

S-parts of values of univariate polynomials, binary forms and decomposable forms at integral points

Yann Bugeaud

IRMA U.M.R. 7501, Université de Strasbourg et CNRS
7, rue René Descartes, F-67084 Strasbourg cedex, France
E-mail: bugeaud@math.unistra.fr

Jan-Hendrik Evertse

Universiteit Leiden, Mathematisch Instituut
Postbus 9512, 2300 RA Leiden, The Netherlands
E-mail: evertse@math.leidenuniv.nl

Kálmán Györy

Institute of Mathematics, University of Debrecen
H-4002 Debrecen, P.O. Box 400, Hungary
E-mail: gyory@science.unideb.hu

To Robert Tijdeman on his 75-th birthday

Abstract

Let $S = \{p_1, \dots, p_s\}$ be a finite, non-empty set of distinct prime numbers. For a non-zero integer m , write $m = p_1^{a_1} \dots p_s^{a_s} b$, where a_1, \dots, a_s are non-negative integers and b is an integer relatively prime to $p_1 \dots p_s$. Then we define the S -part $[m]_S$ of m by $[m]_S := p_1^{a_1} \dots p_s^{a_s}$. In 2013, Gross and Vincent proved that if $f(X)$ is a polynomial with integral coefficients with at least two distinct roots, then there exist effectively computable positive numbers κ_1 and κ_2 , depending only on $f(X)$ and S , such that (*) $[f(x)]_S < \kappa_2 |f(x)|^{1-\kappa_1}$.

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for every integer x with $f(x) \neq 0$. Their proof uses a Baker-type estimate for logarithmic forms. Under the additional hypotheses that $f(X)$ has degree $n \geq 2$ and no multiple roots, we deduce an ineffective analogue of (*), with instead of $1 - \kappa_1$ an exponent $\frac{1}{n} + \epsilon$ for every $\epsilon > 0$ and instead of κ_2 an ineffective number depending on $f(X)$, S and ϵ . This is in fact an easy application of the p -adic Thue-Siegel-Roth Theorem. We show that the exponent $\frac{1}{n}$ is best possible. Lastly, we give an estimate for the density of the set of integers x for which $[f(x)]_S$ is large, i.e., for every small $\epsilon > 0$ we estimate in terms of B the number of integers x with $|x| \leq B$ such that $[f(x)]_S \geq |f(x)|^\epsilon$.

We considerably extend both the result of Gross and Vincent, its ineffective analogue, and the density result by proving similar results for the S -parts of values of homogeneous binary forms and, more generally, of values of decomposable forms at integer points, under suitable assumptions.

1 Introduction

Let $S = \{p_1, \dots, p_s\}$ be a finite, non-empty set of distinct prime numbers. For a non-zero integer m , write $m = p_1^{a_1} \dots p_s^{a_s} b$, where a_1, \dots, a_s are non-negative integers and b is an integer relatively prime to $p_1 \dots p_s$. Then we define the S -part $[m]_S$ of m by

$$[m]_S := p_1^{a_1} \dots p_s^{a_s}.$$

The motivation of the present paper was given by the following result, established in 2013 by Gross and Vincent [10].

Theorem A. *Let $f(X)$ be a polynomial with integral coefficients with at least two distinct roots and S a finite, non-empty set of prime numbers. Then there exist effectively computable positive numbers κ_1 and κ_2 , depending only on $f(X)$ and S , such that for every non-zero integer x that is not a root of $f(X)$ we have*

$$[f(x)]_S < \kappa_2 |f(x)|^{1-\kappa_1}.$$

We mention that earlier, Stewart [24] obtained a similar result, but only for polynomials whose zeros are consecutive integers.

Gross and Vincent's proof of Theorem A depends on the theory of linear forms in complex logarithms. Under the additional hypotheses that $f(X)$ has degree $n \geq 2$ and no multiple roots, we deduce an ineffective analogue of Theorem A, with instead of $1 - \kappa_1$ an exponent $\frac{1}{n} + \epsilon$ for every $\epsilon > 0$ and instead of κ_2 an ineffective number depending on $f(X)$, S and ϵ . This is in fact an easy application of the p -adic Thue-Siegel-Roth Theorem. We show

that the exponent $\frac{1}{n}$ is best possible. Lastly, we give an estimate for the density of the set of integers x for which $[f(x)]_S$ is large, i.e., for every small $\epsilon > 0$ we estimate in terms of B the number of integers x with $|x| \leq B$ such that $[f(x)]_S \geq |f(x)|^\epsilon$.

We considerably extend both Theorem A, its ineffective analogue, and the density result by proving similar results for the S -parts of values of homogeneous binary forms and, more generally, of values of decomposable forms at integer points, under suitable assumptions. In addition, in the effective results we give an expression for κ_1 , which is explicit in terms of S . For our extensions to binary forms and decomposable forms, we use the p -adic Thue-Siegel-Roth Theorem and the p -adic Subspace Theorem of Schmidt and Schlickewei for the ineffective estimates for the S -part. The proof of the effective estimates is based on an effective theorem of Győry and Yu [15] on decomposable form equations whose proof depends on estimates for linear forms in complex and in p -adic logarithms. Lastly, the proofs of our density results on the number of integer points of norm at most B at which the value of the binary form or decomposable form under consideration has large S -value are based on a recent general lattice point counting result of Barroero and Widmer [1] and on work in the PhD-thesis of Junjiang Liu [16].

For simplicity, we have restricted ourselves to univariate polynomials, binary forms and decomposable forms with coefficients in \mathbb{Z} . With some extra technical effort, analogous results could have been obtained over arbitrary number fields.

In Section 2 we state our results, in Sections 3–6 we give the proofs, in Sections 7 and 8 we present some applications, and in Section 9 we give some additional comments on Theorem A.

2 Results

2.1 Results for univariate polynomials and binary forms

We use notation $\ll_{a,b,\dots}, \gg_{a,b,\dots}$ to indicate that the constants implied by the Vinogradov symbols depend only on the parameters a, b, \dots . Further, we use the notation $A \asymp_{a,b,\dots} B$ to indicate that both $A \ll_{a,b,\dots} B$ and $B \ll_{a,b,\dots} A$ hold. We prove the following ineffective analogue of Theorem A mentioned in the previous section.

Theorem 2.1. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$ without multiple zeros.*

(i) Let $S = \{p_1, \dots, p_s\}$ be a non-empty set of primes. Then for every $\epsilon > 0$ and for every $x \in \mathbb{Z}$ with $f(x) \neq 0$,

$$[f(x)]_S \ll_{f,S,\epsilon} |f(x)|^{(1/n)+\epsilon}.$$

(ii) There are infinitely many primes p , and for each of these p , there are infinitely many integers x , such that $f(x) \neq 0$ and

$$[f(x)]_{\{p\}} \gg_f |f(x)|^{1/n}.$$

For completeness, we give here also a more precise effective version of Theorem A, which is a consequence of Theorem 2.5 stated below on the S -parts of values of binary forms.

Theorem 2.2. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial with at least two distinct roots and suppose that its splitting field has degree d over \mathbb{Q} . Further, let $S = \{p_1, \dots, p_s\}$ be a non-empty set of primes and put $P := \max(p_1, \dots, p_s)$. Then for every integer x with $f(x) \neq 0$ we have*

$$[f(x)]_S \leq \kappa_2 |f(x)|^{1-\kappa_1},$$

where

$$\kappa_1 = \left(c_1^s (P(\log p_1) \cdots (\log p_s))^d \right)^{-1},$$

and c_1, κ_2 are effectively computable positive numbers that depend only on $f(X)$.

For variations on this result, and related results, we refer to Section 9.

For polynomials $X(X+1)$ and X^2+7 and special sets S , Bennett, Filaseta, and Trifonov [2, 3] have obtained stronger effective results.

As is to be expected, for most integers x , the S -part $[f(x)]_S$ is small compared with $|f(x)|$. This is made more precise in the following result. For any finite set of primes S and any $\epsilon > 0$, $B > 0$, we denote by $N(f, S, \epsilon, B)$ the number of integers x such that

$$(2.1) \quad |x| \leq B, \quad f(x) \neq 0, \quad [f(x)]_S \geq |f(x)|^\epsilon.$$

Denote by $D(f)$ the discriminant of f and for a prime p , denote by g_p the largest integer g such that p^g divides $D(f)$.

Theorem 2.3. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$ with non-zero discriminant. Further, let $0 < \epsilon < 1/n$, and let S be a finite set of primes. Denote by s' the number of primes $p \in S$ such that $f(x) \equiv 0 \pmod{p^{g_p+1}}$ is solvable and assume that this number is positive. Then*

$$N(f, S, \epsilon, B) \asymp_{f,S,\epsilon} B^{1-n\epsilon} (\log B)^{s'-1} \quad \text{as } B \rightarrow \infty.$$

Remarks.

1. If $s' = 0$ then $[f(x)]_S$ is bounded, and so the set of integers x with $[f(x)]_S \geq |f(x)|^\epsilon$ is finite.
2. There are infinitely many primes p such that $f(x) \equiv 0 \pmod{p}$ is solvable. Removing from those the finitely many that divide $D(f)$, there remain infinitely many primes p such that $g_p = 0$ and $f(x) \equiv 0 \pmod{p}$ is solvable.

We now formulate some analogues of the above mentioned results for binary forms. Denote by $\mathbb{Z}_{\text{prim}}^2$ the set of pairs $(x, y) \in \mathbb{Z}^2$ with $\gcd(x, y) = 1$.

Theorem 2.4. *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 2$ with non-zero discriminant.*

- (i) *Let $S = \{p_1, \dots, p_s\}$ be a non-empty set of primes. Then for every $\epsilon > 0$ and every pair $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \neq 0$,*

$$[F(x, y)]_S \ll_{F, S, \epsilon} |F(x, y)|^{(2/n) + \epsilon}.$$

- (ii) *There are finite sets of primes S with the smallest prime in S arbitrarily large, and for every one of these sets S infinitely many pairs $(x, y) \in \mathbb{Z}_{\text{prim}}^2$, such that $F(x, y) \neq 0$ and*

$$[F(x, y)]_S \gg_{F, S, \epsilon} |F(x, y)|^{2/n}.$$

Our next result is an effective analogue of Theorem 2.2 for binary forms. It is an easy consequence of Theorem 2.10 stated below on decomposable forms. The splitting field of a binary form is the smallest extension of \mathbb{Q} over which it factors into linear forms.

Theorem 2.5. *Let $F(X, Y)$ be a binary form of degree $n \geq 3$ with coefficients in \mathbb{Z} and with splitting field K . Suppose that F has at least three pairwise non-proportional linear factors over K . Let again $S = \{p_1, \dots, p_s\}$ be a finite set of primes and $[K : \mathbb{Q}] = d$. Then*

$$[F(x, y)]_S \leq \kappa_4 |F(x, y)|^{1 - \kappa_3}$$

for every $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \neq 0$, where

$$\kappa_3 = \left(c_2^s \left((P(\log p_1) \cdots (\log p_s))^d \right)^{-1} \right)$$

and κ_4, c_2 are effectively computable positive numbers, depending only on F .

We obtain Theorem 2.2 on polynomials $f(X) \in \mathbb{Z}[X]$ by applying Theorem 2.5 to the binary form $Y^{1+\deg f} f(X/Y)$ with $(x, y) = (x, 1) \in \mathbb{Z}_{\text{prim}}^2$.

Let again $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 2$ and of non-zero discriminant. For any finite set of primes S and any $\epsilon > 0$, $B > 0$, we denote by $N(F, S, \epsilon, B)$ the number of pairs $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ such that

$$(2.2) \quad \max(|x|, |y|) \leq B, \quad F(x, y) \neq 0, \quad [F(x, y)]_S \geq |F(x, y)|^\epsilon.$$

Denote by $D(F)$ the discriminant of F and for a prime p , denote by g_p the largest integer g such that p^g divides $D(F)$.

Theorem 2.6. *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 3$ with non-zero discriminant. Further, let $0 < \epsilon < \frac{1}{n}$, and let S be a finite set of primes. Denote by s' the number of primes $p \in S$ such that $F(x, y) \equiv 0 \pmod{p^{g_p+1}}$ has a solution $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ and assume that this number is positive. Then*

$$N(F, S, \epsilon, B) \asymp_{F, S, \epsilon} B^{2-n\epsilon} (\log B)^{s'-1} \quad \text{as } B \rightarrow \infty.$$

Parts (i) of Theorems 2.1 and 2.4 are easy consequences of the p -adic Thue-Siegel-Roth Theorem. Part (ii) of Theorem 2.1 is a consequence of the fact that for a given non-constant polynomial $f(X) \in \mathbb{Z}[X]$ there are infinitely many primes p such that $f(X)$ has a zero in \mathbb{Z}_p . The proof of part (ii) of Theorem 2.4 uses some geometry of numbers.

There are two main tools in the proof of Theorem 2.6. The first is a result of Stewart [25, Thm. 2] on the number of congruence classes x modulo p^k of $f(x) \equiv 0 \pmod{p^k}$ for $f(X)$ a polynomial and p^k a prime power. The second is a powerful lattice point counting result of Barroero and Widmer [1, Thm. 1.3]. The proof of Theorem 2.3 is very similar, but instead of the result of Barroero and Widmer it uses a much more elementary counting argument.

2.2 Ineffective results for decomposable forms

We will state results on the S -parts of values of decomposable forms in m variables at integral points, where $m \geq 2$.

We start with some notation and definitions. Let K be a finite, normal extension of \mathbb{Q} . For a linear form $\ell = \alpha_1 X_1 + \cdots + \alpha_m X_m$ with coefficients in K and for an element σ of the Galois group $\text{Gal}(K/\mathbb{Q})$ we define $\sigma(\ell) := \sigma(\alpha_1)X_1 + \cdots + \sigma(\alpha_m)X_m$ and then for a set of linear forms $\mathcal{L} = \{\ell_1, \dots, \ell_r\}$ with coefficients in K we write $\sigma(\mathcal{L}) := \{\sigma(\ell_1), \dots, \sigma(\ell_r)\}$. A set of linear forms \mathcal{L} with coefficients in K is called $\text{Gal}(K/\mathbb{Q})$ -*symmetric* if $\sigma(\mathcal{L}) = \mathcal{L}$ for

each $\sigma \in \text{Gal}(K/\mathbb{Q})$, and *Gal*(K/\mathbb{Q})-*proper* if for each $\sigma \in \mathcal{L}$ we have either $\sigma(\mathcal{L}) = \mathcal{L}$ or $\sigma(\mathcal{L}) \cap \mathcal{L} = \emptyset$. We denote by $[\mathcal{L}]$ the K -vector space generated by \mathcal{L} , and define $\text{rank } \mathcal{L}$ to be the dimension of $[\mathcal{L}]$ over K . Finally, we define the sum of two vector spaces V_1, V_2 over K by $V_1 + V_2 := \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in V_1, \mathbf{y} \in V_2\}$.

Recall that a decomposable form in $\mathbb{Z}[X_1, \dots, X_m]$ is a homogeneous polynomial that factors into linear forms in X_1, \dots, X_m over some extension of \mathbb{Q} . The smallest extension over which such a factorization is possible is called the *splitting field* of the decomposable form. This is a finite, normal extension of \mathbb{Q} .

Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form of degree $n \geq 3$ with splitting field K . Then we can express F as

$$(2.3) \quad \left\{ \begin{array}{l} F = c \ell_1^{e(\ell_1)} \cdots \ell_r^{e(\ell_r)} \quad \text{with} \\ c \text{ a non-zero rational,} \\ \mathcal{L}_F = \{\ell_1, \dots, \ell_r\} \text{ a Gal}(K/\mathbb{Q})\text{-symmetric set of pairwise} \\ \text{non-proportional linear forms with coefficients in } K, \\ e(\ell_1), \dots, e(\ell_r) \text{ positive integers, with } e(\ell_i) = e(\ell_j) \\ \text{whenever } \ell_j = \sigma(\ell_i) \text{ for some } \sigma \in \text{Gal}(K/\mathbb{Q}). \end{array} \right.$$

Lastly, define $\mathbb{Z}_{\text{prim}}^m$ to be the set of vectors $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}^m$ with $\text{gcd}(x_1, \dots, x_m) = 1$ and define $\|\mathbf{x}\|$ to be the maximum norm of $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$.

Let $S = \{p_1, \dots, p_s\}$ be a finite set of primes, and $F \in \mathbb{Z}[X_1, \dots, X_m]$ a decomposable form. For $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $F(\mathbf{x}) \neq 0$, we can write

$$(2.4) \quad F(\mathbf{x}) = p_1^{a_1} \cdots p_s^{a_s} \cdot b,$$

where a_1, \dots, a_s are non-negative integers and b is an integer coprime with $p_1 \cdots p_s$. Then the S -part $[F(\mathbf{x})]_S$ is $p_1^{a_1} \cdots p_s^{a_s}$. We may view (2.4) as a Diophantine equation in $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ and $a_1, \dots, a_s \in \mathbb{Z}_{\geq 0}$, a so-called *decomposable form equation*. Schlickewei [23] considered (2.4) in the case that F is a norm form (i.e., a decomposable form that is irreducible over \mathbb{Q}) and formulated a criterion in terms of F implying that (2.4) has only finitely many solutions. Evertse and Györy [7] gave another finiteness criterion in terms of F , valid for arbitrary decomposable forms. Recently [8, Chap. 9, Thm. 9.1.1], they refined this as follows. Call an integer S -free if it is non-zero, and coprime with the primes in S .

Theorem B. *Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form with splitting field K , given in the form (2.3), and let \mathcal{L} be a finite set of linear forms in $K[X_1, \dots, X_m]$, containing \mathcal{L}_F . Then the following two assertions are equivalent:*

(i) $\text{rank } \mathcal{L}_F = m$, and for every $\text{Gal}(K/\mathbb{Q})$ -proper subset \mathcal{M} of \mathcal{L}_F with $\emptyset \subsetneq \mathcal{M} \subsetneq \mathcal{L}_F$, we have

$$(2.5) \quad \mathcal{L} \cap \left(\sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} [\sigma(\mathcal{M})] \cap [\mathcal{L}_F \setminus \sigma(\mathcal{M})] \right) \neq \emptyset;$$

(ii) for every finite set of primes $S = \{p_1, \dots, p_s\}$ and every S -free integer b , there are only finitely many $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ and non-negative integers a_1, \dots, a_s such that

$$(2.6) \quad F(\mathbf{x}) = p_1^{a_1} \cdots p_s^{a_s} b, \quad \ell(\mathbf{x}) \neq 0 \text{ for } \ell \in \mathcal{L}.$$

This theorem was deduced from a finiteness theorem of Evertse [5] and van der Poorten and Schlickewei [20, 21] on S -unit equations over number fields.

The following result gives an improvement of (ii). We denote by $|\cdot|_\infty$ the standard archimedean absolute value on \mathbb{Q} , and for a prime p by $|\cdot|_p$ the standard p -adic absolute value, with $|p|_p = p^{-1}$. Further, $\|\mathbf{x}\|$ denotes the maximum norm of $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$.

Theorem 2.7. *Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form in $m \geq 2$ variables with splitting field K and $\mathcal{L} \supseteq \mathcal{L}_F$ a finite set of linear forms in $K[X_1, \dots, X_m]$, satisfying condition (i) of Theorem B. Further, let S be a finite set of primes and let $\epsilon > 0$. Then there are only finitely many $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with*

$$(2.7) \quad \begin{cases} \prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p \leq \|\mathbf{x}\|^{(1/(m-1))-\epsilon}, \\ \ell(\mathbf{x}) \neq 0 \text{ for } \ell \in \mathcal{L}. \end{cases}$$

Chen and Ru [4] proved a similar result with $\mathcal{L} = \mathcal{L}_F$ the set of linear factors of F and with a stronger condition instead of (i), on the other hand they considered decomposable forms with coefficients in an arbitrary number field.

From Theorem 2.7 and Theorem B we deduce the following corollary.

Corollary 2.8. *Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form in $m \geq 2$ variables with splitting field K and $\mathcal{L} \supseteq \mathcal{L}_F$ a finite set of linear forms in $K[X_1, \dots, X_m]$.*

(i) *Assume that F and \mathcal{L} satisfy condition (i) of Theorem B. Suppose F has degree n . Let S be a finite set of primes and let $\epsilon > 0$. Then for every $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $\ell(\mathbf{x}) \neq 0$ for $\ell \in \mathcal{L}$ we have*

$$(2.8) \quad [F(\mathbf{x})]_S \ll_{F, \mathcal{L}, S, \epsilon} |F(\mathbf{x})|^{1-(1/n(m-1))+\epsilon}.$$

(ii) Assume that F and \mathcal{L} do not satisfy condition (i) of Theorem B. Then there are a finite set of primes S and a constant $\gamma > 0$ such that

$$[F(\mathbf{x})]_S \geq \gamma |F(\mathbf{x})|$$

holds for infinitely many $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $\ell(\mathbf{x}) \neq 0$ for all $\ell \in \mathcal{L}$.

Indeed, if F , \mathcal{L} satisfy condition (i) of Theorem B, S is a finite set of primes and $\epsilon > 0$, then

$$\frac{|F(\mathbf{x})|}{[F(\mathbf{x})]_S} = \prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p \gg \|\mathbf{x}\|^{(1/(m-1))-\epsilon} \gg |F(\mathbf{x})|^{(1/n(m-1))-\epsilon/n}$$

holds for all $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $\ell(\mathbf{x}) \neq 0$ for all $\ell \in \mathcal{L}$, where the implied constants depend on F , S and ϵ . This implies part (i) of Corollary 2.8. If on the other hand F and \mathcal{L} do not satisfy condition (i) of Theorem B then there are a finite set of primes S and an S -free integer b such that (2.6) has infinitely many solutions. This yields infinitely many $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ such that $\ell(\mathbf{x}) \neq 0$ for all $\ell \in \mathcal{L}$ and

$$[F(\mathbf{x})]_S = |F(\mathbf{x})|/|b|.$$

Thus, part (ii) of Corollary 2.8 follows.

We can improve on Corollary 2.8 if we assume condition (i) of Theorem B with $\mathcal{L} = \mathcal{L}_F$, i.e.,

(2.9) rank $\mathcal{L}_F = m$, and for every $\text{Gal}(K/\mathbb{Q})$ -proper subset \mathcal{M} of \mathcal{L}_F with $\emptyset \subsetneq \mathcal{M} \subsetneq \mathcal{L}_F$ we have

$$\mathcal{L}_F \cap \left(\sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} [\sigma(\mathcal{M})] \cap [\mathcal{L}_F \setminus \sigma(\mathcal{M})] \right) \neq \emptyset$$

and in addition to this,

(2.10) $F(\mathbf{x}) \neq 0$ for every non-zero $\mathbf{x} \in \mathbb{Q}^m$.

Let D be a non-zero \mathbb{Q} -linear subspace of \mathbb{Q}^m . We say that a non-empty subset \mathcal{M} of \mathcal{L}_F is *linearly dependent on D* if there is a non-trivial K -linear combination of the forms in \mathcal{M} that vanishes identically on D ; otherwise, \mathcal{M} is said to be *linearly independent on D* . Further, for a non-empty subset \mathcal{M} of \mathcal{L}_F we define rank $_D \mathcal{M}$ to be the cardinality of a maximal subset of \mathcal{M} that is linearly independent on D , and then

$$q_D(\mathcal{M}) := \frac{\sum_{\ell \in \mathcal{M}} e(\ell)}{\text{rank}_D \mathcal{M}}.$$

For instance, $\text{rank}_D \mathcal{L}_F = \dim D$, so $q_D(\mathcal{L}_F) = \deg F / \dim D$. Then put

$$q_D(F) := \max\{q_D(\mathcal{M}) : \emptyset \subsetneq \mathcal{M} \subsetneq \mathcal{L}_F, \text{rank}_D \mathcal{M} < \dim D\}.$$

Finally, put

$$(2.11) \quad c(F) := \max_D \frac{q_D(F)}{q_D(\mathcal{L}_F)} = \max_D q_D(F) \cdot \frac{\dim D}{\deg F},$$

where the maximum is taken over all \mathbb{Q} -linear subspaces D of \mathbb{Q}^m with $\dim D \geq 2$. Lemma 5.2, which is stated and proved in Section 5 below, implies that if F satisfies both (2.9) and (2.10), then $c(F) < 1$. We will not consider the problem how to compute $c(F)$, that is, how to determine a subspace D for which $q_D(F)/q_D(\mathcal{L}_F)$ is maximal; this may involve some linear algebra that is beyond the scope of this paper.

Given a decomposable form $F \in \mathbb{Z}[X_1, \dots, X_m]$, a finite set of primes S , and reals $\epsilon > 0$, $B > 0$, we define $N(F, S, \epsilon, B)$ to be the set of $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $[F(\mathbf{x})]_S \geq |F(\mathbf{x})|^\epsilon$ and $\|\mathbf{x}\| \leq B$.

Theorem 2.9. *Let $m \geq 2$ and let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form as in (2.3) satisfying (2.9) and (2.10). Let $c(F)$ be defined as in (2.11). Then $c(F) < 1$ and*

(i) *for every finite set of primes S , every $\epsilon > 0$ and every $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ we have*

$$[F(\mathbf{x})]_S \ll_{F,S,\epsilon} |F(\mathbf{x})|^{c(F)+\epsilon};$$

(ii) *there are infinitely many primes p , and for each of these primes p infinitely many $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$, such that*

$$[F(\mathbf{x})]_{\{p\}} \gg_{F,p} |F(\mathbf{x})|^{c(F)};$$

(iii) *for every finite set of primes S and every ϵ with $0 < \epsilon < 1$ we have*

$$N(F, S, \epsilon, B) \ll_{F,S,\epsilon} B^{m(1-\epsilon)} \quad \text{as } B \rightarrow \infty.$$

Assertions (i) and (iii) follow without too much effort from work in Liu's thesis [16], while (ii) is an application of Minkowski's Convex Body Theorem.

The constants implied by the Vinogradov symbols in Theorems 2.7 and part (i) of Theorem 2.9 cannot be computed effectively from our method of proof. In fact, these constants can be expressed in terms of the heights of the subspaces occurring in certain instances of the p -adic Subspace Theorem,

but for these we can as yet not compute an upper bound. The constant in (ii) can be computed once one knows a subspace D for which the quotient $q_D(F)/q_D(\mathcal{L}_F)$ is equal to $c(F)$. The work of Liu from which part (iii) is derived uses a quantitative version of the p -adic Subspace Theorem, giving an explicit upper bound for the *number* of subspaces. This enables one to compute effectively the constant in part (iii).

Likely, a result of the same type as part (iii) of Theorem 2.9 can be proved in a similar way as Theorem 2.6 using the lattice point counting result of Barroero and Widmer, thereby avoiding Liu's work and the quantitative Subspace Theorem. But such an approach would be less straightforward.

2.3 Effective results for decomposable forms

We consider again S -parts of values $F(\mathbf{x})$, where F is a decomposable form in $\mathbb{Z}[X_1, \dots, X_m]$ and $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$. Under certain stronger conditions on F , we shall give an estimate of the form $[F(\mathbf{x})]_S \leq \kappa_6 |F(\mathbf{x})|^{1-\kappa_5}$, with effectively computable positive κ_5, κ_6 that depend only on F and S . For applications, we make the dependence of κ_5 and κ_6 explicit in terms of S . The decomposable forms with the said stronger conditions include binary forms, and discriminant forms of an arbitrary number of variables.

Let again $S = \{p_1, \dots, p_s\}$ be a finite set of primes and b an integer coprime with $p_1 \cdots p_s$, and consider equation (2.4) in $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ and non-negative integers a_1, \dots, a_s . Under the stronger conditions for the decomposable form F mentioned above, explicit upper bounds were given in Györy [11, 12] for the solutions of (2.4), from which upper bounds can be deduced for $[F(\mathbf{x})]_S$. Later, more general and stronger explicit results were obtained by Györy and Yu [15] on another version of (2.4). These explicit results provided some information on the arithmetical properties of $F(\mathbf{x})$ at points $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$. In this paper, we deduce from the results of Györy and Yu [15] a better bound for $[F(\mathbf{x})]_S$; see Theorem 2.10. This will give more precise information on the arithmetical structure of those non-zero integers h_0 that can be represented by $F(\mathbf{x})$ at integral points \mathbf{x} ; see Corollary 7.1.

To state our results, we introduce some notation and assumptions. Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a non-zero decomposable form. Denote by K its splitting field. We choose a factorization of F into linear forms with coefficients in K as in (2.3), with \mathcal{L}_F a $\text{Gal}(K/\mathbb{Q})$ -symmetric set of pairwise non-proportional linear forms. Denote by $\mathcal{G}(\mathcal{L}_F)$ the graph with vertex set \mathcal{L}_F in which distinct ℓ, ℓ' in \mathcal{L}_F are connected by an edge if $\lambda\ell + \lambda'\ell' + \lambda''\ell'' = 0$ for some $\ell'' \in \mathcal{L}_F$ and some non-zero $\lambda, \lambda', \lambda''$ in K . Let $\mathcal{L}_1, \dots, \mathcal{L}_k$ be the

vertex sets of the connected components of $\mathcal{G}(\mathcal{L}_F)$. When $k = 1$ and \mathcal{L}_F has at least three elements, \mathcal{L}_F is said to be *triangularly connected*; see Gyóry and Papp [14].

In what follows, we assume that F in (2.4) satisfies the following conditions:

$$(2.12) \quad \mathcal{L}_F \text{ has rank } m;$$

$$(2.13) \quad \text{either } k = 1; \text{ or } k > 1 \text{ and } X_m \text{ can be expressed as a } K\text{-linear combination of the forms from } \mathcal{L}_i, \text{ for } i = 1, \dots, k.$$

We note that these conditions are satisfied by binary forms with at least three pairwise non-proportional linear factors, and also discriminant forms, index forms and a restricted class of norm forms in an arbitrary number of variables. As has been explained in [8, Chap. 9], conditions (2.12), (2.13) imply condition (i) of Theorem B.

As before, let $S = \{p_1, \dots, p_s\}$ be a finite set of primes, and put $P := \max_{1 \leq i \leq s} p_i$. Further, let K denote the splitting field of F , and put $d := [K : \mathbb{Q}]$. Then we have

Theorem 2.10. *Under assumptions (2.12), (2.13), we have*

$$(2.14) \quad [F(\mathbf{x})]_S \leq \kappa_6 |F(\mathbf{x})|^{1-\kappa_5}$$

for every $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_{\text{prim}}^m$ with $F(\mathbf{x}) \neq 0$, and with $x_m \neq 0$ if $k > 1$, where

$$\kappa_5 = (c_3^s ((P(\log p_1) \cdots (\log p_s))^d)^{-1} \geq (c_3^s (2P(\log P)^s)^d)^{-1}$$

and κ_6, c_3 are effectively computable positive numbers, depending only on F .

It is easy to check that if $F \in \mathbb{Z}[X, Y]$ is a binary form with at least three pairwise non-proportional linear factors over its splitting field, then it satisfies (2.12), (2.13) with $m = 2$ and $k = 1$. Thus, Theorem 2.5 follows at once from Theorem 2.10.

We shall deduce Theorem 2.10 from a special case of Theorem 3 of Gyóry and Yu [15]. The constants κ_5, κ_6, c_3 could have been made explicit by using the explicit version of this theorem of Gyóry and Yu [15]. Further, Theorem 2.10 could be proved more generally, over number fields and for a larger class of decomposable forms.

Weaker versions of Theorem 2.10 can be deduced from the results of Gyóry [11, 12].

3 Proofs of Theorems 2.1, 2.3, 2.4, 2.6

Let again $S = \{p_1, \dots, p_s\}$ be a finite, non-empty set of primes. We denote by $|\cdot|_\infty$ the ordinary absolute value, and by $|\cdot|_p$ the p -adic absolute value with $|p|_p = p^{-1}$ for a prime number p . Further, we set $\mathbb{Q}_\infty := \mathbb{R}$, $\overline{\mathbb{Q}_\infty} := \mathbb{C}$.

The following result is a very well-known consequence of the p -adic Thue-Siegel-Roth Theorem. The only reference we could find for it is [18, Chap.IX, Thm.3]. For convenience of the reader we recall the proof.

Proposition 3.1. *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 2$ and of non-zero discriminant. Then*

$$\frac{|F(x, y)|}{[F(x, y)]_S} \gg_{F, S, \epsilon} \max(|x|, |y|)^{n-2-\epsilon}$$

for all $\epsilon > 0$ and all $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \neq 0$.

Proof. We assume that $F(1, 0) \neq 0$. This is no loss of generality. For if this is not the case, there is an integer b of absolute value at most n with $F(1, b) \neq 0$ and we may proceed with the binary form $F(X, bX + Y)$. Our assumption implies that for each $p \in S \cup \{\infty\}$ we have a factorization $F(X, Y) = a \prod_{i=1}^n (X - \beta_{ip} Y)$ with $a \in \mathbb{Z}$ and $\beta_{ip} \in \overline{\mathbb{Q}_p}$ algebraic over \mathbb{Q} for $i = 1, \dots, n$. For every $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \neq 0$ we have

$$\begin{aligned} \frac{|F(x, y)|}{[F(x, y)]_S \cdot (\max(|x|, |y|))^n} &= \left(\prod_{p \in S \cup \{\infty\}} |F(x, y)|_p \right) / \max(|x|, |y|)^n \\ &\gg_{F, S} \prod_{p \in S} \min_{1 \leq i \leq n} \frac{|x - \beta_{ip} y|_p}{\max(|x|_p, |y|_p)} \\ &\gg_{F, S} \prod_{p \in S \cup \{\infty\}} \min \left(1, \left| \frac{x}{y} - \beta_{1p} \right|_p, \dots, \left| \frac{x}{y} - \beta_{np} \right|_p \right). \end{aligned}$$

The latter is $\gg_{F, S, \epsilon} \max(|x|, |y|)^{-2-\epsilon}$ for every $\epsilon > 0$ by the p -adic Thue-Siegel-Roth Theorem. Proposition 3.1 follows. \square

Proof of Theorem 2.1. Let $f(X) \in \mathbb{Z}[X]$ be the polynomial from Theorem 2.1.

(i). The binary form $F(X, Y) := Y^{n+1} f(X/Y)$ has degree $n + 1$ and non-zero discriminant. Now by Proposition 3.1, we have for every $\epsilon > 0$ and every sufficiently large integer x ,

$$\frac{|f(x)|}{[f(x)]_S} \gg_{f, S, \epsilon} |x|^{n-1-n\epsilon} \gg_{f, S, \epsilon} |f(x)|^{(n-1-n\epsilon)/n},$$

implying $[f(x)]_S \ll_{f,S,\epsilon} |f(x)|^{(1/n)+\epsilon}$.

(ii). There are infinitely many primes p such that $f(x) \equiv 0 \pmod{p}$ is solvable. Excluding the finitely many primes dividing the leading coefficient or the discriminant of $f(X)$, there remain infinitely many primes. Take such a prime p . By Hensel's Lemma, there is for every positive integer k an integer x_k such that $f(x_k) \equiv 0 \pmod{p^k}$. We may choose such an integer with $p^k \leq x_k < 2p^k$. Then clearly, $x_1 < x_2 < \dots$ and for k sufficiently large, $f(x_k) \neq 0$ and $f(x_k) \equiv 0 \pmod{p^k}$. Consequently,

$$[f(x_k)]_{\{p\}} \geq p^k \geq \frac{1}{2}|x_k| \gg_f |f(x_k)|^{1/n}.$$

This proves Theorem 2.1. \square

Proof of Theorem 2.4. Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be the binary form from Theorem 2.4.

(i) By Proposition 3.1, we have for every $\epsilon > 0$ and every pair $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \neq 0$ and $\max(|x|, |y|)$ sufficiently large,

$$\frac{|F(x, y)|}{[F(x, y)]_S} \gg_{F,S,\epsilon} \max(|x|, |y|)^{n-2-n\epsilon} \gg_{F,S,\epsilon} |F(x, y)|^{1-(2/n)-\epsilon}.$$

(ii) We assume that $F(1, 0) \neq 0$ which, similarly as in the proof of Proposition 3.1, is no loss of generality. By Chebotarev's Density Theorem, there are infinitely many primes p such that F splits into linear factors over \mathbb{Q}_p . From these, we exclude the finitely many primes that divide $D(F)$ or $F(1, 0)$. Let \mathcal{P} be the infinite set of remaining primes. Then for every $p \in \mathcal{P}$, we can express $F(X, Y)$ as

$$F(X, Y) = a \prod_{i=1}^n (X - \beta_{ip} Y)$$

with $a \in \mathbb{Z}$ with $|a|_p = 1$, $\beta_{ip} \in \mathbb{Z}_p$ for $i = 1, \dots, n$ and $|\beta_{ip} - \beta_{jp}|_p = 1$ for $i, j = 1, \dots, n$ with $i \neq j$.

We distinguish two cases. First assume that F does not split into linear factors over \mathbb{Q} . Take $p \in \mathcal{P}$. Then without loss of generality, $\beta_{1p} \notin \mathbb{Q}$. Let k be a positive integer. By Minkowski's Convex Body Theorem, there is a non-zero pair $(x, y) \in \mathbb{Z}^2$ such that

$$|x - \beta_{1p}y|_p \leq p^{-k}, \quad \max(|x|, |y|) \leq p^{k/2}.$$

We may assume without loss of generality that $\gcd(x, y)$ is not divisible by any prime other than p . Assume that $\gcd(x, y) = p^u$ with $u \geq 0$, and let $x_k := p^{-u}x$, $y_k := p^{-u}y$. Then $(x_k, y_k) \in \mathbb{Z}_{\text{prim}}^2$ and

$$|x_k - \beta_{1p}y_k|_p \leq p^{u-k}, \quad \max(|x_k|, |y_k|) \leq p^{(k/2)-u}.$$

This clearly implies $u \leq k/2$. We observe that if we let $k \rightarrow \infty$ then (x_k, y_k) runs through an infinite subset of $\mathbb{Z}_{\text{prim}}^2$. Indeed, otherwise we would have a pair $(x_0, y_0) \in \mathbb{Z}_{\text{prim}}^2$ with $|x_0 - \beta_{1p}y_0|_p \leq p^{-k/2}$ for infinitely many k which is impossible since $\beta_{1p} \notin \mathbb{Q}$. Next we have $F(x_k, y_k) \neq 0$ for all k . Indeed, suppose that $F(x_k, y_k) = 0$ for some k . Then $x_k/y_k = \beta_{ip}$ for some $i \geq 2$. Since $\beta_{ip} \in \mathbb{Z}_p$ we necessarily have $|y_k|_p = 1$. But then $|x_k - \beta_{1p}y_k|_p = |\beta_{ip} - \beta_{1p}|_p = 1$, which is again impossible. Finally, since clearly $|x_k - \beta_{ip}y_k|_p \leq 1$ for $i = 2, \dots, n$, we derive that for each positive integer k ,

$$\begin{aligned} [F(x_k, y_k)]_{\{p\}} &= |F(x_k, y_k)|_p^{-1} \geq p^{k-u} \geq \max(|x_k|, |y_k|)^2 \\ &\gg_{F,p} |F(x_k, y_k)|^{2/n}. \end{aligned}$$

Next, we assume that $F(X, Y)$ splits into linear factors over \mathbb{Q} . Then $F(X, Y) = a \prod_{i=1}^n (X - \beta_i Y)$ with $a \in \mathbb{Z}$, $|a|_p = 1$ for $p \in \mathcal{P}$, $\beta_i \in \mathbb{Q}$ and $|\beta_i|_p \leq 1$ for $p \in \mathcal{P}$, $i = 1, \dots, n$, and $|\beta_i - \beta_j|_p = 1$ for $p \in \mathcal{P}$, $i, j = 1, \dots, n$, $i \neq j$. Pick distinct $p, q \in \mathcal{P}$ and let $S = \{p, q\}$. Then there is an integer u , coprime with pq , such that $u\beta_1$, $u\beta_2$ and $u/(\beta_2 - \beta_1)$ are all integers. Choose positive integers k, l . Then

$$x := \frac{u(\beta_2 p^k - \beta_1 q^l)}{\beta_2 - \beta_1}, \quad y := \frac{u(p^k - q^l)}{\beta_1 - \beta_2}$$

are integers satisfying $x - \beta_1 y = up^k$, $x - \beta_2 y = uq^l$. By our choice of $p, q \in \mathcal{P}$ and by direct substitution, it follows that the numbers $x - \beta_i y$ ($i = 3, \dots, n$) have p -adic and q -adic absolute values equal to 1. Thus, $|F(x, y)|_p = p^{-k}$, $|F(x, y)|_q = q^{-l}$ and so $[F(x, y)]_S = p^k q^l$.

Clearly, $g := \gcd(x, y)$ is coprime with pq . Let $x_{k,l} := x/g$, $y_{k,l} := y/g$ so that $(x_{k,l}, y_{k,l}) \in \mathbb{Z}_{\text{prim}}^2$. Then clearly, $[F(x_{k,l}, y_{k,l})]_S = p^k q^l$. We now choose k, l such that p^k, q^l are approximately equal, say $p^k < q^l < q \cdot p^k$. Then $\max(|x_{k,l}|, |y_{k,l}|) \leq \max(|x|, |y|) \ll_{F,S} (p^k q^l)^{1/2}$ and thus,

$$[F(x_{k,l}, y_{k,l})]_S \gg_{F,S} \max(|x_{k,l}|, |y_{k,l}|)^2 \gg_{F,S} |F(x_{k,l}, y_{k,l})|^{2/n}.$$

□

In the proofs of Theorems 2.3 and 2.6 we need a few auxiliary results.

Lemma 3.2. *Let $f(X) \in \mathbb{Z}[X]$ be a polynomial of non-zero discriminant and a an integer and p a prime. Denote by g_p the largest non-negative integer g such that p^g divides the discriminant $D(f)$ of f . For $k > 0$ denote by $r(f, a, p^k)$ the number of congruence classes x modulo p^k with $f(x) \equiv 0 \pmod{p^k}$, $x \equiv a \pmod{p}$. Then $r(f, a, p^k) = r(f, a, p^{g_p+1})$ for $k \geq g_p + 1$.*

Proof. This is a consequence of [25, Thm. 2]. \square

Given a positive integer h , we say that two pairs $(x_1, y_1), (x_2, y_2) \in \mathbb{Z}_{\text{prim}}^2$ are congruent modulo h if $x_1 y_2 \equiv x_2 y_1 \pmod{h}$. With this notion, for a given binary form $F(X, Y) \in \mathbb{Z}[X, Y]$ we can divide the solutions $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ of $F(x, y) \equiv 0 \pmod{h}$ into congruence classes modulo h .

Lemma 3.3. *Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 2$ and of non-zero discriminant and p a prime. Denote by g_p the largest non-negative integer g such that p^g divides the discriminant $D(F)$ of F . For $k > 0$ denote by $r(F, p^k)$ the number of congruence classes modulo p^k of $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \equiv 0 \pmod{p^k}$. Then $r(F, p^k) = r(F, p^{g_p+1})$ for $k \geq g_p + 1$.*

Proof. Neither the number of congruence classes under consideration, nor the discriminant of F , changes if we replace $F(X, Y)$ by $F(aX + bY, cX + dY)$ for some matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$. After such a replacement, we can achieve that $F(1, 0)F(0, 1) \neq 0$, so we assume this henceforth. Let $f(X) := F(X, 1)$ and $f^*(X) := F(1, X)$. The map $(x, y) \mapsto x \cdot y^{-1} \pmod{p^k}$ gives a bijection between the congruence classes modulo p^k of pairs $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \equiv 0 \pmod{p^k}$ and $y \not\equiv 0 \pmod{p}$ and the congruence classes modulo p^k of integers z with $f(z) \equiv 0 \pmod{p}$. Likewise, the map $(x, y) \mapsto y \cdot x^{-1} \pmod{p^k}$ establishes a bijection between the congruence classes modulo p^k of $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \equiv 0 \pmod{p^k}$ and $y \equiv 0 \pmod{p}$ and the congruence classes modulo p^k of integers z with $f^*(z) \equiv 0 \pmod{p^k}$ and $z \equiv 0 \pmod{p}$. Further, our assumption $F(1, 0)F(0, 1) \neq 0$ implies that $D(F) = D(f) = D(f^*)$. Now an application of Lemma 3.2 yields that $r(F, p^k) = \sum_{a=0}^{p-1} r(f, a, p^k) + r(f^*, 0, p^k)$ is constant for $k \geq g_p + 1$. \square

For a binary form $F(X, Y) \in \mathbb{R}[X, Y]$ and for positive reals B, M , we denote by $V_F(B, M)$ the set of pairs $(x, y) \in \mathbb{R}^2$ with $\max(|x|, |y|) \leq B$ and $|F(x, y)| \leq M$, and by $\mu_F(B, M)$ the area (two-dimensional Lebesgue measure) of this set.

Our next lemma is a consequence of a general lattice point counting result of Barroero and Widmer [1, Thm. 1.3].

Lemma 3.4. *let n be an integer ≥ 2 . Then there is a constant $c(n) > 0$ such that for every non-zero binary form $F(X, Y) \in \mathbb{R}[X, Y]$ of degree n , every lattice $\Lambda \subseteq \mathbb{Z}^2$ and all positive reals B, M ,*

$$\left| \#(V_F(B, M) \cap \Lambda) - \frac{\mu_F(B, M)}{\det \Lambda} \right| \leq c(n) \max(1, B/m(\Lambda)),$$

where $m(\Lambda)$ is the length of the shortest non-zero vector of Λ .

Proof. We write points in $\mathbb{R}^{n+3} \times \mathbb{R}^2$ as $(z_0, \dots, z_n, u, v, x, y)$. The set $Z \subseteq \mathbb{R}^{n+3} \times \mathbb{R}^2$ given by the inequalities

$$|z_0x^n + z_1x^{n-1}y + \dots + z_ny^n| \leq v, \quad |x| \leq u, \quad |y| \leq u$$

is a definable family in the sense of [1], parametrized by the tuple $T = (z_0, \dots, z_n, u, v)$. By substituting for this tuple the coefficients of F , respectively B and M , we obtain the set $V_F(B, M)$ as defined above. The sum of the one-dimensional volumes of the orthogonal projections of $V_F(B, M)$ on the x -axis and y -axis is at most $4B$, and the first minimum of Λ is $m(\Lambda)$. Now Lemma 3.4 follows directly from [1, Thm. 3.1]. \square

A lattice $\Lambda \subseteq \mathbb{Z}^2$ is called *primitive* if it contains points $(x, y) \in \mathbb{Z}_{\text{prim}}^2$.

Lemma 3.5. *Let again n be an integer ≥ 2 . Then there is a constant $c'(n) > 0$ such that for every binary form $F \in \mathbb{Z}[X, Y]$ of degree n , every primitive lattice $\Lambda \subseteq \mathbb{Z}^2$, and all reals $B, M > 1$,*

$$\left| \#(V_F(B, M) \cap \Lambda \cap \mathbb{Z}_{\text{prim}}^2) - \left(\frac{6}{\pi^2} \prod_{p|\det \Lambda} (1 + p^{-1})^{-1} \right) \cdot \frac{\mu_F(B, M)}{\det \Lambda} \right| \leq c'(n)B \log 3B.$$

Proof. In the proof below, p, p_i denote primes.

Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form, $\Lambda \subseteq \mathbb{Z}^2$ a primitive lattice, and B, M reals > 1 . Put $d := \det \Lambda$. For a positive integer h , define the lattice $\Lambda_h := \Lambda \cap h\mathbb{Z}^2$. Since Λ is primitive, there is a basis $\{\mathbf{a}, \mathbf{b}\}$ of \mathbb{Z}^2 such that $\{\mathbf{a}, d\mathbf{b}\}$ is a basis of Λ . Hence $\{h\mathbf{a}, \text{lcm}(h, d)\mathbf{b}\}$ is a basis of Λ_h , and so

$$(3.1) \quad \det \Lambda_h = h \cdot \text{lcm}(h, d) = d \cdot \frac{h^2}{\text{gcd}(h, d)}.$$

Further, the shortest non-zero vector of Λ_h has length

$$(3.2) \quad m(\Lambda_h) \geq h.$$

We define $\rho(h) := \#(V_F(B, M) \cap \Lambda_h)$. Then by the rule of inclusion and exclusion,

$$\begin{aligned} & \#(V_F(B, M) \cap \Lambda \cap \mathbb{Z}_{\text{prim}}^2) \\ &= \rho(1) - \sum_{p \leq B} \rho(p) + \sum_{p_1 < p_2: p_1 p_2 \leq B} \rho(p_1 p_2) - \dots \\ &= \sum_{h \leq B} \mu(h) \rho(h), \end{aligned}$$

where $\mu(h)$ denotes the Möbius function. The previous lemma together with (3.1), (3.2) implies

$$\begin{aligned} & \left| \# \left(V_F(B, M) \cap \Lambda \cap \mathbb{Z}_{\text{prim}}^2 \right) - \frac{\mu_F(B, M)}{d} \cdot \sum_{h \leq B} \mu(h) \cdot \frac{\gcd(d, h)}{h^2} \right| \\ & \leq c(n)B \cdot \sum_{h \leq B} \frac{|\mu(h)|}{h}, \end{aligned}$$

hence

$$\begin{aligned} & \left| \# \left(V_F(B, M) \cap \Lambda \cap \mathbb{Z}_{\text{prim}}^2 \right) - \frac{\mu_F(B, M)}{d} \cdot \sum_{h=1}^{\infty} \mu(h) \cdot \frac{\gcd(d, h)}{h^2} \right| \\ & \leq \frac{\mu_F(B, M)}{d} \cdot \sum_{h > B} |\mu(h)| \frac{\gcd(d, h)}{h^2} + c(n)B \cdot \sum_{h \leq B} \frac{|\mu(h)|}{h} \\ & \leq c'(n)B \log 3B, \end{aligned}$$

where we have used $\sum_{h > B} |\mu(h)| \frac{\gcd(d, h)}{h^2} \leq 2d/B$, $\mu_F(B, M) \leq 4B^2$, and $\sum_{h \leq B} \frac{|\mu(h)|}{h} \leq \log 3B$. Now the proof is finished by observing that

$$\sum_{h=1}^{\infty} \mu(h) \cdot \frac{\gcd(d, h)}{h^2} = \prod_{p|d} (1 - p^{-1}) \cdot \prod_{p \nmid d} (1 - p^{-2}) = \frac{6}{\pi^2} \cdot \prod_{p|d} (1 + p^{-1})^{-1}.$$

□

Lemma 3.6. *Let $\alpha_1, \dots, \alpha_t$ be positive reals. Denote by $N(A)$ the number of tuples of non-negative integers (u_1, \dots, u_t) with*

$$(3.3) \quad A \leq \alpha_1 u_1 + \dots + \alpha_t u_t \leq A + 2(\alpha_1 + \dots + \alpha_t).$$

Then

$$N(A) \asymp_{t, \alpha_1, \dots, \alpha_t} A^{t-1} \quad \text{as } A \rightarrow \infty.$$

Proof. Constants implied by the Vinogradov symbols \ll, \gg will depend on $t, \alpha_1, \dots, \alpha_t$.

For $\mathbf{u} = (u_1, \dots, u_t) \in \mathbb{Z}^t$, denote by $\mathcal{C}_{\mathbf{u}}$ the cube in \mathbb{R}^t consisting of the points $\mathbf{y} = (y_1, \dots, y_t)$ with $u_i \leq y_i < u_i + 1$ for $i = 1, \dots, t$. Let \mathcal{C} be the union of the cubes $\mathcal{C}_{\mathbf{u}}$ over all points \mathbf{u} with non-negative integer coordinates satisfying (3.3). Put $\alpha := \alpha_1 + \dots + \alpha_t$. Then $\mathcal{C}_1 \subseteq \mathcal{C} \subseteq \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2$ are the subsets of \mathbb{R}^s given by

$$\begin{aligned} A + \alpha & \leq \alpha_1 y_1 + \dots + \alpha_t y_t \leq A + 2\alpha, & y_1 \geq 0, \dots, y_t \geq 0, \\ A & \leq \alpha_1 y_1 + \dots + \alpha_t y_t \leq A + 3\alpha, & y_1 \geq 0, \dots, y_t \geq 0, \end{aligned}$$

respectively. Clearly $N(A)$ is estimated from below and above by the measures of \mathcal{C}_1 and \mathcal{C}_2 , the first being $\gg (A + 2\alpha)^t - (A + \alpha)^t \gg A^{t-1}$, the second being $\ll A^{t-1}$. The lemma follows. \square

We first give the complete proof of Theorem 2.6. The proof of Theorem 2.3 is then obtained by making a few modifications.

Proof of Theorem 2.6. Let $F(X, Y) \in \mathbb{Z}[X, Y]$ be a binary form of degree $n \geq 3$ with non-zero discriminant, ϵ a real with $0 < \epsilon < \frac{1}{n}$ and $S = \{p_1, \dots, p_s\}$ a finite set of primes. Let $S' = \{p_1, \dots, p_{s'}\}$ be the set of $p \in S$ such that $F(x, y) \equiv 0 \pmod{p^{q_p+1}}$ has a solution in $\mathbb{Z}_{\text{prim}}^2$, and let $S'' = \{p_{s'+1}, \dots, p_s\}$ be the set of remaining primes. In what follows, constants implied by Vinogradov symbols \ll, \gg and by the Landau O -symbol will depend only on F, S and ϵ .

We first prove that

$$N(F, S, \epsilon, B) \ll_{F, S, \epsilon} B^{2-n\epsilon} (\log B)^{s'-1} \quad \text{as } B \rightarrow \infty.$$

The set of pairs (x, y) under consideration can be partitioned into sets \mathcal{N}_h , where h runs through the set of positive integers composed of primes from S , and \mathcal{N}_h is the set of pairs $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with

$$\max(|x|, |y|) \leq B, \quad [F(x, y)]_S = h, \quad |F(x, y)| \leq h^{1/\epsilon}.$$

We first estimate $\#\mathcal{N}_h$ from above by means of Lemma 3.5 where h is any positive integer composed of primes from S . Notice that for $(x, y) \in \mathcal{N}_h$ we have $F(x, y) \equiv 0 \pmod{h}$. By Lemma 3.3 and the Chinese Remainder Theorem, the set of these (x, y) lies in $\ll 1$ congruence classes modulo h . Each of these congruence classes is precisely the set of primitive points in a set of the shape

$$\{(x, y) \in \mathbb{Z}^2 : y_0 x \equiv x_0 y \pmod{h}\}$$

with $(x_0, y_0) \in \mathbb{Z}_{\text{prim}}^2$, which is a primitive lattice of determinant h . So \mathcal{N}_h is contained in $\ll 1$ primitive lattices of determinant h .

We next estimate the area $\mu_F(B, h^{1/\epsilon})$ of $V(B, h^{1/\epsilon})$. There is a constant $c_F > 0$ such that

$$(3.4) \quad |F(x, y)| \leq c_F (\max(|x|, |y|))^n \quad \text{for } (x, y) \in \mathbb{R}^2.$$

If $h \geq (c_F B^n)^\epsilon$ then the condition $|F(x, y)| \leq h^{1/\epsilon}$ is already implied by $\max(|x|, |y|) \leq B$, and so $\mu_F(B, h^{1/\epsilon}) = 4B^2$. On the other hand, if $h <$

$(c_F B^n)^\epsilon$, we have, denoting by μ the area,

$$\begin{aligned} \mu_F(B, h^{1/\epsilon}) &\leq \mu(\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leq h^{1/\epsilon}\}) \\ &= h^{2/n\epsilon} \cdot \mu(\{(x, y) \in \mathbb{R}^2 : |F(x, y)| \leq 1\}) \ll h^{2/n\epsilon}, \end{aligned}$$

since the set of $(x, y) \in \mathbb{R}^2$ with $|F(x, y)| \leq 1$ has finite area (see for instance [17]). Now invoking Lemma 3.5, we infer

$$(3.5) \quad \#\mathcal{N}_h \ll \begin{cases} B^2/h + O(B \log B) & \text{if } h \geq (c_F B^n)^\epsilon, \\ h^{(2/n\epsilon)-1} + O(B \log B) & \text{if } h < (c_F B^n)^\epsilon. \end{cases}$$

Finally, from (3.4) it is clear that $\mathcal{N}_h = \emptyset$ if $h > c_F B^n$.

Let $\alpha := \log(p_1 \cdots p_{s'})$. For $j \in \mathbb{Z}$, let \mathcal{M}_j be the union of the sets \mathcal{N}_h with

$$(3.6) \quad e^{2j\alpha}(c_F B^n)^\epsilon \leq h < e^{(2j+2)\alpha}(c_F B^n)^\epsilon.$$

We restrict ourselves to j with

$$(3.7) \quad e^{2j\alpha}(c_F B^n)^\epsilon \leq c_F B^n, \quad e^{(2j+2)\alpha}(c_F B^n)^\epsilon > 1,$$

since for the remaining j the set \mathcal{M}_j is empty. Thus,

$$(3.8) \quad N(F, S, \epsilon, B) \ll \sum_j \#\mathcal{M}_j,$$

where the summation is over j with (3.7).

We estimate the number of h with (3.6). Write $h = h' p_1^{u_1} \cdots p_{s'}^{u_{s'}}$ where h' is composed of primes from S'' . Then h' divides $\prod_{p \in S''} p^{g_p}$, so we have $\ll 1$ possibilities for h' . By applying Lemma 3.6 with $t = s'$, $A = e^{2j\alpha}(c_F B^n)^\epsilon/h'$, $\alpha_i = \log p_i$ for $i = 1, \dots, s'$, we infer from Lemma 3.6 that for given h' the number of possibilities for $(u_1, \dots, u_{s'})$ is $\ll (\log B)^{s'-1}$. Hence the number of h with (3.6) is $\ll (\log B)^{s'-1}$. Now from (3.5) it follows that for j with (3.7),

$$\#\mathcal{M}_j \ll \begin{cases} e^{-2j\alpha} B^{2-n\epsilon} (\log B)^{s'-1} + O(B(\log B)^{s'}) & \text{if } j \geq 0, \\ e^{-2|j|\alpha((2/n\epsilon)-1)} B^{2-n\epsilon} (\log B)^{s'-1} + O(B(\log B)^{s'}) & \text{if } j < 0. \end{cases}$$

Finally, from these estimates and (3.8) we deduce, taking into consideration that the number of j with (3.7) is $\ll \log B$, and also our assumption $0 < \epsilon < \frac{1}{n}$,

$$\begin{aligned} N(F, S, \epsilon, B) &\ll \left(\sum_{j \geq 0} e^{-2j\alpha} + \sum_{j < 0} e^{-2|j|\alpha((2/n\epsilon)-1)} \right) \cdot B^{2-n\epsilon} (\log B)^{s'-1} \\ &\quad + O(B(\log B)^{s'+1}) \\ &\ll B^{2-n\epsilon} (\log B)^{s'-1}. \end{aligned}$$

We next prove that

$$N(F, S, \epsilon, B) \gg B^{2-n\epsilon} (\log B)^{s'-1} \quad \text{as } B \rightarrow \infty.$$

For $i = s' + 1, \dots, s$, let a_i be the largest integer u such that $F(x, y) \equiv 0 \pmod{p_i^u}$ is solvable in $(x, y) \in \mathbb{Z}_{\text{prim}}^2$. Let for the moment h be any integer of the shape $h = p_1^{u_1} \cdots p_s^{u_s}$ where $u_i \geq g_{p_i} + 1$ for $i = 1, \dots, s'$ and $u_i = a_i$ for $i = s' + 1, \dots, s$, and where $h \geq (c_F B^n)^\epsilon$. By Lemma 3.5 and the Chinese Remainder Theorem, the number of congruence classes modulo h of $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $F(x, y) \equiv 0 \pmod{h}$ is

$$r := \prod_{i=1}^{s'} r(F, p_i^{g_{p_i}+1}) \cdot \prod_{i=s'+1}^s r(F, p_i^{a_i}),$$

which is independent of h . As mentioned above, each of these congruence classes is just the set of primitive points in a primitive lattice of determinant h . Furthermore, since these lattices arise from different residue classes modulo h of points in $\mathbb{Z}_{\text{prim}}^2$, the intersection of any two of these lattices does not contain points from $\mathbb{Z}_{\text{prim}}^2$ anymore. Since moreover by our assumption $h \geq (c_F B^n)^\epsilon$ the set $V(B, h^{1/\epsilon})$ has area $(4B)^2$, an application of Lemma 3.5 yields that the set of $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ with $\max(|x|, |y|) \leq B$, $|F(x, y)| \leq h^{1/\epsilon}$ and $F(x, y) \equiv 0 \pmod{h}$ has cardinality

$$cr \cdot \frac{(4B)^2}{h} + O(B \log B),$$

where $c = (6/\pi^2) \prod_{p \in S_0} (1 + p^{-1})^{-1}$, with S_0 the set obtained from S by removing those primes p_i from S'' for which $a_i = 0$. By the rule of inclusion and exclusion, the set \mathcal{N}_h , i.e., the set of $(x, y) \in \mathbb{Z}_{\text{prim}}^2$ as above with $F(x, y)$ divisible by h but not by hp for $p \in S'$, has cardinality

$$\begin{aligned} (3.9) \quad & cr \cdot \frac{(4B)^2}{h} - \sum_{p \in S'} cr \cdot \frac{(4B)^2}{ph} + \sum_{p, q \in S', p < q} cr \cdot \frac{(4B)^2}{pqh} - \dots \\ & \qquad \qquad \qquad + O(B \log B) \\ & = cr \prod_{p \in S'} (1 - p^{-1}) \cdot \frac{(4B)^2}{h} + O(B \log B) \gg \frac{B^2}{h} + O(B \log B). \end{aligned}$$

We now consider the set of integers h of the shape $p_1^{u_1} \cdots p_s^{u_s}$ with $u_i \geq g_{p_i} + 1$ for $i = 1, \dots, s'$ and $u_i = a_i$ for $i = s' + 1, \dots, s$, and with $(c_F B^n)^\epsilon \leq h \leq e^{2\alpha} (c_F B^n)^\epsilon$, where again $\alpha = \log(p_1 \cdots p_{s'})$. By Lemma 3.6, there are $\gg (\log B)^{s'-1}$ such integers. Using again $0 < \epsilon < \frac{1}{n}$, it follows that

$$N(F, S, \epsilon, B) \geq \sum_h \#\mathcal{N}_h \gg B^{2-n\epsilon} (\log B)^{s'-1}.$$

This completes the proof of Theorem 2.6. □

Proof of Theorem 2.3. Let $f \in \mathbb{Z}[X]$ be a polynomial of degree $n \geq 2$ with non-zero discriminant, ϵ a real with $0 < \epsilon < \frac{1}{n}$ and $S = \{p_1, \dots, p_s\}$ a finite set of primes. Similarly as above $S' = \{p_1, \dots, p_{s'}\}$ is the set of $p \in S$ such that $f(x) \equiv 0 \pmod{p^{g_p+1}}$ is solvable in \mathbb{Z} and $S'' = \{p_{s'+1}, \dots, p_s\}$.

The proof is the same as that of Theorem 2.6 except from a few small modifications. The main difference is that instead of Lemma 3.5 we use the simple observation that if $V_f(B, M)$ is the set of $x \in \mathbb{R}$ with $|x| \leq B$ and $|f(x)| \leq M$ and $\mu_f(B, M)$ is the one-dimensional measure of this set, then for all $a, h \in \mathbb{Z}$ with $h > 0$, the number of integers $x \in V_f(B, M)$ with $x \equiv a \pmod{h}$ is

$$(3.10) \quad \mu_f(B, M)/h + \text{error term}, \quad \text{with } |\text{error term}| \leq c(n)$$

for some quantity $c(n)$ depending only on $n = \deg f$.

We first prove that

$$(3.11) \quad N(f, S, \epsilon, B) \ll_{f, S, \epsilon} B^{1-n\epsilon} (\log B)^{s'-1} \quad \text{as } B \rightarrow \infty.$$

Let c_f be a constant such that $|f(x)| \leq c_f |x|^n$ for $x \in \mathbb{R}$. Consider the set \mathcal{N}_h of integers x with $|x| \leq B$, $[f(x)]_S = h$ and $|f(x)| \leq h^{1/\epsilon}$. Then if $h \geq (c_f B^n)^\epsilon$ we have $\mu_f(B, h^{1/\epsilon}) = 2B$, while otherwise, $\mu_f(B, h^{1/\epsilon}) \ll h^{1/n\epsilon}$, since $|f(x)| \gg |x|^n$ if $|x| \gg 1$. Now a similar computation as in the proof of Theorem 2.6, using Lemma 3.2 instead of Lemma 3.3, gives instead of (3.5),

$$\#\mathcal{N}_h \ll \begin{cases} B/h + O(1) & \text{if } h \geq (c_f B^n)^\epsilon, \\ h^{(1/n\epsilon)-1} + O(1) & \text{if } h < (c_f B^n)^\epsilon, \end{cases}$$

and then the proof of (3.11) is completed in exactly the same way as in the proof of Theorem 2.6.

The proof of

$$(3.12) \quad N(f, S, \epsilon, B) \gg_{f, S, \epsilon} B^{1-n\epsilon} (\log B)^{s-1} \quad \text{as } B \rightarrow \infty$$

follows the same lines as that of Theorem 2.6. For $i = s' + 1, \dots, s$ let a_i be the largest integer u such that $f(x) \equiv 0 \pmod{p_i^{a_i}}$ is solvable. Let $h = p_1^{u_1} \cdots p_s^{u_s}$ with $u_i \geq g_{p_i} + 1$ for $i = 1, \dots, s'$ and $u_i = a_i$ for $i = s' + 1, \dots, s$, and with $h \geq (c_f B^n)^\epsilon$. Then by combining (3.10) with Lemma 3.2 one obtains that the set of integers x with $|x| \leq B$, $f(x) \equiv 0 \pmod{h}$ and $|f(x)| \leq h^{1/\epsilon}$ has cardinality

$$rB/h + O(1)$$

with $r > 0$ depending only on f , and then an inclusion and exclusion argument gives

$$\#\mathcal{N}_h \gg B/h + O(1).$$

Again, an argument completely similar to that in the proof of Theorem 2.6 gives (3.12). \square

4 Proof of Theorem 2.7

The theorem can be proved by modifying the arguments from [4]. We prefer to follow [6, §8], which already contains the basic ideas. Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form of degree n with splitting field K . We take a factorization of F as in (2.3). Assume that F satisfies condition (i) of Theorem B.

Let D be a linear subspace of \mathbb{Q}^m of dimension ≥ 2 . Denote by D^* the K -vector space of linear forms in $K[X_1, \dots, X_m]$ that vanish identically on D . Then a set of linear forms in $K[X_1, \dots, X_m]$ is linearly dependent on D if some non-trivial K -linear combination of these forms belongs to D^* and linearly independent on D if no such linear combination exists. The D -rank $\text{rank}_D \mathcal{M}$ of a set of linear forms $\mathcal{M} \subset K[X_1, \dots, X_m]$, is the maximal number of linear forms in \mathcal{M} that are linearly independent on D . We have $\text{rank}_D \mathcal{L}_F = \dim D$.

We call a subset \mathcal{I} of \mathcal{L}_F *minimally linearly dependent on D* , if \mathcal{I} itself is linearly dependent on D , but every proper, non-empty subset of \mathcal{I} is linearly independent on D . We define a(n undirected) graph \mathcal{G}_D as follows. The set of vertices of \mathcal{G}_D is \mathcal{L}_F ; and $\{\ell, \ell'\}$ is an edge of \mathcal{G}_D if there is a subset of \mathcal{L}_F that is minimally linearly dependent on D and contains both ℓ and ℓ' . Clearly, if $\{\ell, \ell'\}$ is an edge of \mathcal{G}_D , then so is $\{\sigma(\ell), \sigma(\ell')\}$ for each $\sigma \in \text{Gal}(K/\mathbb{Q})$, i.e., each σ acts on \mathcal{G}_D as an automorphism.

Lemma 4.1. *Let D be a linear subspace of \mathbb{Q}^m of dimension ≥ 2 such that none of the linear forms in \mathcal{L} vanishes identically on D . Then \mathcal{G}_D is connected.*

Proof. Assume that \mathcal{G}_D is not connected. Let \mathcal{M} be the vertex set of a connected component of \mathcal{G}_D . Then $\emptyset \subsetneq \mathcal{M} \subsetneq \mathcal{L}_F$. Clearly, for each $\sigma \in \text{Gal}(K/\mathbb{Q})$, $\sigma(\mathcal{M})$ is also the vertex set of a connected component of \mathcal{G}_D , hence either $\sigma(\mathcal{M}) = \mathcal{M}$, or $\sigma(\mathcal{M}) \cap \mathcal{M} = \emptyset$. That is, \mathcal{M} is $\text{Gal}(K/\mathbb{Q})$ -proper.

By assumption (i) of Theorem B, the K -vector space

$$\sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} [\sigma(\mathcal{M})] \cap [\mathcal{L}_F \setminus \sigma(\mathcal{M})]$$

contains a linear form from \mathcal{L} , which, by assumption, does not lie in D^* . Hence there is $\sigma \in \text{Gal}(K/\mathbb{Q})$ such that $[\sigma(\mathcal{M})] \cap [\mathcal{L}_F \setminus \sigma(\mathcal{M})]$ contains a linear form outside D^* . But since D^* is defined over \mathbb{Q} , we have $\sigma(D^*) = D^*$ and so $[\mathcal{M}] \cap [\mathcal{L}_F \setminus \mathcal{M}]$ contains a linear form not in D^* , say ℓ_0 . Take maximal subsets $\mathcal{M}_1, \mathcal{M}_2$ of \mathcal{M} and $\mathcal{L}_F \setminus \mathcal{M}$, respectively, that are both linearly independent on D . Then there are $\lambda_\ell \in K$ for $\ell \in \mathcal{M}_1 \cup \mathcal{M}_2$ such that

$$\sum_{\ell \in \mathcal{M}_1} \lambda_\ell \ell \equiv \sum_{\ell \in \mathcal{M}_2} \lambda_\ell \ell \equiv \ell_0 \pmod{D^*}.$$

This implies that $\mathcal{M}_1 \cup \mathcal{M}_2$ is linearly dependent on D . We can take a subset of $\mathcal{M}_1 \cup \mathcal{M}_2$ that is minimally linearly dependent on D . This set necessarily must have elements with both \mathcal{M}_1 and \mathcal{M}_2 in common. But then there would be an edge connecting an element of \mathcal{M} with one of $\mathcal{L}_F \setminus \mathcal{M}$, which contradicts that \mathcal{M} is the vertex set of a connected component of \mathcal{G}_D . \square

Lemma 4.2. *Let D be a linear subspace of \mathbb{Q}^m of dimension $d \geq 2$ and \mathcal{M} a non-empty subset of \mathcal{L}_F with $\text{rank}_D \mathcal{M} < d$. Then there is a subset \mathcal{I} of \mathcal{L}_F that is minimally linearly dependent on D , such that $\mathcal{M} \cap \mathcal{I} \neq \emptyset$ and $\text{rank}_D \mathcal{M} \cup \mathcal{I} > \text{rank}_D \mathcal{M}$.*

Proof. Let \mathcal{M}' consist of all linear forms in \mathcal{L}_F that are linear combinations of the linear forms in \mathcal{M} and of the linear forms in D^* . Then $\text{rank}_D \mathcal{M}' = \text{rank}_D \mathcal{M} < d$, hence $\emptyset \subsetneq \mathcal{M}' \subsetneq \mathcal{L}_F$. Take a maximal subset \mathcal{M}_1 of \mathcal{M} that is linearly independent on D ; then it is also a maximal subset of \mathcal{M}' that is linearly independent on D . Let \mathcal{M}_2 be a maximal subset of $\mathcal{L}_F \setminus \mathcal{M}'$ that is linearly independent on D .

By Lemma 4.1 there is a set $\mathcal{J} \subseteq \mathcal{L}_F$ that is minimally linearly dependent on D and contains elements of both \mathcal{M}' and $\mathcal{L}_F \setminus \mathcal{M}'$. This gives a linear combination $\sum_{\ell \in \mathcal{J}} \lambda_\ell \ell \in D^*$, with $\ell_0 := \sum_{\ell \in \mathcal{J} \cap \mathcal{M}'} \lambda_\ell \ell \notin D^*$. Writing the linear forms in $\mathcal{J} \cap \mathcal{M}'$ as linear combinations modulo D^* of the linear forms in \mathcal{M}_1 , and the linear forms in $\mathcal{J} \cap (\mathcal{L}_F \setminus \mathcal{M}')$ as linear combinations modulo D^* of the linear forms in \mathcal{M}_2 , we obtain a relation $\sum_{\ell \in \mathcal{M}_1 \cup \mathcal{M}_2} \mu_\ell \ell \in D^*$, with $\sum_{\ell \in \mathcal{M}_1} \mu_\ell \ell \equiv \ell_0 \not\equiv 0 \pmod{D^*}$. Hence $\mathcal{M}_1 \cup \mathcal{M}_2$ is linearly dependent on D . Take a subset \mathcal{I} of $\mathcal{M}_1 \cup \mathcal{M}_2$ that is minimally linearly dependent on D . We have $\mathcal{I} \cap \mathcal{M}_1 \neq \emptyset$ and $\mathcal{I} \cap \mathcal{M}_2 \neq \emptyset$ since \mathcal{M}_1 and \mathcal{M}_2 are linearly independent on D . This implies $\mathcal{I} \cap \mathcal{M} \neq \emptyset$. Further, $\mathcal{M}_2 \cap \mathcal{M}' = \emptyset$, therefore each of the linear forms in \mathcal{M}_2 is linearly independent on D of the linear forms in \mathcal{M} . Hence $\text{rank}_D \mathcal{M} \cup \mathcal{I} > \text{rank}_D \mathcal{M}$. \square

Denote by M_K the set of places of K . We choose normalized absolute values $|\cdot|_v$ ($v \in M_K$) in such a way that if v lies above $p \in \{\infty\} \cup \{\text{primes}\}$,

then $|x|_v = |x|_p^{|Kv:\mathbb{Q}_p|/|K:\mathbb{Q}|}$ for $x \in \mathbb{Q}$. These absolute values satisfy the product formula $\prod_{v \in M_K} |x|_v = 1$ for $x \in K^*$. For a vector $\mathbf{y} = (y_1, \dots, y_r) \in K^r$, we define

$$\|\mathbf{y}\|_v := \max_{1 \leq i \leq r} |y_i|_v \quad (v \in M_K), \quad H(\mathbf{y}) := \prod_{v \in M_K} \|\mathbf{y}\|_v.$$

By the product formula, $H(\lambda \mathbf{y}) = H(\mathbf{y})$ for $\mathbf{y} \in K^r$, $\lambda \in K^*$.

For $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ and a subset \mathcal{I} of \mathcal{L}_F , we define

$$H_{\mathcal{I}}(\mathbf{x}) := \prod_{v \in M_K} \max_{\ell \in \mathcal{I}} |\ell(\mathbf{x})|_v.$$

Lemma 4.3. *Let $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $\ell(\mathbf{x}) \neq 0$ for $\ell \in \mathcal{L}_F$ and let \mathcal{I}, \mathcal{J} be subsets of \mathcal{L}_F with $\mathcal{I} \cap \mathcal{J} \neq \emptyset$. Then*

$$H_{\mathcal{I} \cup \mathcal{J}}(\mathbf{x}) \leq H_{\mathcal{I}}(\mathbf{x}) \cdot H_{\mathcal{J}}(\mathbf{x}).$$

Proof. Let $\ell_0 \in \mathcal{I} \cap \mathcal{J}$. Then by the product formula,

$$\begin{aligned} H_{\mathcal{I} \cup \mathcal{J}}(\mathbf{x}) &= \prod_{v \in M_K} \max_{\ell \in \mathcal{I} \cup \mathcal{J}} |\ell(\mathbf{x})/\ell_0(\mathbf{x})|_v \\ &\leq \left(\prod_{v \in M_K} \max_{\ell \in \mathcal{I}} |\ell(\mathbf{x})/\ell_0(\mathbf{x})|_v \right) \cdot \left(\prod_{v \in M_K} \max_{\ell \in \mathcal{J}} |\ell(\mathbf{x})/\ell_0(\mathbf{x})|_v \right) \\ &= H_{\mathcal{I}}(\mathbf{x}) \cdot H_{\mathcal{J}}(\mathbf{x}). \end{aligned}$$

□

Lemma 4.4. *Let D be a linear subspace of \mathbb{Q}^m of dimension ≥ 2 on which none of the linear forms in \mathcal{L} vanishes identically. Then for every $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ with $\ell(\mathbf{x}) \neq 0$ for $\ell \in \mathcal{L}_F$, there is a subset \mathcal{I} of \mathcal{L}_F that is minimally linearly dependent on D such that $H_{\mathcal{I}}(\mathbf{x}) \gg_{F,D} \|\mathbf{x}\|^{1/(m-1)}$.*

Proof. Let $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ with $\ell(\mathbf{x}) \neq 0$ for $\ell \in \mathcal{L}_F$. Start with a linear form $\ell_0 \in \mathcal{L}_F$. By Lemma 4.2, there is a subset \mathcal{I}_1 of \mathcal{L}_F that is minimally linearly dependent on D , that contains ℓ_0 and for which $\text{rank}_D \mathcal{I}_1 \geq 2$. Using Lemma 4.2, we choose inductively subsets $\mathcal{I}_2, \mathcal{I}_3, \dots$ of \mathcal{L}_F that are minimally linearly dependent on D as follows: if $\text{rank}_D \mathcal{I}_1 \cup \dots \cup \mathcal{I}_t < \dim D$, we choose \mathcal{I}_{t+1} such that $\mathcal{I}_{t+1} \cap (\mathcal{I}_1 \cup \dots \cup \mathcal{I}_t) \neq \emptyset$ and $\text{rank}_D \mathcal{I}_1 \cup \dots \cup \mathcal{I}_{t+1} > \text{rank}_D \mathcal{I}_1 \cup \dots \cup \mathcal{I}_t$. It is clear that for some $s \leq \dim D - 1 \leq m - 1$ we get $\text{rank}_D \mathcal{I}_1 \cup \dots \cup \mathcal{I}_s = \dim D$. Then X_1, \dots, X_m can be expressed as linear combinations modulo D^* of the linear forms in $\mathcal{I}_1 \cup \dots \cup \mathcal{I}_s$, implying $\|\mathbf{x}\| = H(\mathbf{x}) \ll_{F,D} H_{\mathcal{I}_1 \cup \dots \cup \mathcal{I}_s}(\mathbf{x})$. Now from Lemma 4.3 we infer

$$\|\mathbf{x}\| \ll_{F,D} \prod_{i=1}^s H_{\mathcal{I}_i}(\mathbf{x}) \leq \max_{1 \leq i \leq s} H_{\mathcal{I}_i}(\mathbf{x})^s \leq \max_{1 \leq i \leq s} H_{\mathcal{I}_i}(\mathbf{x})^{m-1}.$$

□

Proof of Theorem 2.7. Without loss of generality, we assume that the linear forms in (2.3) have their coefficients in the ring of integers O_K of K . We prove by induction on d that if D is a linear subspace of \mathbb{Q}^m of dimension d on which none of the linear forms in \mathcal{L} vanishes identically, then (2.7) has only finitely many solutions in $\mathbb{Z}_{\text{prim}}^m \cap D$. For $d = 1$ this is clear.

Assume that $d \geq 2$ and that our assertion holds true for all linear subspaces of \mathbb{Q}^m of dimension smaller than d . Let D be a linear subspace of \mathbb{Q}^m of dimension d on which none of the linear forms in \mathcal{L} vanishes identically and let $0 < \epsilon < \frac{1}{m-1}$. Take $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ satisfying (2.7). Choose a subset \mathcal{I} of \mathcal{L}_F that is minimally linearly dependent on D such that

$$(4.1) \quad H_{\mathcal{I}}(\mathbf{x}) \gg_{F,D} \|\mathbf{x}\|^{1/(m-1)}.$$

Let T be the set of places of K lying above the places in $S \cup \{\infty\}$. For $v \in T$, choose $\ell_v \in \mathcal{I}$ such that $|\ell_v(\mathbf{x})|_v = \max_{\ell \in \mathcal{I}} |\ell(\mathbf{x})|_v$, and let $\mathcal{I}_v := \mathcal{I} \setminus \{\ell_v\}$. We have $|\ell_v(\mathbf{x})|_v \ll_F 1$ for $\ell \in \mathcal{L}_F$, $v \in M_K \setminus T$ since $\mathbf{x} \in \mathbb{Z}^m$. So by the product formula, $\prod_{v \in T} |\ell(\mathbf{x})|_v \gg_F 1$ for $\ell \in \mathcal{L}_F$. Together with (4.1) this implies

$$\begin{aligned} \prod_{v \in T} \prod_{\ell \in \mathcal{I}} |\ell(\mathbf{x})|_v &\ll_F \prod_{v \in T} |F(\mathbf{x})|_v \leq \|\mathbf{x}\|^{(1/(m-1))-\epsilon} \\ &\ll_{F,D} H_{\mathcal{I}}(\mathbf{x})^{1-(m-1)\epsilon}, \end{aligned}$$

and subsequently, dividing both sides by $\prod_{v \in T} |\ell_v(\mathbf{x})|_v$,

$$(4.2) \quad \prod_{v \in T} \prod_{\ell \in \mathcal{I}_v} |\ell(\mathbf{x})|_v \ll_{F,D} H_{\mathcal{I}}(\mathbf{x})^{-(m-1)\epsilon}.$$

Write $\mathcal{I} = \{\ell_0, \dots, \ell_u\}$. Then $\ell_0 \equiv \beta_1 \ell_1 + \dots + \beta_u \ell_u \pmod{D^*}$ with $\beta_i \neq 0$ for $i = 1, \dots, u$. Put $y_i := \ell_i(\mathbf{x})$ for $i = 1, \dots, u$, and $\mathbf{y} = (y_1, \dots, y_u)$. Then $\mathbf{y} \in O_K^u$. We can express $\ell(\mathbf{x})$ ($\ell \in \mathcal{I}_v$) as u linearly independent linear forms in \mathbf{y} , say $\ell_{1,v}(\mathbf{y}), \dots, \ell_{u,v}(\mathbf{y})$, taken from the set $\{Y_1, \dots, Y_u, \beta_1 Y_1 + \dots + \beta_u Y_u\}$. Now (4.2) translates into

$$\prod_{v \in T} \prod_{i=1}^u |\ell_{i,v}(\mathbf{y})|_v \ll_{F,D} H(\mathbf{y})^{-(m-1)\epsilon}, \quad \mathbf{y} \in O_K^u.$$

Thus, we can apply the p -adic Subspace Theorem [22], and conclude that the vectors \mathbf{y} lie in finitely many proper linear subspaces of K^u . It follows that the solutions $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ of (2.7), corresponding to the same sets \mathcal{I}_v ($v \in T$) in (4.2), lie in finitely many proper linear subspaces of D . Since there are only finitely many possibilities for the sets \mathcal{I}_v , it follows

that the solutions $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ altogether lie in only finitely many proper linear subspaces of D . By applying the induction hypothesis to each of these spaces, it follows that (2.7) has only finitely many solutions in $\mathbb{Z}_{\text{prim}}^m \cap D$. This completes our proof. \square

5 Proof of Theorem 2.9

Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form in $m \geq 2$ variables with a factorization as in (2.3), satisfying (2.9) and (2.10). Our first goal is to prove that $c(F) < 1$. We have used some arguments from [16, §3.3]. We start with some preparations.

For a subset \mathcal{M} of \mathcal{L}_F we put $|\mathcal{M}| := \sum_{\ell \in \mathcal{M}} e(\ell)$. Let D be a linear subspace of \mathbb{Q}^m of dimension $d \geq 2$. A subset \mathcal{M} of \mathcal{L}_F is called *D-critical* if $q_D(\mathcal{M})$ is maximal among all non-empty subsets of \mathcal{L}_F . A *D-critical* subset is called *minimal* if none of its proper subsets is *D-critical*.

Lemma 5.1. *Let $\mathcal{M}_1, \mathcal{M}_2$ be two D-critical subsets of \mathcal{L}_F .*

(i) *Assume that $\mathcal{M}_1, \mathcal{M}_2$ are minimal and $\mathcal{M}_1 \neq \mathcal{M}_2$.
Then $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$.*

(ii) *Assume that $\mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset$. Then $\mathcal{M}_1 \cup \mathcal{M}_2$ is D-critical.*

Proof. We use that for any two subsets $\mathcal{N}_1, \mathcal{N}_2$ of \mathcal{L}_F we have

$$(5.1) \quad \begin{cases} \text{rank}_D \mathcal{N}_1 \cap \mathcal{N}_2 + \text{rank}_D \mathcal{N}_1 \cup \mathcal{N}_2 \leq \text{rank}_D \mathcal{N}_1 + \text{rank}_D \mathcal{N}_2, \\ |\mathcal{N}_1 \cap \mathcal{N}_2| + |\mathcal{N}_1 \cup \mathcal{N}_2| = |\mathcal{N}_1| + |\mathcal{N}_2|. \end{cases}$$

(i) Let $q_0 := \max_{\mathcal{M}} q_D(\mathcal{M})$, where the maximum is taken over all non-empty subsets \mathcal{M} of \mathcal{L}_F . Assume $\mathcal{M}_1 \cap \mathcal{M}_2 \neq \emptyset$. Then by (5.1),

$$\begin{aligned} & \text{rank}_D \mathcal{M}_1 \cap \mathcal{M}_2 \\ & \leq \text{rank}_D \mathcal{M}_1 + \text{rank}_D \mathcal{M}_2 - \text{rank}_D \mathcal{M}_1 \cup \mathcal{M}_2 \\ & = q_0^{-1} |\mathcal{M}_1| + q_0^{-1} |\mathcal{M}_2| - q_D(\mathcal{M}_1 \cup \mathcal{M}_2)^{-1} |\mathcal{M}_1 \cup \mathcal{M}_2| \\ & \leq q_0^{-1} (|\mathcal{M}_1| + |\mathcal{M}_2| - |\mathcal{M}_1 \cup \mathcal{M}_2|) = q_0^{-1} |\mathcal{M}_1 \cap \mathcal{M}_2|, \end{aligned}$$

implying $q_D(\mathcal{M}_1 \cap \mathcal{M}_2) \geq q_0$. This is clearly impossible.

(ii) Again by (5.1),

$$\begin{aligned} \text{rank}_D \mathcal{M}_1 \cup \mathcal{M}_2 & \leq \text{rank}_D \mathcal{M}_1 + \text{rank}_D \mathcal{M}_2 \\ & = q_0^{-1} (|\mathcal{M}_1| + |\mathcal{M}_2|) = q_0^{-1} |\mathcal{M}_1 \cup \mathcal{M}_2|, \end{aligned}$$

which implies $q_D(\mathcal{M}_1 \cup \mathcal{M}_2) \geq q_0$. Hence $\mathcal{M}_1 \cup \mathcal{M}_2$ is *D-critical*. \square

Lemma 5.2. *We have $c(F) < 1$.*

Proof. We have to prove that for every \mathbb{Q} -linear subspace D of \mathbb{Q}^m of dimension ≥ 2 we have $q_D(F) < q_D(\mathcal{L}_F) = n/d$, where $d = \dim D$ and $n = \deg F$. Assume that for some of these subspaces D we have $q_D(F) \geq n/d$, i.e., there is a subset \mathcal{M}_1 of \mathcal{L}_F with $\text{rank}_D \mathcal{M}_1 < d$ and $q_D(\mathcal{M}_1) \geq n/d$. Without loss of generality, we take for \mathcal{M}_1 a minimal D -critical subset of \mathcal{L}_F .

Let D^* be the K -vector space of linear forms in $K[X_1, \dots, X_m]$ that vanish identically on D . Then for each $\sigma \in \text{Gal}(K/\mathbb{Q})$, $\sigma(\mathcal{M}_1)$ is also a minimal D -critical set since $\text{rank}_D \sigma(\mathcal{M}_1) = \text{rank}_D \mathcal{M}_1$ and $|\sigma(\mathcal{M}_1)| = |\mathcal{M}_1|$, and so by Lemma 5.1, we have either $\sigma(\mathcal{M}_1) = \mathcal{M}_1$ or $\sigma(\mathcal{M}_1) \cap \mathcal{M}_1 = \emptyset$. That is, \mathcal{M}_1 is $\text{Gal}(K/\mathbb{Q})$ -proper. Let $\mathcal{M}_1, \dots, \mathcal{M}_t$ be the distinct sets among the $\sigma(\mathcal{M}_1)$, $\sigma \in \text{Gal}(K/\mathbb{Q})$. We first prove that

$$(5.2) \quad \mathcal{L}_F = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \dots \cup \mathcal{M}_t.$$

Suppose the contrary, i.e., $\mathcal{M}_0 := \mathcal{M}_1 \cup \dots \cup \mathcal{M}_t \subsetneq \mathcal{L}_F$. By Lemma 5.1, the set \mathcal{M}_0 is D -critical, hence $q_D(\mathcal{M}_0) \geq d/n$, which implies $\text{rank}_D \mathcal{M}_0 < d$. This, together with the fact that \mathcal{M}_0 is $\text{Gal}(K/\mathbb{Q})$ -symmetric, implies that there is a non-zero $\mathbf{x} \in D$ with $\ell(\mathbf{x}) = 0$ for $\ell \in \mathcal{M}_0$. This clearly contradicts (2.10). So indeed, (5.2) holds. By Lemma 5.1 (ii), any non-empty union \mathcal{M} of some of the sets \mathcal{M}_i is D -critical, implying $q_D(\mathcal{M}) = q_D(\mathcal{L}_F) = d/n$.

As observed above, the set \mathcal{M}_1 is $\text{Gal}(K/\mathbb{Q})$ -proper. So by assumption (2.9), the K -vector space

$$\sum_{\sigma \in \text{Gal}(K/\mathbb{Q})} [\sigma(\mathcal{M}_1)] \cap [\mathcal{L}_F \setminus \sigma(\mathcal{M}_1)] = \sum_{i=1}^t [\mathcal{M}_i] \cap [\mathcal{L}_F \setminus \mathcal{M}_i]$$

contains a linear form from \mathcal{L}_F . By assumption (2.10), this form does not lie in D^* . Hence there is $i \in \{1, \dots, t\}$ such that $[\mathcal{M}_i] \cap [\mathcal{L}_F \setminus \mathcal{M}_i]$ contains a linear form not in D^* . Moreover, $q_D(\mathcal{M}_i) = q_D(\mathcal{L}_F \setminus \mathcal{M}_i) = d/n$. Hence

$$\begin{aligned} d = \text{rank}_D \mathcal{L}_F &< \text{rank}_D \mathcal{M}_i + \text{rank}_D (\mathcal{L}_F \setminus \mathcal{M}_i) \\ &= \frac{d}{n} (|\mathcal{M}_i| + |\mathcal{L}_F \setminus \mathcal{M}_i|) = d, \end{aligned}$$

which is impossible. Thus, our assumption that $q_D(F) \geq n/d$ is false. \square

We need a few other, much deeper auxiliary results, which are taken from the literature. We keep the notation and assumptions from Theorem 2.9. For each $p \in S \cup \{\infty\}$, we choose an extension of $|\cdot|_p$ to the splitting field K of F .

Lemma 5.3. *Let D be a linear subspace of \mathbb{Q}^m of dimension $d \geq 2$. Then for every $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$, there are subsets \mathcal{L}_p ($p \in S \cup \{\infty\}$) of \mathcal{L}_F of cardinality d that are linearly independent on D , such that*

$$(5.3) \quad \prod_{p \in S \cup \{\infty\}} \prod_{\ell \in \mathcal{L}_p} |\ell(\mathbf{x})|_p \ll_{F,S,D} \left(\prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p \cdot \|\mathbf{x}\|^{-(n-dq_D(F))} \right)^{1/q_D(F)}.$$

Proof. In the case $D = \mathbb{Q}^m$ this is a special case of [16, Lemma 2.2.4]. The case of arbitrary D can be reduced to this by choosing a \mathbb{Z} -basis $\{\mathbf{a}_1, \dots, \mathbf{a}_d\}$ of $\mathbb{Z}^m \cap D$ and working with the decomposable form $F(\varphi(\mathbf{y}))$, where $\varphi(\mathbf{y}) = \sum_{i=1}^d y_i \mathbf{a}_i$. Note that φ establishes a bijection between $\mathbb{Z}_{\text{prim}}^d$ and $\mathbb{Z}_{\text{prim}}^m \cap D$. \square

Lemma 5.4 (*p*-adic Minkowski). *Let p be a prime number. Further, let ℓ_1, \dots, ℓ_m be linearly independent linear forms in m variables with real coefficients and $\ell_{1,p}, \dots, \ell_{m',p}$ ($m' \leq m$) linearly independent linear forms in m variables with coefficients in \mathbb{Q}_p . Then there are constants $\gamma_1, \gamma_2 > 1$, depending only on $p, m, \ell_1, \dots, \ell_m, \ell_{1,p}, \dots, \ell_{m',p}$, such that if $A_1, \dots, A_m, B_1, \dots, B_{m'}$ are any positive reals with*

$$(5.4) \quad A_1 \cdots A_m B_1 \cdots B_{m'} \geq \gamma_1, \quad B_i \leq \gamma_2^{-1} \text{ for } i = 1, \dots, m'$$

then there is a non-zero $\mathbf{x} \in \mathbb{Z}^m$ with

$$(5.5) \quad |\ell_i(\mathbf{x})| \leq A_i \text{ for } i = 1, \dots, m, \quad |\ell_{i,p}(\mathbf{x})|_p \leq B_i \text{ for } i = 1, \dots, m'.$$

Proof. We augment $\ell_{1,p}, \dots, \ell_{m',p}$ to a linearly independent set of m linear forms $\ell_{1,p}, \dots, \ell_{m,p}$ with coefficients in \mathbb{Q}_p . Let \mathcal{C} be the symmetric convex body consisting of those $\mathbf{x} \in \mathbb{R}^m$ with

$$|\ell_i(\mathbf{x})| \leq A_i \text{ for } i = 1, \dots, m$$

and Λ the lattice consisting of those $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Q}^m$ such that

$$\begin{aligned} |\ell_{i,p}(\mathbf{x})|_p &\leq B_i \text{ for } i = 1, \dots, m', \\ |\ell_{i,p}(\mathbf{x})|_p &\leq \gamma_2^{-1} \text{ for } i = m' + 1, \dots, m, \\ |x_i|_q &\leq 1 \text{ for } i = 1, \dots, m \text{ and all primes } q \neq p, \end{aligned}$$

with γ_2 yet to be chosen. By choosing γ_2 sufficiently large, we can guarantee that $\Lambda \subseteq \mathbb{Z}^m$ for all $B_1, \dots, B_{m'} \leq \gamma_2^{-1}$ and by choosing γ_1 sufficiently large, we can guarantee that $\text{vol}(\mathcal{C})/\det \Lambda \geq 2^m$ for all A_1, \dots, A_m with $A_1 \cdots A_m B_1 \cdots B_{m'} \geq \gamma_1$. Minkowski's Theorem implies that for such A_i, B_i there is a non-zero $\mathbf{x} \in \mathcal{C} \cap \Lambda$. This \mathbf{x} satisfies (5.5) and lies in \mathbb{Z}^m . \square

Proposition 5.5. *Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form of degree n with (2.9) and (2.10). Then the number of $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ such that $\prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p \leq M$ is $\ll_{n,S} M^{m/n}$ as $M \rightarrow \infty$.*

Proof. Liu proved this in his thesis for all decomposable forms F with $c(F) < 1$ and with (2.10), see [16, Theorem 2.1.3]. As observed in Lemma 5.2, the condition $c(F) < 1$ follows from (2.9) and (2.10). Liu's theorem and its proof are a p -adic generalization of Thunder's theorem [27, Theorem 2] and its proof. \square

Proof of Theorem 2.9. (i). Let $0 < \epsilon < 1 - c(F)$. We prove by induction on d that if D is any d -dimensional \mathbb{Q} -linear subspace of \mathbb{Q}^m , then $[F(\mathbf{x})]_S \ll_{F,S,D} |F(\mathbf{x})|^{c(F)+\epsilon}$ for all $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$. For $d = 1$ this is clear. Let $d \geq 2$, and assume the assertion is true for all linear subspaces of \mathbb{Q}^m of dimension $< d$. Let D be a \mathbb{Q} -linear subspace of \mathbb{Q}^m of dimension d . Take $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ for which

$$(5.6) \quad [F(\mathbf{x})]_S \geq |F(\mathbf{x})|^{c(F)+\epsilon}.$$

Then

$$\prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p = \frac{|F(\mathbf{x})|}{[F(\mathbf{x})]_S} \leq |F(\mathbf{x})|^{1-c(F)-\epsilon} \ll_{F,S,D} \|\mathbf{x}\|^{n(1-c(F)-\epsilon)}.$$

Take subsets \mathcal{L}_p ($p \in S \cup \{\infty\}$) of \mathcal{L}_F as in Lemma 5.4 and insert the above inequality into (5.3). Then since $c(F) \geq q_D(F) \cdot d/n$,

$$\begin{aligned} \prod_{p \in S \cup \{\infty\}} \prod_{\ell \in \mathcal{L}_p} |\ell(\mathbf{x})|_p &\ll_{F,S,D} \left(\|\mathbf{x}\|^{n(1-c(F)-\epsilon)} \cdot \|\mathbf{x}\|^{-(n-dq_D(F))} \right)^{1/q_D(F)} \\ &\ll_{F,S,D} \|\mathbf{x}\|^{-n\epsilon/q_D(F)}. \end{aligned}$$

By the p -adic Subspace Theorem, the points $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ with (5.6) lie in finitely many proper linear subspaces of D . By applying the induction hypothesis with each of these subspaces, we infer that for the points $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m \cap D$ with (5.6) we have $[F(\mathbf{x})]_S \ll_{F,S,D} |F(\mathbf{x})|^{c(F)+\epsilon}$. This completes our induction step, and hence the proof of (i).

(ii). Let $K = \mathbb{Q}(\theta)$. By Chebotarev's Density Theorem there are infinitely many primes p such that the minimal polynomial of θ over \mathbb{Q} has all its roots in \mathbb{Q}_p . Take such a prime p . Then in the factorization (2.3) we may assume that the linear forms in \mathcal{L}_F have their coefficients in \mathbb{Q}_p . Let D be a linear subspace of \mathbb{Q}^m of dimension $d \geq 2$, and \mathcal{M} a subset of \mathcal{L}_F with $\text{rank}_D \mathcal{M} =: d' < d$ for which $q_D(\mathcal{M}) \cdot d/n = c(F)$. Choose a subset \mathcal{M}'

of \mathcal{M} of cardinality d' that is linearly independent over D . By Lemma 5.4 there is for every sufficiently large Q a non-zero point $\mathbf{x} \in \mathbb{Z}^m \cap D$ such that

$$\|\mathbf{x}\| \ll Q, \quad |\ell(\mathbf{x})|_p \ll Q^{-d/d'} \quad \text{for } \ell \in \mathcal{M}',$$

where here and below, the constants implied by \ll depend on F, D and p and in fact only on F and p since D depends on F . Without loss of generality, we may assume that the greatest common divisor of the coordinates of \mathbf{x} does not contain factors coprime with p . Let p^k be the greatest common divisor of the coordinates of \mathbf{x} and put $\mathbf{x}' := p^{-k}\mathbf{x}$. Then $\mathbf{x}' \in \mathbb{Z}_{\text{prim}}^m \cap D$, $p^k \ll Q$, and

$$(5.7) \quad \begin{aligned} \|\mathbf{x}'\| &\ll p^{-k}Q, \\ |\ell(\mathbf{x}')|_p &\ll p^k Q^{-d/d'} = (p^{-k}Q)^{-d/d'} (p^k)^{1-(d/d')} \quad \text{for } \ell \in \mathcal{M}'. \end{aligned}$$

Now if we let $Q \rightarrow \infty$, then \mathbf{x}' runs through an infinite set. Indeed, otherwise there were a non-zero $\mathbf{x}' \in \mathbb{Z}_{\text{prim}}^m \cap D$ such that (5.7) holds for arbitrarily large Q . But by letting $Q \rightarrow \infty$, we can make $\max(p^{-k}Q, p^k)$ arbitrarily large and thus $|\ell(\mathbf{x}')|_p$ arbitrarily small for every $\ell \in \mathcal{M}'$. But then it would follow that $\ell(\mathbf{x}') = 0$ for $\ell \in \mathcal{M}'$, which is however excluded by assumption (2.10).

From the above we conclude that there are infinitely many $\mathbf{x}' \in \mathbb{Z}_{\text{prim}}^m \cap D$ such that

$$|\ell(\mathbf{x}')|_p \ll \|\mathbf{x}'\|^{-d/d'} \quad \text{for } \ell \in \mathcal{M}'.$$

Since the other linear forms in \mathcal{M} are linear combinations modulo D^* of the linear forms in \mathcal{M}' , these \mathbf{x}' satisfy

$$|\ell(\mathbf{x}')|_p \ll \|\mathbf{x}'\|^{-d/d'} \quad \text{for } \ell \in \mathcal{M},$$

and moreover, trivially, $|\ell(\mathbf{x}')|_p \ll 1$ for $\ell \in \mathcal{L}_F \setminus \mathcal{M}$. Using the decomposition (2.3), it follows that these \mathbf{x}' satisfy

$$|F(\mathbf{x}')|_p \ll \|\mathbf{x}'\|^{-(d/d')|\mathcal{M}|} = \|\mathbf{x}'\|^{-dq_D(\mathcal{M})} = \|\mathbf{x}'\|^{-nc(F)},$$

hence

$$[F(\mathbf{x}')]_{\{p\}} = |F(\mathbf{x}')|_p^{-1} \gg |F(\mathbf{x}')|^{c(F)}.$$

This proves (ii).

(iii) Let $0 < \epsilon < 1$ and $B > 1$. Then every $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $[F(\mathbf{x})]_S \geq |F(\mathbf{x})|^\epsilon$ and $\|\mathbf{x}\| \leq B$ satisfies

$$\prod_{p \in S \cup \{\infty\}} |F(\mathbf{x})|_p = \frac{|F(\mathbf{x})|}{[F(\mathbf{x})]_S} \leq |F(\mathbf{x})|^{1-\epsilon} \ll_{F,\epsilon} B^{n(1-\epsilon)},$$

where $n := \deg F$. Hence $N(F, S, \epsilon, B)$ is at most the number of solutions in $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ of this last inequality. Now Proposition 5.5 implies

$$N(F, S, \epsilon, B) \ll_{F, S, \epsilon} (B^{n(1-\epsilon)})^{m/n} \ll_{F, S, \epsilon} B^{m(1-\epsilon)}$$

as $B \rightarrow \infty$. This proves (iii). \square

6 Proof of Theorem 2.10

Theorem 2.10 will be deduced from Proposition 6.1 below, which is a special case of a non-explicit version of Theorem 3 of Gyóry and Yu [15]. Its proof is based on effective results of Gyóry and Yu [15] for unit equations, and ultimately depends on Baker's method, more precisely on explicit estimates of Matveev [19] concerning linear forms in complex logarithms of algebraic numbers and similar such estimates by Yu [28] for p -adic logarithms.

Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form, $S = \{p_1, \dots, p_s\}$ a finite non-empty set of primes, and b a non-zero integer. Let $\mathbb{Z}_S := \mathbb{Z}[(p_1 \cdots p_s)^{-1}]$ be the ring of S -integers in \mathbb{Q} , and consider the equation

$$(6.1) \quad F(\mathbf{x}) = b \quad \text{in } \mathbf{x} \in \mathbb{Z}_S^m.$$

Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ be the prime ideals in K that divide p_1, \dots, p_s , and let $P' = \max_{1 \leq i \leq t} N(\mathfrak{p}_i)$, where $N(\mathfrak{a}) := \#O_K/\mathfrak{a}$ denotes the absolute norm of a non-zero ideal \mathfrak{a} of O_K . Further, denote by h the absolute logarithmic height.

Proposition 6.1. *Let F be a decomposable form as above with properties (2.12) and (2.13). With the above notation, every solution $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_S^m$ of (6.1) with $x_m \neq 0$ if $k > 1$ satisfies*

$$(6.2) \quad \max_{1 \leq j \leq m} h(x_j) < c_4^t (P'/\log P') \prod_{i=1}^t \log N(\mathfrak{p}_i) \cdot (c_5 + \log N(\mathfrak{p}_1 \cdots \mathfrak{p}_t) + h(b)),$$

where c_4, c_5 are effectively computable positive numbers that depend only on F .

We mention that Theorem 3 of [15] implies Proposition 6.1 with explicit expressions for c_4, c_5 in terms of the heights of the coefficients of F and the degree and regulator of the splitting field K of F .

We now prove Theorem 2.10 by means of Proposition 6.1.

Proof of Theorem 2.10. Let $\mathbf{x} \in \mathbb{Z}_{\text{prim}}^m$ with $F(\mathbf{x}) \neq 0$, and define $b := F(\mathbf{x})/[F(\mathbf{x})]_S$. Then $F(\mathbf{x}) = p_1^{a_1} \cdots p_s^{a_s} b$ for certain non-negative integers a_1, \dots, a_s . We can write $a_i = na'_i + a''_i$ with $a'_i, a''_i \in \mathbb{Z}_{\geq 0}$ such that $a''_i < n$ for each i . Then (2.4) implies that

$$(6.3) \quad F(\mathbf{x}') = b',$$

where

$$(6.4) \quad \mathbf{x}' = \mathbf{x}/p_1^{a'_1} \cdots p_s^{a'_s} \quad \text{and} \quad b' = bp_1^{a''_1} \cdots p_s^{a''_s}.$$

Here $\mathbf{x}' = (x'_1, \dots, x'_m)$ is a solution of (6.3) in \mathbb{Z}_S^m .

We apply now Proposition 6.1 to the equation (6.3). Let $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ and P' as above. Then we get

$$\max_{1 \leq j \leq m} h(x'_j) < C_1,$$

for every solution $\mathbf{x} = (x'_1, \dots, x'_m) \in \mathbb{Z}_S^m$ of (6.3) with $x'_m \neq 0$ if $k > 1$, where C_1 denotes the upper bound occurring in (6.2) but with b replaced by b' .

Since $t \leq sd$, $P' \leq P^d$ where $d = [K : \mathbb{Q}]$, and $h(b') \leq ns \log P + \log |b|$, we infer that

$$(6.5) \quad \max_{1 \leq j \leq m} h(x'_j) < C_2(c_6 + \log |b|),$$

where $C_2 = c_7^s (P(\log p_1) \cdots (\log p_s))^d$ and c_6, c_7 are effectively computable positive numbers that depend only on F . It is easy to deduce from (6.5) and (6.4) that

$$p_1^{a'_1} \cdots p_s^{a'_s} \leq C_3 |b|^{mC_2},$$

where $C_3 = e^{mC_2c_6}$. This gives

$$p_1^{a_1} \cdots p_s^{a_s} < (p_1 \cdots p_s)^n (p_1^{a'_1} \cdots p_s^{a'_s})^n \leq C_4 |b|^{mnC_2}$$

with $C_4 = P^{sn} C_3^n$. Multiplying both sides by $(p_1^{a_1} \cdots p_s^{a_s})^{mnC_2}$ and then raising to the power $1/(mnC_2 + 1)$, we infer that

$$[F(\mathbf{x})]_S \leq (P^s C_3)^{\frac{1}{mC_2}} |F(\mathbf{x})|^{1 - \frac{1}{mnC_2 + 1}}.$$

But $(P^s C_3)^{\frac{1}{mC_2}} \leq \kappa_6$, while $mnC_2 + 1 \leq c_3^s (P(\log p_1) \cdots (\log p_s))^d$ with effectively computable κ_6, c_3 depending only on F . This gives (2.14). \square

7 Lower bound for the greatest prime factors of decomposable forms at integral points

We now deduce over \mathbb{Z} an improved and more explicit version of Corollary 5 of Győry and Yu [15] on the greatest prime factors of decomposable forms at integral points. We note that in Győry and Yu [15] it was more complicated to deduce Corollary 5 from Theorem 3 of that paper. The next corollary gives some useful information about those non-zero integers that can be represented by decomposable forms of the above type.

For a positive integer a we denote by $P(a)$ and $\omega(a)$ the greatest prime factor and the number of distinct prime factors of a with the convention that $P(1) = 1$, $\omega(1) = 0$. Further, we denote by \log_i the i -th iterated logarithm.

Corollary 7.1. *Let $F(X_1, \dots, X_m) \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form as in Theorem 2.10, and let h_0 be a non-zero integer that can be represented by $F(\mathbf{x})$ with some $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_{\text{prim}}^m$ with $x_m \neq 0$ if $k > 1$. Then*

$$(7.1) \quad (P(\log P)^{2\omega})^d > \log |h_0|$$

and

$$(7.2) \quad P > \begin{cases} (\log |h_0|)^{1/3d} & \text{if } \omega \leq \log P / \log_2 P, \\ C_5 \log_2 |h_0| \cdot \log_3 |h_0| / \log_4 |h_0| & \text{otherwise,} \end{cases}$$

provided that $|h_0| \geq C_6$, where $P = P(h_0)$, $\omega = \omega(h_0)$. Here C_5 , C_6 are effectively computable positive numbers that depend only on F .

Proof. Let h_0 be a non-zero integer such that $h_0 = F(\mathbf{x})$ for some $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_{\text{prim}}^m$ with $x_m \neq 0$ if $k > 1$. Write

$$F(\mathbf{x}) = p_1^{a_1} \cdots p_s^{a_s}$$

with distinct primes p_1, \dots, p_s . Then $P = P(h_0) = \max_{1 \leq i \leq s} p_i$ and $\omega = \omega(h_0) = s$. Put $S := \{p_1, \dots, p_s\}$. In this case $[F(\mathbf{x})]_S = |F(\mathbf{x})|$. Now (2.14) immediately gives

$$|h_0| \leq \kappa_6 |h_0|^{1-\kappa_5}$$

with κ_5 , κ_6 specified in Theorem 2.10. This implies that

$$|h_0| \leq \kappa_6^{1/\kappa_5},$$

whence

$$\log |h_0| \leq c_8^s (2P(\log P)^s)^d$$

with an effectively computable positive c_8 that depends only on F .

We know from prime number theory that $s < \frac{2P}{\log P}$. Hence, if $|h_0| \geq C_7$ with a large and effectively computable $C_7 = C_7(F) > 0$, then P must be also large and so $(c_8(\log P)^d)^s \leq (\log P)^{2ds}$ and (7.1) follows.

If $s \leq \frac{\log P}{\log_2 P}$ then it follows from (7.1) that

$$\log_2 |h_0| < d \log P + 2ds \log_2 P \leq 3d \log P,$$

which gives the first inequality in (7.2), provided that C_7 is sufficiently large. Otherwise, we deduce from (7.1) that

$$\log_2 |h_0| < d \log P + 4d \frac{P}{\log P} \log_2 P,$$

which gives the second inequality in (7.2), provided that C_7 is sufficiently large. \square

8 Applications to discriminants of algebraic integers

As was mentioned above, Theorem 2.10 and its corollaries can be applied to discriminant forms, index forms and a large class of norm forms. We now present some applications to discriminants of algebraic integers. Similar consequences can be obtained for indices of algebraic integers.

Let L be a number field of degree $n \geq 3$ with ring of integers O_L , and suppose that K is the normal closure of L over \mathbb{Q} . Further, let $S = \{p_1, \dots, p_s\}$ be a finite, non-empty set of primes. We define the discriminant of an algebraic integer to be the discriminant of its monic minimal polynomial over \mathbb{Z} . Consider the *discriminant equation*

$$(8.1) \quad D_{L/\mathbb{Q}}(\alpha) = p_1^{a_1} \cdots p_s^{a_s} \cdot b \text{ in } \alpha \in O_L, \quad a_1, \dots, a_s \in \mathbb{Z}_{\geq 0},$$

where b is an S -free integer, i.e., coprime with p_1, \dots, p_s . Clearly, α and $\alpha + a$ with $a \in \mathbb{Z}$ have the same discriminant. Such elements of O_L are called *equivalent*. Denote by \mathcal{S} the set of positive integers composed of primes from S . We claim that any solution of (8.1) can be derived from one which is not equivalent to any element of O_L that is divisible in O_L by any $\eta > 1$ from \mathcal{S} . Indeed, if α satisfies (8.1) then, by Theorem 3 of Győry [13], α can be written in the form

$$\alpha = \eta\alpha' + a$$

with some $a \in \mathbb{Z}$, $\eta \in \mathcal{S}$ and $\alpha' \in O_L$. This representation is not necessarily unique. For fixed α , choose η , α' , a such that η is maximal. Since $D_{L/\mathbb{Q}}(\alpha) = \eta^{n(n-1)} D_{L/\mathbb{Q}}(\alpha')$, α' is also a solution of (8.1) with other a_1, \dots, a_s . Further, by the choice of η , the number α' cannot be equivalent to any $\eta'\alpha''$ in O_L with $\alpha'' \in O_L$ and $\eta' \in \mathcal{S}$ with $\eta' > 1$, since otherwise α would be equivalent to $\eta\eta'\alpha''$ with $\eta\eta' > \eta$. This proves our claim.

Note that in the representation (8.1), the S -part of the discriminant of α is

$$[D_{L/\mathbb{Q}}(\alpha)]_S = p_1^{a_1} \cdots p_s^{a_s}.$$

As a consequence of Theorem 2.10, we want to estimate $[D_{L/\mathbb{Q}}(\alpha)]_S$ from above in terms of $|D_{L/\mathbb{Q}}(\alpha)|^{1-\kappa_7}$ for some constant $\kappa_7 > 0$. In view of the above we require that α not be equivalent to any element of the form $\eta\alpha'$ where $\alpha' \in O_L$ and η is an integer from \mathcal{S} with $\eta > 1$.

Corollary 8.1. *Assume that α in (8.1) is not equivalent to any element of O_L that is divisible in O_L by an $\eta \in \mathcal{S}$ greater than 1. Then*

$$(8.2) \quad [D_{L/\mathbb{Q}}(\alpha)]_S \leq \kappa_8 \cdot |D_{L/\mathbb{Q}}(\alpha)|^{1-\kappa_7},$$

where

$$\kappa_7 = (c_9^s ((P(\log p_1) \cdots (\log p_s))^d)^{-1} \geq (c_9^s (2P(\log P)^s)^d)^{-1},$$

and κ_8, c_9 are effectively computable positive numbers depending only on L .

Proof. If L is effectively given in the sense of e.g., Evertse and Gyóry [9, §3.7], an integral basis of O_L of the form $\{1, \omega_2, \dots, \omega_n\}$ can be effectively determined. Then we can write $\alpha = a + x_2\omega_2 + \cdots + x_n\omega_n$ with appropriate integers a, x_2, \dots, x_n . Using the fact that $D_{L/\mathbb{Q}}(\alpha) = D_{L/\mathbb{Q}}(\alpha - a)$ we get

$$D_{L/\mathbb{Q}}(\alpha) = D_{L/\mathbb{Q}}(x_2\omega_2 + \cdots + x_n\omega_n).$$

Our assumption on α implies that p_1, \dots, p_s do not divide $\gcd(x_2, \dots, x_n)$. Moreover, we may assume without loss of generality that $\gcd(x_2, \dots, x_n) = 1$. The discriminant form $D_{L/\mathbb{Q}}(\omega_2 X_2 + \cdots + \omega_n X_n)$ satisfies (2.12) and (2.13) with $k = 1$, see e.g. Gyóry and Yu [15], so we can apply Theorem 2.10 with this discriminant form. By observing that the dependence of the constants in Theorem 2.10 can be replaced by a dependence on L , Corollary 8.1 follows. \square

Corollary 8.1 has similar consequences as Theorem 2.10 for arithmetical properties of non-zero integers D_0 that are discriminants of some $\alpha \in O_L$,

but are not the discriminants of any $k\beta$ with $\beta \in O_L$ and rational integer $k > 1$. Then it follows from Theorem 2.10 that

$$(P(\log P)^{2\omega})^d > |D_0|$$

provided that $|D_0| \geq C(L)$, where $P = P(D_0)$, $\omega = \omega(D_0)$ and $C(L)$ is effectively computable in terms of L . We can get also inequalities similar to (7.2). We note that more general but weaker results of this type can be found in Győry [13] and Evertse and Győry [9].

9 Additional comments

Let $f(X)$ be an integer polynomial with at least two distinct roots, and $S = \{p_1, \dots, p_s\}$ a finite set of primes. According to the result of Gross and Vincent [10] quoted as Theorem A in the Introduction, we have

$$[f(x)]_S \leq \kappa_2 |f(x)|^{1-\kappa_1} \quad \text{for every } x \in \mathbb{Z} \text{ with } f(x) \neq 0,$$

where κ_1, κ_2 are positive numbers, effectively computable in terms of f and S . As mentioned in Theorem 2.2, in this estimate we can take

$$\kappa_1 = (c_1^s (((\max_i p_i)(\log p_1) \cdots (\log p_s))^d)^{-1},$$

where d is the degree of the splitting field of f and c_1 depends only on f . The factor $\max_i p_i$ comes from the use of linear forms in p -adic logarithms in our argument. If we follow instead the proof of [10], by applying a result of Matveev [19] replacing the older and less sharp estimate for linear forms in logarithms due to Alan Baker that was used by Gross and Vincent, we would have obtained an estimate of the above type with

$$\kappa_1 = (c_2^s ((\log p_1) \cdots (\log p_s))^{c_3})^{-1},$$

where c_2, c_3 (as well as the other constants c_4, c_5, \dots, c_8 below) are effectively computable in terms of f . Taking for p_1, \dots, p_s the first s prime numbers, an easy computation using the Prime Number Theorem shows that, for every positive ϵ , we have

$$P(f(x)) \geq (1 - \epsilon) \log_2 x \cdot \log_3 x / \log_4 x,$$

for $x \in \mathbb{Z}$ with $f(x) \neq 0$ and $|x|$ sufficiently large in terms of ϵ .

For a positive integer a we denote by $Q(a)$ its greatest square-free factor. Let again x be an integer with $f(x) \neq 0$ and p_1, \dots, p_s the prime divisors

of $f(x)$. Proceeding as in [10], but applying a result of Matveev [19] instead of one of Baker, we get

$$\log |x| \leq c_4^s ((\log p_1) \cdots (\log p_s))^{c_5}.$$

Using the arithmetico-geometric inequality as in Stewart's paper [26], we deduce that

$$\frac{\log \log |x|}{s} \leq c_6 \left(1 + \log \left(\frac{\log Q(f(x))}{s} \right) + \frac{\log_3 Q(f(x))}{s} \right).$$

We then conclude that

$$\log Q(f(x)) \geq c_7 \log_2 |x| \cdot \log_3 |x| / \log_4 |x|.$$

With the approach followed in the present paper, we would only get that

$$\log Q(f(x)) \geq c_8 \log_2 |x|,$$

a result that was already known.

Let $F \in \mathbb{Z}[X_1, \dots, X_m]$ be a decomposable form as in Theorem 2.10, and let h_0 be a non-zero integer that can be represented by $F(\mathbf{x})$ with some $\mathbf{x} = (x_1, \dots, x_m) \in \mathbb{Z}_{\text{prim}}^m$ with $x_m \neq 0$ if $k > 1$. We are not able to prove the existence of effectively computable positive numbers c_9, c_{10} , which depend only on F , such that

$$\log Q(h_0) > c_9 \log_2 |h_0| \cdot \log_3 |h_0| / \log_4 |h_0|,$$

provided that $|h_0| > c_{10}$.

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