.1) allows us to establish the mirror formula

$$\frac{q_{\ell-1}}{q_\ell} = [0; a_\ell, a_{\ell-1}, \ldots, a_1], \quad \ell \geq 1. \tag{2.4}$$

The main tool for the proof of Theorem 1.3 is the Schmidt Subspace Theorem.

Theorem 2.1 (W. M. Schmidt). *Let $m \geq 2$ be an integer. Let $L_1, \ldots, L_m$ be linearly independent linear forms in $\mathbf{x} = (x_1, \ldots, x_m)$ with algebraic coefficients. Let ε be a positive real number. Then, the set of solutions $\mathbf{x} = (x_1, \ldots, x_m)$ in $\mathbf{Z}^m$ to the inequality*

$$|L_1(\mathbf{x}) \ldots L_m(\mathbf{x})| \leq (\max\{|x_1|, \ldots, |x_m|\})^{-\varepsilon}$$

lies in finitely many proper linear subspaces of $\mathbf{Q}^m$.

Proof. See e.g. [28, 29]. □

3. Transcendence criterion for stammering continued fractions

In this section we establish the part of Theorem 1.3 dealing with stammering continued fractions. Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition ($\spadesuit$) if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n \geq 1}$, $(V_n)_{n \geq 1}$ and $(W_n)_{n \geq 1}$ such that:

(i) For every $n \geq 1$, the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$;

(ii) The sequence $(|V_n|/|U_n|)_{n \geq 1}$ is bounded from above;

(iii) The sequence $(|W_n|/|U_n|)_{n \geq 1}$ is bounded from above;

(iv) The sequence $(|U_n|)_{n \geq 1}$ is increasing.

Theorem 3.1. *Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition ($\spadesuit$), then α is transcendental.

Theorem 3.1 improves Theorem 2 from [1] and Theorem 3.1 from [9]. Furthermore, it contains Theorem 3.2 from [4].

Theorem 3.1 is the exact analogue of the combinatorial transcendence criterion for expansions to integer bases proved in [10]. Although its proof is very close to that of Theorem 2 of [1], we have decided to write it completely. The new ingredient is estimate (3.4) below.

Proof. Throughout, the constants implied in $\ll$ depend only on α. Assume that the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ occurring in the definition of Condition ($\spadesuit$) are fixed. For $n \geq 1$, set $u_n = |U_n|$, $v_n = |V_n|$ and $w_n = |W_n|$. We assume that the real number $\alpha := [0; a_1, a_2, \ldots]$ is algebraic of degree at least three. Set $p_{-1} = q_0 = 1$ and $q_{-1} = p_0 = 0$.

We observe that α admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for every positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$
\begin{aligned}
b_h^{(n)} &= a_h \quad \text{for } 1 \leq h \leq w_n + u_n + v_n, \\
b_{w_n+h+j(u_n+v_n)}^{(n)} &= a_{w_n+h} \quad \text{for } 1 \leq h \leq u_n + v_n \text{ and } j \geq 0.
\end{aligned}
$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is ultimately periodic, with preperiod W_n and with period $U_n V_n$. Set

$$\alpha_n = [0; b_1^{(n)}, b_2^{(n)}, \ldots, b_k^{(n)}, \ldots]$$

and note that, since the first $w_n + u_n + v_n + u_n$ partial quotients of α and of α_n are the same, it follows from (2.2) that

$$\left|\alpha - \frac{p_{w_n+2u_n+v_n}}{q_{w_n+2u_n+v_n}}\right| < \frac{1}{q^2_{w_n+2u_n+v_n}} \quad \text{and} \quad \left|\alpha_n - \frac{p_{w_n+2u_n+v_n}}{q_{w_n+2u_n+v_n}}\right| < \frac{1}{q^2_{w_n+2u_n+v_n}},$$

thus,

$$|\alpha - \alpha_n| \leq 2q^{-2}_{w_n+2u_n+v_n}. \tag{3.1}$$

Furthermore, an elementary computation (see e.g. [26] on page 71) shows that α_n is root of the quadratic polynomial

$$
\begin{aligned}
P_n(X) := {} & (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})X^2 \\
& - (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1} + p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1})X \\
& + (p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}).
\end{aligned}
$$

Since α_n lies in $(0, 1)$, we have $p_\ell \leq q_\ell$ for $\ell \geq 1$ and the height $H(P_n)$ of the polynomial $P_n(X)$ (the height $H(P)$ of an integer polynomial $P(X)$ is the maximum of the absolute values of its coefficients) is at most equal to $2q_{w_n}q_{w_n+u_n+v_n}$. By (2.2), we have

$$
\begin{aligned}
& |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1})| \\
& \quad \leq \; q_{w_n-1}|q_{w_n+u_n+v_n}\alpha - p_{w_n+u_n+v_n}| + q_{w_n}|q_{w_n+u_n+v_n-1}\alpha - p_{w_n+u_n+v_n-1}| \\
& \quad \leq 2 \; q_{w_n} \, q^{-1}_{w_n+u_n+v_n}
\end{aligned}
\tag{3.2}
$$

and, likewise,

$$
\begin{aligned}
& |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1})| \\
& \quad \leq \; q_{w_n+u_n+v_n}|q_{w_n-1}\alpha - p_{w_n-1}| + q_{w_n+u_n+v_n-1}|q_{w_n}\alpha - p_{w_n}| \\
& \quad \leq 2 \; q^{-1}_{w_n} \, q_{w_n+u_n+v_n}.
\end{aligned}
\tag{3.3}
$$

Using (3.1), (3.2), and (3.3), we then get

$$\begin{aligned}
&|P_n(\alpha)| = |P_n(\alpha) - P_n(\alpha_n)| \\
&\ = |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})(\alpha-\alpha_n)(\alpha+\alpha_n) \\
&- (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1} + p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1})(\alpha-\alpha_n)| \\
&\ = |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}) \\
&+ (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1}) \\
&+ (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})(\alpha_n-\alpha)| \cdot |\alpha-\alpha_n| \\
&\ \ll |\alpha-\alpha_n| \cdot \big(q_{w_n}\, q_{w_n+u_n+v_n}^{-1} + q_{w_n}^{-1}\, q_{w_n+u_n+v_n} + q_{w_n}q_{w_n+u_n+v_n}|\alpha-\alpha_n|\big) \\
&\ \ll |\alpha-\alpha_n| q_{w_n}^{-1}\, q_{w_n+u_n+v_n} \\
&\ \ll q_{w_n}^{-1}\, q_{w_n+u_n+v_n}\, q_{w_n+2u_n+v_n}^{-2}.
\end{aligned} \tag{3.4}$$

This estimate is more precise than the upper bound

$$|P_n(\alpha)| \ll H(P_n) \cdot |\alpha-\alpha_n| \ll q_{w_n}\, q_{w_n+u_n+v_n}\, q_{w_n+2u_n+v_n}^{-2}$$

used in [1]; namely, we gain a factor $q_{w_n}^{-2}$. This improvement is crucial when w_n is large.

We consider the four linearly independent linear forms:

$$\begin{aligned}
L_1(X_1,X_2,X_3,X_4) &= \alpha^2 X_1 - \alpha(X_2+X_3) + X_4, \\
L_2(X_1,X_2,X_3,X_4) &= \alpha X_1 - X_2, \\
L_3(X_1,X_2,X_3,X_4) &= \alpha X_1 - X_3, \\
L_4(X_1,X_2,X_3,X_4) &= X_1.
\end{aligned}$$

Evaluating them on the quadruple

$$\begin{aligned}
\underline{v}_n := (&q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1}, q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}, \\
&p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1}, p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}),
\end{aligned}$$

it follows from (3.2), (3.3), (3.4), and (2.3) that

$$\begin{aligned}
\prod_{1\le j\le 4} |L_j(\underline{v}_n)| &\ll q_{w_n+u_n+v_n}^2\, q_{w_n+2u_n+v_n}^{-2} \\
&\ll 2^{-u_n} \\
&\ll (q_{w_n}q_{w_n+u_n+v_n})^{-\delta u_n/(2w_n+u_n+v_n)},
\end{aligned}$$

if n is sufficiently large, where we have set

$$M = 1 + \limsup_{\ell\to+\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.$$

Since $\mathbf{a}$ satisfies Condition (♠), we have

$$\liminf_{n\to+\infty} \frac{u_n}{2w_n+u_n+v_n} > 0.$$

Consequently, there exists $\varepsilon > 0$ such that

$$\prod_{1\le j\le 4} |L_j(\underline{v}_n)| \ll (q_{w_n}\, q_{w_n+u_n+v_n})^{-\varepsilon}$$

holds for any sufficiently large integer n.

It then follows from Theorem 2.1 that the points $\underline{v}_n$ lie in a finite union of proper linear subspaces of $\mathbf{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set $\mathcal{N}_1$ of distinct positive integers such that

$$\begin{aligned} & x_1(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1}) + x_2(q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}) \\ & + x_3(p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1}) + x_4(p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}) \\ & = 0, \end{aligned} \tag{3.5}$$

for any n in $\mathcal{N}_1$.

• First case: we assume that there exist an integer ℓ and infinitely many integers n in $\mathcal{N}_1$ with $w_n = \ell$.

By extracting an infinite subset of $\mathcal{N}_1$ if necessary and by considering the real number $[0; a_{\ell+1}, a_{\ell+2}, \ldots]$ instead of α, we may without loss of generality assume that $w_n = \ell = 0$ for any n in $\mathcal{N}_1$.

Then, recalling that $q_{-1} = p_0 = 0$ and $q_0 = p_{-1} = 1$, we deduce from (3.5) that

$$x_1 q_{u_n+v_n-1} + x_2 p_{u_n+v_n-1} - x_3 q_{u_n+v_n} - x_4 p_{u_n+v_n} = 0, \tag{3.6}$$

for any n in $\mathcal{N}_1$. Observe that $(x_1, x_2) \neq (0, 0)$, since, otherwise, by letting n tend to infinity along $\mathcal{N}_1$ in (3.6), we would get that the real number α is rational. Dividing (3.6) by $q_{u_n+v_n}$, we obtain

$$x_1 \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} + x_2 \frac{p_{u_n+v_n-1}}{q_{u_n+v_n-1}} \cdot \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} - x_3 - x_4 \frac{p_{u_n+v_n}}{q_{u_n+v_n}} = 0. \tag{3.7}$$

By letting n tend to infinity along $\mathcal{N}_1$ in (3.7), we get that

$$\beta := \lim_{\mathcal{N}_1 \ni n \to +\infty} \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} = \frac{x_3 + x_4\alpha}{x_1 + x_2\alpha}.$$

Furthermore, observe that, for any sufficiently large integer n in $\mathcal{N}_1$, we have

$$\left| \beta - \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} \right| = \left| \frac{x_3 + x_4\alpha}{x_1 + x_2\alpha} - \frac{x_3 + x_4 p_{u_n+v_n}/q_{u_n+v_n}}{x_1 + x_2 p_{u_n+v_n-1}/q_{u_n+v_n-1}} \right| \ll \frac{1}{q_{u_n+v_n-1}q_{u_n+v_n}}, \tag{3.8}$$

by (2.2). Since the rational number $q_{u_n+v_n-1}/q_{u_n+v_n}$ is under its reduced form and u_n+v_n tends to infinity when n tends to infinity along $\mathcal{N}_1$, we see that, for every positive real number η and every positive integer N, there exists a reduced rational number a/b such that $b > N$ and $|\beta - a/b| \le \eta/b$. This implies that β is irrational.

Consider now the three linearly independent linear forms

$$L'_1(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \quad L'_2(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \quad L'_3(Y_1, Y_2, Y_3) = Y_2.$$

Evaluating them on the triple $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})$ with $n \in \mathcal{N}_1$, we infer from (2.2) and (3.8) that

$$\prod_{1\le j\le 3} |L'_j(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})| \ll q_{u_n+v_n}^{-1}.$$

It then follows from Theorem 2.1 that the points $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})$ with $n \in \mathcal{N}_1$ lie in a finite union of proper linear subspaces of $\mathbf{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2 \subset \mathcal{N}_1$ such that

$$y_1 q_{u_n+v_n} + y_2 q_{u_n+v_n-1} + y_3 p_{u_n+v_n} = 0, \tag{3.9}$$

for any n in $\mathcal{N}_2$. Dividing (3.9) by $q_{u_n+v_n}$ and letting n tend to infinity along $\mathcal{N}_2$, we get

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{3.10}$$

To obtain another equation linking α and β, we consider the three linearly independent linear forms

$$L''_1(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \quad L''_2(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \quad L''_3(Z_1, Z_2, Z_3) = Z_2.$$

Evaluating them on the triple $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})$ with n in $\mathcal{N}_1$, we infer from (2.2) and (3.8) that

$$\prod_{1\le j\le 3} |L''_j(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})| \ll q_{u_n+v_n}^{-1}.$$

It then follows from Theorem 2.1 that the points $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})$ with $n \in \mathcal{N}_1$ lie in a finite union of proper linear subspaces of $\mathbf{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_3 \subset \mathcal{N}_2$ such that

$$z_1 q_{u_n+v_n} + z_2 q_{u_n+v_n-1} + z_3 p_{u_n+v_n-1} = 0, \tag{3.11}$$

for any n in $\mathcal{N}_3$. Dividing (3.11) by $q_{u_n+v_n-1}$ and letting n tend to infinity along $\mathcal{N}_3$, we get

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0. \tag{3.12}$$

We infer from (3.10) and (3.12) that

$$(z_3\alpha + z_2)(y_3\alpha + y_1) = y_2 z_1.$$

Since β is irrational, we get from (3.10) and (3.12) that $y_3 z_3 \neq 0$. This shows that α is an algebraic number of degree at most two, which is a contradiction with our assumption that α is algebraic of degree at least three.

• Second case: extracting an infinite subset $\mathcal{N}_4$ of $\mathcal{N}_1$ if necessary, we assume that $(w_n)_{n\in\mathcal{N}_4}$ tends to infinity.

In particular $(p_{w_n}/q_{w_n})_{n\in\mathcal{N}_4}$ and $(p_{w_n+u_n+v_n}/q_{w_n+u_n+v_n})_{n\in\mathcal{N}_4}$ both tend to α as n tends to infinity.

We make the following observation. Let a be a letter and U, V, W be three finite words (V may be empty) such that $\mathbf{a}$ begins with $WUVU$ and a is the last letter of W and of UV. Then, writing $W = W'a$, $V = V'a$ if V is non-empty, and $U = U'a$ if V is empty, we see that $\mathbf{a}$ begins with $W'(aU)V'(aU)$ if V is non-empty and with $W'(aU')(aU')$ if V is empty. Consequently, by iterating this remark if necessary, we can assume that for any n in $\mathcal{N}_4$, the last letter of the word U_nV_n differs from the last letter of the word W_n. Said differently, we have $a_{w_n} \neq a_{w_n+u_n+v_n}$ for any n in $\mathcal{N}_4$.

Divide (3.5) by $q_{w_n}\, q_{w_n+u_n+v_n-1}$ and write

$$Q_n := (q_{w_n-1} q_{w_n+u_n+v_n})/(q_{w_n} q_{w_n+u_n+v_n-1}).$$

We then get

$$\begin{aligned} &x_1(Q_n - 1) + x_2\left(Q_n \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) + x_3\left(Q_n \frac{p_{w_n-1}}{q_{w_n-1}} - \frac{p_{w_n}}{q_{w_n}}\right) \\ &+ x_4\left(Q_n \frac{p_{w_n-1}}{q_{w_n-1}} \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n}}{q_{w_n}} \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) = 0, \end{aligned} \tag{3.13}$$

for any n in $\mathcal{N}_4$. To shorten the notation, for any $\ell \geq 1$, we put $R_\ell := \alpha - p_\ell/q_\ell$ and rewrite (3.13) as

$$\begin{aligned} &x_1(Q_n - 1) + x_2\big(Q_n(\alpha - R_{w_n+u_n+v_n}) - (\alpha - R_{w_n+u_n+v_n-1})\big) \\ &+ x_3\big(Q_n(\alpha - R_{w_n-1}) - (\alpha - R_{w_n})\big) \\ &+ x_4\big(Q_n(\alpha - R_{w_n-1})(\alpha - R_{w_n+u_n+v_n}) - (\alpha - R_{w_n})(\alpha - R_{w_n+u_n+v_n-1})\big) = 0. \end{aligned}$$

This yields

$$\begin{aligned} &(Q_n - 1)\big(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2\big) \\ &= x_2 Q_n R_{w_n+u_n+v_n} - x_2 R_{w_n+u_n+v_n-1} + x_3 Q_n R_{w_n-1} - x_3 R_{w_n} \\ &\quad - x_4 Q_n R_{w_n-1} R_{w_n+u_n+v_n} + x_4 R_{w_n} R_{w_n+u_n+v_n-1} \\ &\quad + \alpha(x_4 Q_n R_{w_n-1} + x_4 Q_n R_{w_n+u_n+v_n} - x_4 R_{w_n} - x_4 R_{w_n+u_n+v_n-1}). \end{aligned} \tag{3.14}$$

Observe that

$$|R_\ell| \leq q_\ell^{-1} q_{\ell+1}^{-1}, \quad \ell \geq 1, \tag{3.15}$$

by (2.2).

We use (3.14), (3.15) and the assumption that $a_{w_n} \neq a_{w_n+u_n+v_n}$ for any n in $\mathcal{N}_4$ to establish the following claim.

Claim. *We have*

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0.$$

Proof of the Claim. If there are arbitrarily large integers n in $\mathcal{N}_4$ such that $Q_n \geq 2$ or $Q_n \leq 1/2$, then the claim follows from (3.14) and (3.15).

Assume that $1/2 \leq Q_n \leq 2$ holds for every large n in $\mathcal{N}_4$. We then derive from (3.14) and (3.15) that

$$|(Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2)| \ll |R_{w_n - 1}| \ll q_{w_n-1}^{-1} q_{w_n}^{-1}.$$

If $x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 \neq 0$, then we get

$$|Q_n - 1| \ll q_{w_n-1}^{-1} q_{w_n}^{-1}. \tag{3.16}$$

On the other hand, observe that, by (2.4), the rational number Q_n is the quotient of the two continued fractions $[a_{w_n+u_n+v_n}; a_{w_n+u_n+v_n-1}, \ldots, a_1]$ and $[a_{w_n}; a_{w_n-1}, \ldots, a_1]$. Since $a_{w_n+u_n+v_n} \neq a_{w_n}$, we have either $a_{w_n+u_n+v_n} - a_{w_n} \geq 1$ or $a_{w_n} - a_{w_n+u_n+v_n} \geq 1$. In the former case, we see that

$$Q_n \geq \frac{a_{w_n+u_n+v_n}}{a_{w_n} + \cfrac{1}{1 + \cfrac{1}{a_{w_n-2}+1}}} \geq \frac{a_{w_n}+1}{a_{w_n} + \cfrac{a_{w_n-2}+1}{a_{w_n-2}+2}} \geq 1 + \frac{1}{(a_{w_n}+1)(a_{w_n-2}+2)}.$$

In the latter case, we have

$$\frac{1}{Q_n} \geq \frac{a_{w_n} + \cfrac{1}{a_{w_n-1}+1}}{a_{w_n+u_n+v_n}+1} \geq 1 + \frac{1}{(a_{w_n-1}+1)(a_{w_n+u_n+v_n}+1)} \geq 1 + \frac{1}{(a_{w_n-1}+1)a_{w_n}}.$$

Consequently, in any case, we have

$$|Q_n - 1| \gg a_{w_n}^{-1} \min\{a_{w_n-2}^{-1}, a_{w_n-1}^{-1}\} \gg a_{w_n}^{-1}\, q_{w_n-1}^{-1}.$$

Combined with (3.16), this gives

$$a_{w_n} \gg q_{w_n} \gg a_{w_n} q_{w_n-1},$$

which implies that n is bounded, a contradiction. This proves the Claim. $\square$

Since α is irrational and not quadratic, we deduce from the Claim that $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, x_2 is non-zero and, by (3.5), we have, for any n in $\mathcal{N}_4$,

$$q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1} = p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1}.$$

Thus, the polynomial $P_n(X)$ can simply be expressed as

$$\begin{aligned} P_n(X) &:= (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})X^2 \\ &\quad - 2(q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1})X + (p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}). \end{aligned}$$

Consider now the three linearly independent linear forms

$$\begin{aligned} L_1'''(T_1, T_2, T_3) &= \alpha^2 T_1 - 2\alpha T_2 + T_3, \\ L_2'''(T_1, T_2, T_3) &= \alpha T_1 - T_2, \\ L_3'''(T_1, T_2, T_3) &= T_1. \end{aligned}$$

Evaluating them on the triple

$$\begin{aligned} \underline{v}'_n := (q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1}, q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1}, \\ p_{w_n-1} p_{w_n+u_n+v_n} - p_{w_n} p_{w_n+u_n+v_n-1}), \end{aligned}$$

for n in $\mathcal{N}_4$, it follows from (3.2) and (3.4) that

$$\prod_{1 \le j \le 3} |L_j'''(\underline{v}'_n)| \ll q_{w_n}\, q_{w_n+u_n+v_n}\, q_{w_n+2u_n+v_n}^{-2} \ll (q_{w_n}\, q_{w_n+u_n+v_n})^{-\varepsilon},$$

with the same ε as above, if n is sufficiently large.

We then deduce from Theorem 2.1 that the points $\underline{v}'_n$, $n \in \mathcal{N}_4$, lie in a finite union of proper linear subspaces of $\mathbf{Q}^3$. Thus, there exist a non-zero integer triple (t_1, t_2, t_3) and an infinite set of distinct positive integers $\mathcal{N}_5$ included in $\mathcal{N}_4$ such that

$$\begin{aligned} t_1(q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1}) + t_2(q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1}) \\ + t_3(p_{w_n-1} p_{w_n+u_n+v_n} - p_{w_n} p_{w_n+u_n+v_n-1}) = 0, \end{aligned} \tag{3.17}$$

for any n in $\mathcal{N}_5$.

We proceed exactly as above. Divide (3.17) by $q_{w_n}\, q_{w_n+u_n+v_n-1}$ and set

$$Q_n := (q_{w_n-1} q_{w_n+u_n+v_n})/(q_{w_n} q_{w_n+u_n+v_n-1}).$$

We then get

$$\begin{aligned} t_1(Q_n - 1) + t_2\left(Q_n \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}} \right) \\ + t_3\left(Q_n \frac{p_{w_n-1}}{q_{w_n-1}} \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n}}{q_{w_n}} \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}} \right) = 0, \end{aligned} \tag{3.18}$$

for any n in $\mathcal{N}_5$. We argue as after (3.13). Since p_{w_n}/q_{w_n} and $p_{w_n+u_n+v_n}/q_{w_n+u_n+v_n}$ tend to α as n tends to infinity along $\mathcal{N}_5$, we derive from (3.18) that

$$t_1 + t_2\alpha + t_3\alpha^2 = 0,$$

a contradiction since α is irrational and not quadratic. Consequently, α must be transcendental. This concludes the proof of the theorem. $\square$

4. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1.

Let $\mathbf{a} = a_1 a_2 \ldots$ be an infinite word on the alphabet $\mathbf{Z}_{\geq 1}$. Assume that (1.1) does not hold. Then, there exist an integer $C \geq 2$ and an infinite set $\mathcal{N}$ of positive integers such that

$$p(n, \mathbf{a}) \leq Cn, \quad \text{for every } n \text{ in } \mathcal{N}. \tag{4.1}$$

This implies in particular that $\mathbf{a}$ is written over a finite alphabet, thus, by (2.1), the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded.

Let n be in $\mathcal{N}$. By (4.1) and the *Schubfachprinzip*, there exists (at least) one block X_n of length n having (at least) two occurrences in the prefix of length $(C+1)n$ of $\mathbf{a}$. Thus, there are words W_n, W'_n, B_n and B'_n such that $|W_n| < |W'_n|$ and

$$a_1 \ldots a_{(C+1)n} = W_n X_n B_n = W'_n X_n B'_n.$$

If $|W_n X_n| \leq |W'_n|$, then define V_n by the equality $W_n X_n V_n = W'_n$. Observe that

$$a_1 \ldots a_{(C+1)n} = W_n X_n V_n X_n B'_n \tag{4.2}$$

and

$$\frac{|V_n| + |W_n|}{|X_n|} \leq C. \tag{4.3}$$

Set $U_n := X_n$.

If $|W'_n| < |W_n X_n|$, then, recalling that $|W_n| < |W'_n|$, we define X'_n by $W'_n = W_n X'_n$. Since $X_n B_n = X'_n X_n B'_n$ and $|X'_n| < |X_n|$, the word X'_n is a prefix strict of X_n and X_n is a rational power of X'_n. Thus, there are a positive integer x_n and a rational number y_n such that $0 \leq y_n < 2$ and

$$X'_n X_n = {X'_n}^{1+|X_n|/|X'_n|} = {X'_n}^{2x_n + y_n} = ({X'_n}^{x_n})^2 {X'_n}^{y_n}.$$

Here and below, for a positive integer k, we write Z^k for the word $Z \ldots Z$ (k times repeated concatenation of the word Z). More generally, for any positive rational number r such that $r|Z|$ is an integer, we denote by Z^r the word $Z^{[r]} Z'$, where Z' is the prefix of Z of length $(r - [r])|Z|$.

Observe that

$$2x_n |X'_n| + 2|X'_n| \geq |X'_n X_n|,$$

thus

$$n = |X_n| \leq (2x_n + 1)|X'_n| \leq 3x_n |X'_n|.$$

Consequently, $W_n ({X'_n}^{x_n})^2$ is a prefix of $\mathbf{a}$ such that

$$|{X'_n}^{x_n}| \geq n/3$$

and

$$\frac{|W_n|}{|{X'_n}^{x_n}|} \leq \frac{3}{n} \cdot \big((C+1)n - 2|{X'_n}^{x_n}|\big) \leq 3C+1. \tag{4.4}$$

Set $U_n := {X'_n}^{x_n}$ and let V_n be the empty word.

It then follows from (4.2), (4.3), and (4.4) that, for every n in the infinite set $\mathcal{N}$,

$$W_n U_n V_n U_n \quad \text{is a prefix of } \mathbf{a}$$

with

$$|W_n| + |V_n| \leq (3C+1)\,|U_n|.$$

This shows that $\mathbf{a}$ satisfies Condition (♠). Applying Theorem 3.1, we get that the real number $[0; a_1, a_2, \ldots]$ is transcendental. This proves the theorem. □

Proof of Theorem 1.4.

If the sequence $(r_k)_{k\geq 0}$ is bounded, then Theorem 1.4 is Corollary 3.3 of [4]. Thus, we assume that $(r_k)_{k\geq 0}$ is unbounded and we consider the infinite set $\mathcal{K}$ composed of the positive integers k such that $r_k > \max\{r_0, \ldots, r_{k-1}\}$. By the assumption (1.2), there exist $\varepsilon > 0$ and k_0 such that $\lambda_{k_0} > 2$ and $\lambda_{k+1} > (1+\varepsilon)\lambda_k$ for $k \geq k_0$. Let k be in $\mathcal{K}$ with $k > k_0$. Set

$$W_k = a_1 a_2 \ldots a_{n_k - 1}$$

and

$$U_k = (a_{n_k} \ldots a_{n_k + r_k - 1})^{[\lambda_k/2]}.$$

Observe that $\mathbf{a}$ begins with $W_k U_k^2$. Furthermore, setting

$$n'_0 = n_0 + \sum_{h=0}^{k_0 - 1} \lambda_h r_h,$$

we have

$$\begin{aligned} |W_k| &\leq n'_0 + \sum_{h=k_0}^{k-1} \lambda_h r_h \\ &\leq n'_0 + r_k \lambda_k \left(\frac{1}{1+\varepsilon} + \ldots + \frac{1}{(1+\varepsilon)^{k-k_0}} \right) \\ &\leq n'_0 + r_k \lambda_k / \varepsilon \leq 2 r_k \lambda_k / \varepsilon \end{aligned}$$

and

$$|U_k| \geq \frac{(\lambda_k - 1) r_k}{2} \geq \frac{\lambda_k r_k}{4} \geq \frac{\varepsilon}{8} |W_k|,$$

for every sufficiently large k in $\mathcal{K}$. Consequently, the word $\mathbf{a} = a_1 a_2 \ldots$ satisfies Condition (♠). We conclude by applying Theorem 3.1. □

5. Transcendence criterion for quasi-palindromic continued fractions

In this section, we establish the part of Theorem 1.3 dealing with quasi-palindromic continued fractions. Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition ($\clubsuit$) if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

(i) For every $n \geq 1$, the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;
(iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;
(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Theorem 5.1. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \ldots, a_\ell, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition ($\clubsuit$), then α is transcendental.

Theorem 5.1 improves Theorem 2.4 from [5].

Proof. Throughout, the constants implied in $\ll$ are absolute. We content ourselves to explain which changes should be made to the proof of Theorem 2.4 from [5] in order to establish Theorem 5.1. We keep the notation of that paper.

Assume that the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ are fixed. Set $r_n = |W_n|$, $s_n = |W_n U_n|$ and $t_n = |W_n U_n V_n \overline{U_n}|$, for $n \geq 1$. Assume that the real number $\alpha := [0; a_1, a_2, \ldots]$ is algebraic of degree at least three.

For $n \geq 1$, consider the rational number P_n/Q_n defined by

$$\frac{P_n}{Q_n} := [0; W_n U_n V_n \overline{U_n}\, \overline{W_n}]$$

and denote by P'_n/Q'_n the last convergent to P_n/Q_n which is different from P_n/Q_n. It has been proved in [5] that

$$|Q_n\alpha - P_n| < Q_n q_{t_n}^{-2}, \quad |Q'_n\alpha - P'_n| < Q_n q_{t_n}^{-2}, \tag{5.1}$$

$$|Q_n\alpha - Q'_n| < Q_n q_{s_n}^{-2}, \tag{5.2}$$

and

$$Q_n \leq 2q_{r_n} q_{t_n} \leq 2q_{s_n} q_{t_n}. \tag{5.3}$$

Inequality (5.2) is a consequence of the mirror formula (2.4) which is a key ingredient for the proof of the combinatorial transcendence criteria for quasi-palindromic continued fractions. Since

$$\begin{aligned}\alpha(Q_n\alpha - P_n) - (Q'_n\alpha - P'_n) &= \alpha Q_n\left(\alpha - \frac{P_n}{Q_n}\right) - Q'_n\left(\alpha - \frac{P'_n}{Q'_n}\right)\\ &= (\alpha Q_n - Q'_n)\left(\alpha - \frac{P_n}{Q_n}\right) + Q'_n\left(\frac{P'_n}{Q'_n} - \frac{P_n}{Q_n}\right),\end{aligned}$$

it follows from (5.1), (5.2) and (5.3) that

$$\begin{aligned} |\alpha^2 Q_n - \alpha Q'_n - \alpha P_n + P'_n| &\ll Q_n q_{s_n}^{-2} q_{t_n}^{-2} + Q_n^{-1} \\ &\ll Q_n^{-1}. \end{aligned} \tag{5.4}$$

Together with the four linearly independent linear forms with algebraic coefficients

$$\begin{aligned} L_1(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3, \\ L_2(X_1, X_2, X_3, X_4) &= \alpha X_2 - X_4, \\ L_3(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2, \\ L_4(X_1, X_2, X_3, X_4) &= X_2, \end{aligned}$$

introduced in [5], we consider the linear form

$$L_5(X_1, X_2, X_3, X_4) = \alpha^2 X_1 - \alpha X_2 - \alpha X_3 + X_4,$$

and we deduce from (5.1), (5.2), (5.3) and (5.4) that

$$\prod_{2 \le j \le 5} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^2 \, q_{t_n}^{-2} \, q_{s_n}^{-2} \ll q_{r_n}^2 \, q_{s_n}^{-2}.$$

By (2.3) and (5.3), we have

$$q_{r_n}^2 \, q_{s_n}^{-2} \ll 2^{-|U_n|} \ll Q_n^{-\delta(u_n+v_n-r_n)/(r_n+t_n)},$$

if n is sufficiently large, where we have set

$$M = 1 + \limsup_{\ell \to +\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.$$

Since $\mathbf{a}$ satisfies Condition (♣), we have

$$\limsup_{n \to +\infty} \frac{r_n}{s_n} < 1 \quad \text{and} \quad \limsup_{n \to +\infty} \frac{r_n + t_n}{s_n} < +\infty,$$

thus,

$$\liminf_{n \to +\infty} \frac{u_n + v_n - r_n}{r_n + t_n} > 0.$$

Consequently, there exists $\varepsilon > 0$ such that

$$\prod_{2 \le j \le 5} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^{-\varepsilon},$$

for every sufficiently large n.

Following the proof from [5], we apply a first time Theorem 2.1. It implies that the points (Q_n, Q'_n, P_n, P'_n) lie in a finite union of proper linear subspaces of $\mathbf{Q}^4$. As in [5], we deduce that there exists an infinite set of distinct positive integers $\mathcal{N}$ such that $Q'_n = P_n$ for n in $\mathcal{N}$. Thus, for n in $\mathcal{N}$, we have

$$|\alpha^2 Q_n - 2\alpha Q'_n + P'_n| \ll Q_n^{-1}, \tag{5.5}$$

instead of (5.4). Consider now the three linearly independent linear forms

$$\begin{aligned}
L'_1(X_1, X_2, X_3) &= \alpha^2 X_1 - 2\alpha X_2 + X_3,\\
L'_2(X_1, X_2, X_3) &= \alpha X_2 - X_3,\\
L'_3(X_1, X_2, X_3) &= X_1.
\end{aligned}$$

Evaluating them on the triple (Q_n, Q'_n, P'_n) for n in $\mathcal{N}$, it follows from (5.1), (5.3) and (5.5) that

$$\prod_{1\le j\le 3} |L'_j(Q_n, Q'_n, P'_n)| \ll Q_n q_{t_n}^{-2} \ll q_{r_n} q_{t_n}^{-1} \ll q_{r_n} q_{s_n}^{-1} \ll Q_n^{-\varepsilon/2},$$

with the same ε as above, if n is sufficiently large.

We then apply again Theorem 2.1 and we continue as in the proof of Theorem 2.4 from [5]. We omit the details. □

6. Concluding remarks

It is likely that we are now able to get the analogues for continued fraction expansions to all the transcendence results established recently for expansions to an integer base and whose proofs ultimately rest on the Schmidt Subspace Theorem. For instance, combining the arguments of [11] with Theorem 1.3, it is easy to prove that if $1 \le m < M$ are integers and $\mathbf{a} = a_1 a_2 \ldots$ is a word over $\{m, M\}$ such that $[0; a_1, a_2, \ldots, a_\ell, \ldots]$ is algebraic, then there are arbitrarily large (finite) blocks U such that $U^{7/3}$ occurs in $\mathbf{a}$.

Recent developments have shown that the use of quantitative versions of the Schmidt Subspace Theorem allows us often to strengthen or to complement results established by means of the qualitative Schmidt Subspace Theorem; see for instance the survey [16]. In particular, by combining ideas from [6, 7, 8] with new arguments, we have obtained in [17] transcendence measures for transcendental real numbers whose sequence of partial quotients $\mathbf{a}$ is such that $n \mapsto p(n, \mathbf{a})/n$ is bounded.

Furthermore, proceeding as in [15] and in [18], it seems to be possible to prove that if $\mathbf{a} = a_1 a_2 \ldots$ is an infinite word over $\mathbf{Z}_{\ge 1}$ such that $[0; a_1, a_2, \ldots, a_\ell, \ldots]$ is algebraic of degree at least three, then there exists $\delta > 0$ such that

$$\limsup_{n\to+\infty} \frac{p(n, \mathbf{a})}{n(\log n)^\delta} = +\infty, \tag{6.1}$$

and there exists an effectively computable positive constant M such that

$$p(n, \mathbf{a}) \geq \left(1 + \frac{1}{M}\right) n, \quad \text{for } n \geq 1.$$

More details will be given in a subsequent note. Observe that a statement like (6.1) does not contain Theorem 1.1 since there exist infinite words $\mathbf{w}$ such that

$$\liminf_{n \to \infty} \frac{p(n, \mathbf{w})}{n} = 2 \quad \text{and} \quad \limsup_{n \to \infty} \frac{p(n, \mathbf{w})}{n^t} = +\infty, \quad \text{for any } t > 1;$$

see [21].

Acknowledgements. I am very thankful to the referees for their careful reading and numerous comments, which help me to improve the presentation of the paper.

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