Variations around a problem of Mahler and Mendès France

Yann Bugeaud

To the memory of Alf van der Poorten

Abstract. *We discuss the following general question and some of its extensions. Let $(\varepsilon_k)_{k\ge1}$ be a sequence taking its values in $\{0,1\}$, which is not ultimately periodic. Define $\xi := \sum_{k\ge1} \varepsilon_k/2^k$ and $\xi' := \sum_{k\ge1} \varepsilon_k/3^k$. Let $\mathcal{P}$ be a property valid for almost all real numbers. Is it true that at least one among ξ and ξ' satisfies $\mathcal{P}$?*

1. Introduction

The main motivation for the present note comes from the following problem appeared at the end of a paper of Mendès France [15]. According to him (see the discussion in [4], page 403) it was proposed by Mahler; however, we were unable to find any mention of it in Mahler's works.

Problem (Mahler–Mendès France). *For an arbitrary infinite sequence $(\varepsilon_k)_{k\ge1}$ of 0's and 1's, prove that the real numbers*

$$\sum_{k=1}^{+\infty} \frac{\varepsilon_k}{2^k} \quad \text{and} \quad \sum_{k=1}^{+\infty} \frac{\varepsilon_k}{3^k}$$

are algebraic if, and only if, both are rational.

The resolution of this problem seems to be far beyond our current knowledge. Nonetheless, in this note, we discuss the following more general question. Throughout, 'almost all' always refers to the Lebesgue measure.

Problem 1. *Let $\mathcal{P}$ be a property valid for almost all real numbers. Let $b \ge 2$ be an integer. Let b_1 and b_2 be distinct integers, at least equal to b. Let $(\varepsilon_k)_{k\ge1}$ be a sequence*

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taking its values in $\{0, 1, \ldots, b-1\}$*, which is not ultimately periodic. Is it true that at least one among the numbers*

$$\xi_1 := \sum_{k \geq 1} \frac{\varepsilon_k}{b_1^k} \quad \textit{and} \quad \xi_2 := \sum_{k \geq 1} \frac{\varepsilon_k}{b_2^k}$$

satisfies property $\mathcal{P}$*?*

If $\mathcal{P}$ is the property 'being transcendental', $b = b_1 = 2$ and $b_2 = 3$, then to give a positive answer to Problem 1 is equivalent to solve the Mahler–Mendès France problem. The aim of this note is to discuss Problem 1 for other properties $\mathcal{P}$, including 'not being a Liouville number' or 'not being badly approximable'.

Recall that the irrationality exponent of an irrational real number ξ, denoted by $\mu(\xi)$, is the supremum of the real numbers μ for which the inequality $|\xi - p/q| < q^{-\mu}$ has infinitely many solutions in rational numbers p/q with $q \geq 1$. A real number ξ is a Liouville number if, and only if, $\mu(\xi)$ is infinite. The irrationality exponent of every irrational real number is at least equal to 2. Recall also that an irrational real number ξ for which there exists a positive real number c such that $|\xi - p/q| \geq c/q^2$ for every pair (p, q) of integers with $q \geq 1$ is called a badly approximable number.

For both properties mentioned above, the answer to Problem 1 is negative. Indeed, it is easy to check that

$$\sum_{k \geq 1} \frac{1}{b^{k!}}$$

is a Liouville number for every integer $b \geq 2$. Furthermore, Shallit [20] has shown that

$$\sum_{k \geq 1} \frac{1}{b^{2^k}}$$

is a badly approximable number for every integer $b \geq 2$. For this, he used a version of the Folding Lemma for continued fractions, which was first established by Mendès France [14] and then rediscovered by several authors [11, 12, 17, 21, 18, 19] (this list is not exhaustive).

Furthermore, it has been proved recently [6], again using the Folding Lemma, that

$$\mu\left(\sum_{k \geq 1} \frac{1}{b^{\lfloor c^k \rfloor}}\right) = c,$$

for every real number $c \geq 2$ and every integer $b \geq 2$. Here and below, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

More generally, most of the recent Diophantine results on real numbers expressed as

$$\sum_{k \geq 1} \frac{\varepsilon_k}{b^k}$$

depend on combinatorial properties of the sequence of digits $(\varepsilon_k)_{k \geq 1}$, but are independent of the integer base b (here, it is assumed that ε_k is in $\{0, 1, \ldots, b-1\}$ for $k \geq 1$), see for example [1, 2, 7, 8, 9].

All these results motivate the following problems.

Problem 2. *Let $b \geq 2$ be an integer. Let b_1 and b_2 be distinct integers, at least equal to b. Does there exist a sequence $(\varepsilon_k)_{k\geq 1}$ taking its values in $\{0, 1, \ldots, b-1\}$ such that*

$$\mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\biggr) = +\infty \quad \textit{and} \quad \mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\biggr) \textit{is finite?}$$

Since the set of Liouville numbers has zero Hausdorff dimension, metric arguments do not seem to help to solve Problem 2.

Problem 3. *Let $b \geq 2$ be an integer. Let b_1 and b_2 be distinct integers, at least equal to b. To find explicitly a sequence $(\varepsilon_k)_{k\geq 1}$ taking its values in $\{0, 1, \ldots, b-1\}$ such that*

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}$$

is badly approximable, while

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}$$

is not badly approximable.

With the notation of Problem 3, the Hausdorff dimension of the set of badly approximable real numbers of the form

$$\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k},$$

with ε_k in $\{0, 1, \ldots, b-1\}$ for $k \geq 1$, is equal to the Hausdorff dimension of the set of real numbers of this form, that is, to $(\log b)/(\log b_1)$ (see for example [10]). This implies that sequences $(\varepsilon_k)_{k\geq 1}$ satisfying the properties requested in Problem 3 do exist when b_1 is less than b_2. The difficult point is to provide an explicit construction of such a sequence.

Problem 4. *Let $b \geq 2$ be an integer. Let b_1 and b_2 be distinct integers, at least equal to b. Let μ_1 and μ_2 be real numbers at least equal to 2. Does there exist a sequence $(\varepsilon_k)_{k\geq 1}$ taking its values in $\{0, 1, \ldots, b-1\}$ such that*

$$\mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\biggr) = \mu_1 \quad \textit{and} \quad \mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\biggr) = \mu_2?$$

Surprisingly, it even does not seem to be easy to construct a sequence $(\varepsilon_k)_{k\geq 1}$ taking its values in $\{0, 1, \ldots, b-1\}$ such that

$$\mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}\biggr) \neq \mu\biggl(\sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}\biggr),$$

see Theorem 1 below for a contribution to this question.

Problem 4 is difficult since, in most cases, knowing the b-ary expansion of a real number gives no information on its irrationality exponent, see [5]. However, if the sequence

$(\varepsilon_k)_{k\geq 1}$ contains long repetitions which occur unexpectedly early, then, by truncating and completing by periodicty, one can construct very good rational approximations to $\sum_{k\geq 1} \varepsilon_k/b_1^k$ of the form $P(b_1)/(b_1^r(b_1^s-1))$, for r, s positive integers and $P(X)$ an integral polynomial. However, it is not clear at all whether $P(b_1)/(b_1^r(b_1^s-1))$ is written under its reduced form. Furthermore, $P(b_2)/(b_2^r(b_2^s-1))$ is then a good rational approximation to $\sum_{k\geq 1} \varepsilon_k/b_2^k$, but we as well do not know whether it is written under its reduced form. Such an information is crucial when one wishes to determine the exact value of the irrationality exponent. Otherwise, we get only a lower bound for it. A related question has been discussed by Mahler [13].

We conclude this section with an extension of the Mahler–Mendès France problem.

Problem 5. *Let $b \geq 2$ be an integer. Let b_1 and b_2 be distinct integers, at least equal to b. Let $(\varepsilon_k)_{k\geq 1}$ be a sequence taking its values in $\{0, 1, \dots, b-1\}$, which is not ultimately periodic. Are the real numbers*

$$\xi_1 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k} \quad \text{and} \quad \xi_2 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_2^k}$$

algebraically independent?

Under strong additional assumptions on the sequence $(\varepsilon_k)_{k\geq 1}$, a positive answer to Problem 5 has been given using the so-called Mahler method, see for example Chapter 3 of Nishioka's monograph [16]. In particular, the real numbers

$$\sum_{k\geq 1} \frac{1}{b^{2^k}}, \quad b \geq 2,$$

are algebraically independent.

2. Result

Our small contribution towards Problem 4 is the following result.

Theorem 1. *Let $b \geq 2$ and $b_1 > b$ be integers with $b_1 \neq b^2$. Let a be a real number and w an integer such that $a \geq 3$ and $w \geq 3a$. For $k \geq 1$, set $n_k = \lfloor (aw)^k \rfloor$. Let $(\varepsilon_k)_{k\geq 1}$ be the sequence of integers defined as follows. We set $\varepsilon_k = b$ if, and only if, there exist $h \geq 1$ and $m = 0, 1, \dots, w-1$ such that $k = n_h + 1 + m(2n_h + 1)$. We set $\varepsilon_k = 1$ if, and only if, there exist $h \geq 1$ and $m = 1, 2, \dots, w$ such that $k = m(2n_h + 1)$. Otherwise, we set $\varepsilon_k = 0$. Define*

$$\xi := \sum_{k\geq 1} \frac{\varepsilon_k}{(b^2)^k} \quad \text{and} \quad \xi_1 := \sum_{k\geq 1} \frac{\varepsilon_k}{b_1^k}$$

Then we have

$$\mu(\xi) = \frac{a(2w+1)}{a+2}$$

and

$$\mu(\xi_1) = \frac{a(2w+1)}{2(a+1)}.$$

The conclusion of Theorem 1 also holds if the real number a satisfies $a > 2 + 1/w$ with a sufficiently large integer w.

Observe that for every real number μ sufficiently large, there are integers w, w_1 and real numbers a, a_1 with $3 \le a \le w/3$, $3 \le a_1 \le w_1/3$ such that

$$\mu = \frac{a(2w+1)}{a+2} = \frac{a_1(2w_1+1)}{2(a_1+1)}.$$

The proof of Theorem 1 is elementary. The basic idea is to truncate the b^2-ary expansion to ξ (resp. the b_1-ary expansion to ξ_1) and then to complete by periodicity to construct good rational approximants to ξ (resp. to ξ_1). The denominators of these rationals, when written under their lowest form, are essentially of the form $b^r(b^s-1)$ (resp. $b_1^r(b_1^s-1)$), where r and s are positive integers.

Proof. The key point is the observation that

$$\frac{b \times b^{2n}+1}{b^{2(2n+1)}-1} = \frac{1}{b^{2n+1}-1},$$

while the fraction

$$\frac{b \times b_1^n + 1}{b_1^{2n+1}-1}$$

is nearly written under its reduced form.

To be more precise, observe that

$$(b \times b_1^n + 1)(b_1^{n+1}-1) = b(b_1^{2n+1}-1) + (b_1 - b)b_1^n + b - 1,$$

thus $\gcd(b \times b_1^n + 1, b_1^{2n+1}-1)$ divides $(b_1-b)b_1^n + b - 1$. Since

$$(b_1 - b)(b \times b_1^n + 1) - b((b_1-b)b_1^n + b - 1) = b_1 - b^2,$$

we get that $\gcd(b \times b_1^n + 1, b_1^{2n+1}-1)$ divides $b_1 - b^2$, hence, this greatest common divisor is bounded indepedently of n.

Observe that

$$\xi = \sum_{k\ge1} \big(b(b^2)^{-n_k-1} + (b^2)^{-2n_k-1}\big)\big(1 + (b^2)^{-2n_k-1} + \ldots + (b^2)^{-(w-1)(2n_k+1)}\big)$$

and

$$\xi_1 = \sum_{k\ge1} \big(b \times b_1^{-n_k-1} + b_1^{-2n_k-1}\big)\big(1 + b_1^{-2n_k-1} + \ldots + b_1^{-(w-1)(2n_k+1)}\big).$$

To construct good rational approximants to ξ (resp. to ξ_1), we simply truncate the summation and complete by periodicity. For $K \ge 2$, define

$$\begin{aligned}\xi_K := \sum_{k=1}^{K-1} \big(b(b^2)^{-n_k-1} + (b^2)^{-2n_k-1}\big)\big(1 + (b^2)^{-2n_k-1} + \ldots + (b^2)^{-(w-1)(2n_k+1)}\big)& \\ + \frac{b(b^2)^{-n_K-1} + (b^2)^{-2n_K-1}}{1-(b^2)^{-2n_K-1}}& \\ = \frac{m_K}{(b^2)^{w(2n_{K-1}+1)}} + \frac{b(b^2)^{n_K}+1}{(b^2)^{2n_K+1}-1}&\end{aligned}$$

and

$$\xi_{1,K} := \sum_{k=1}^{K-1} \big(b \times b_1^{-n_k-1} + b_1^{-2n_k-1}\big)\big(1 + b_1^{-2n_k-1} + \ldots + b_1^{-(w-1)(2n_k+1)}\big) + \frac{bb_1^{-n_K-1} + b_1^{-2n_K-1}}{1 - b_1^{-2n_K-1}}$$

$$= \frac{m_{1,K}}{(b_1)^{w(2n_{K-1}+1)}} + \frac{b(b_1)^{n_K} + 1}{(b_1)^{2n_K+1} - 1},$$

for some integers m_K and $m_{1,K}$. It follows from the key point explained at the beginning of the proof that there exist an integer p_K such that

$$\xi_K = \frac{p_K}{(b^2)^{w(2n_{K-1}+1)}(b^{2n_K+1} - 1)},$$

in its lowest form, and an integer $p_{1,K}$ such that

$$\xi_{1,K} = \frac{p_{1,K}}{b_1^{w(2n_{K-1}+1)}(b_1^{2n_K+1} - 1)},$$

and the greatest common divisor of $p_{1,K}$ and $b_1^{w(2n_{K-1}+1)}(b_1^{2n_K+1} - 1)$ is bounded independently of K.

Since $a > 2 + 1/w$, the inequality $n_{K+1} + 1 > n_K + 1 + w(2n_K + 1)$ is satisfied if K is sufficiently large. If this is the case, then we check that

$$b(b^2)^{-n_K-1-w(2n_K+1)} \leq |\xi - \xi_K| \leq 2b(b^2)^{-n_K-1-w(2n_K+1)} \tag{2.1}$$

and

$$bb_1^{-n_K-1-w(2n_K+1)} \leq |\xi_1 - \xi_{1,K}| \leq 2bb_1^{-n_K-1-w(2n_K+1)}. \tag{2.2}$$

Since we know, up to a bounded numerical constant, the reduced form of the rational numbers ξ_K and $\xi_{1,K}$, it then follows from (2.1) and (2.2) that

$$|\xi - \xi_K| \asymp (\mathrm{den}(\xi_K))^{-2(n_K+1+w(2n_K+1))/(2w(2n_{K-1}+1)+2n_K+1)}$$

and

$$|\xi_1 - \xi_{1,K}| \asymp (\mathrm{den}(\xi_{1,K}))^{-(n_K+1+w(2n_K+1))/(w(2n_{K-1}+1)+2n_K+1)},$$

where the notation $A_K \asymp B_K$ means that the ratio A_K/B_K is bounded from above and from below by positive constants independent of K.

This gives the lower bounds

$$\mu(\xi) \geq \frac{a(2w+1)}{a+2} \quad \text{and} \quad \mu(\xi_1) \geq \frac{a(2w+1)}{2(a+1)}. \tag{2.3}$$

It remains to show that the inequalities in (2.3) are indeed equalities. To do this, we use a classical lemma whose proof is based on triangle inequalities (see for example Lemma 4.1 of [3]).

Lemma 2. *Let ξ be a real number such that there exist positive real numbers c_1, c_2, μ, θ and reduced rational numbers $(p_k/q_k)_{k\ge 1}$ such that*

$$\frac{c_1}{q_k^{\mu}} \le \left| \xi - \frac{p_k}{q_k} \right| \le \frac{c_2}{q_k^{\mu}}, \quad k \ge 1,$$

and

$$q_k \le q_{k+1} \le q_k^{\theta}.$$

If $\theta \le (\mu-1)^2$, then the irrationality exponent of ξ is equal to μ.

We check that

$$\lim_{K\to+\infty} \left| \frac{\log \mathrm{den}(\xi_{K+1})}{\log \mathrm{den}(\xi_K)} - \frac{2w(2n_K+1)+2n_{K+1}+1}{2w(2n_{K-1}+1)+2n_K+1} \right| = 0$$

and

$$\lim_{K\to+\infty} \left| \frac{\log \mathrm{den}(\xi_{1,K+1})}{\log \mathrm{den}(\xi_{1,K})} - \frac{w(2n_K+1)+2n_{K+1}+1}{w(2n_{K-1}+1)+2n_K+1} \right| = 0.$$

Consequently, by the definition of $(n_k)_{k\ge 1}$, the sequences

$$(\log \mathrm{den}(\xi_{K+1})/\log \mathrm{den}(\xi_K))_{K\ge 1} \quad \text{and} \quad (\log \mathrm{den}(\xi_{1,K+1})/\log \mathrm{den}(\xi_{1,K}))_{K\ge 1}$$

both tend to aw as K tends to infinity. Since

$$aw \le \left(\frac{a(2w+1)}{2(a+1)} - 1 \right)^2$$

for $a \ge 3$ and $w \ge 3a$, the theorem follows from Lemma 2. □

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Yann Bugeaud
Université de Strasbourg
Mathématiques
7, rue René Descartes
67084 STRASBOURG (FRANCE)

bugeaud@math.unistra.fr