Uniform Diophantine approximation related to b-ary and β-expansions

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Abstract. *Let $b \geq 2$ be an integer and $\hat{v}$ a real number. Among other results, we compute the Hausdorff dimension of the set of real numbers ξ with the property that, for every sufficiently large integer N, there exists an integer n such that $1 \leq n \leq N$ and the distance between $b^n \xi$ and its nearest integer is at most equal to $b^{-\hat{v}N}$. We further solve the same question when replacing $b^n \xi$ by $T_\beta^n \xi$, where T_β denotes the classical β-transformation.*

1. Introduction and results

Throughout this text, $\| \cdot \|$ stands for the distance to the nearest integer. Let ξ be an irrational real number. The well-known Dirichlet Theorem asserts that for *every* real number $X \geq 1$, there exists an integer x with $1 \leq x \leq X$ and

$$\|x\xi\| < X^{-1}. \tag{1.1}$$

This is a uniform statement in the sense that (1.1) has a solution for every sufficiently large real number X (as opposed to 'for arbitrarily large real numbers X'). Following the notation introduced in [5], we denote by $\hat{w}_1(\xi)$ the supremum of the real numbers w such that, for any sufficiently large real number X, the inequality

$$\|x\xi\| < X^{-w}$$

has an integer solution x with $1 \leq x \leq X$. The Dirichlet Theorem implies that $\hat{w}_1(\xi) \geq 1$. In 1926, Khintchine [9] established that, in fact, $\hat{w}_1(\xi) = 1$ always holds. To see this, let $(p_\ell/q_\ell)_{\ell \geq 1}$ denote the sequence of convergents to ξ. If $\hat{w}_1(\xi) > 1$, then there exists a positive real number ε such that, for every sufficiently large integer ℓ, the inequality $\|q\xi\| < (q_\ell - 1)^{-1-\varepsilon}$ has an integer solution q with $1 \leq q < q_\ell$. However, it follows from the theory of continued fractions that $\|q\xi\| \geq \|q_{\ell-1}\xi\| \geq 1/(2q_\ell)$. This gives $q_\ell^{\varepsilon/2} < 2$, thus we have reached a contradiction. Consequently, the set of values taken by the exponent of Diophantine approximation $\hat{w}_1$ is easy to determine.

In the present paper, we first restrict our attention to approximation by rational numbers whose denominator is a power of some given integer $b \geq 2$ and we consider the following exponents of approximation.

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Definition 1.1. *Let ξ be an irrational real number. Let b be an integer with $b \geq 2$. We denote by $v_b(\xi)$ the supremum of the real numbers v for which the equation*

$$\|b^n \xi\| < (b^n)^{-v}$$

has infinitely many solutions in positive integers n. We denote by $\hat{v}_b(\xi)$ the supremum of the real numbers $\hat{v}$ for which, for every sufficiently large integer N, the equation

$$\|b^n \xi\| < (b^N)^{-\hat{v}}$$

has a solution n with $1 \leq n \leq N$.

The exponents v_b have already been introduced in [1]; see also Chapter 7 of [4]. Roughly speaking, the quantity $v_b(\xi)$ measures the maximal lengths of blocks of digits 0 (or of digits $b-1$) in the b-ary expansion of ξ. The exponents $\hat{v}_b$ are, like $\hat{w}_1$, exponents of uniform approximation. Although they occur rather naturally, they do not seem to have been studied until now.

Alternatively, we may consider the quantity $\ell_n(\xi)$, defined as the maximal length of a block of digits 0 or a block of digits $b-1$ among the n first b-ary digits of ξ. Let v and $\hat{v}$ be positive real numbers. We have $v_b(\xi) \geq v$ (resp., $\hat{v}_b(\xi) \geq \hat{v}$) if, and only if, there are arbitrarily large integers n (resp., for every sufficiently large integer n) such that $\ell_n(\xi)/n \geq v/(1+v)$ (resp., $\ell_n(\xi)/n \geq \hat{v}/(1+\hat{v})$).

It follows immediately from the above mentioned result of Khintchine that $\hat{v}_b(\xi) \leq 1$. An easy covering argument shows that the set

$$\{\xi \in \mathbf{R} : v_b(\xi) = 0\}$$

has full Lebesgue measure. Since

$$0 \leq \hat{v}_b(\xi) \leq v_b(\xi), \tag{1.2}$$

almost all real numbers ξ (with respect to the Lebesgue measure) satisfy $\hat{v}_b(\xi) = 0$. However, it is easy to construct a suitable lacunary series $f(x)$ such that $\hat{v}_b(f(1/b))$ has a prescribed value between 0 and 1. Indeed, for any $v > 0$ we have

$$v_b\Big(\sum_{j\geq 1} b^{-(1+v)^j}\Big) = v \quad \text{and} \quad \hat{v}_b\Big(\sum_{j\geq 1} b^{-(1+v)^j}\Big) = \frac{v}{v+1}.$$

Observe also that

$$v_b\Big(\sum_{j\geq 1} b^{-2^{j^2}}\Big) = +\infty \quad \text{and} \quad \hat{v}_b\Big(\sum_{j\geq 1} b^{-2^{j^2}}\Big) = 1.$$

Hence, for every v in $\mathbf{R}_{>0} \cup \{+\infty\}$ and every real number $\hat{v}$ in $[0,1]$, the sets

$$\mathcal{V}_b(v) := \{\xi \in \mathbf{R} : v_b(\xi) = v\} \quad \text{and} \quad \widehat{\mathcal{V}_b}(\hat{v}) := \{\xi \in \mathbf{R} : \hat{v}_b(\xi) = \hat{v}\}$$

are non-empty. Again, an easy covering argument yields that

$$\dim \mathcal{V}_b(+\infty) = 0, \tag{1.3}$$

where dim stands for the Hausdorff dimension.

Let $\hat{v}$ be in $[0,1]$. It follows from (1.2) that

$$\dim \widehat{\mathcal{V}_b}(\hat{v}) \le \dim\{\xi \in \mathbf{R} : \hat{v}_b(\xi) \ge \hat{v}\} \le \dim\{\xi \in \mathbf{R} : v_b(\xi) \ge \hat{v}\}.$$

Combined with

$$\dim\{\xi \in \mathbf{R} : v_b(\xi) \ge \hat{v}\} = \frac{1}{1+\hat{v}}, \tag{1.4}$$

which follows from a general result of Borosh and Frankel [3], this gives

$$\dim \widehat{\mathcal{V}_b}(\hat{v}) \le \frac{1}{1+\hat{v}}.$$

Note that (1.4) is also a special case of Theorem 5 in [13] and that, furthermore, it easily follows from the mass transference principle of Beresnevich and Velani [2]. Moreover, by [2] or by Theorem 7 from [1], we have

$$\dim \mathcal{V}_b(v) = \frac{1}{1+v}, \tag{1.5}$$

for every $v \ge 0$.

Our first result gives the Hausdorff dimension of the set $\widehat{\mathcal{V}_b}(\hat{v})$ for any $\hat{v}$ in $[0,1]$.

Theorem 1.1. *Let $b \ge 2$ be an integer and $\hat{v}$ be a real number in $[0,1]$. Then we have*

$$\dim\{\xi \in \mathbf{R} : \hat{v}_b(\xi) \ge \hat{v}\} = \dim\{\xi \in \mathbf{R} : \hat{v}_b(\xi) = \hat{v}\} = \left(\frac{1-\hat{v}}{1+\hat{v}}\right)^2. \tag{1.6}$$

Theorem 1.1 follows from a more general statement, in which the values of both functions v_b and $\hat{v}_b$ are prescribed.

Theorem 1.2. *Let $b \ge 2$ be an integer. Let θ and $\hat{v}$ be positive real numbers with $\hat{v} < 1$. If $\theta < 1/(1-\hat{v})$, then the set*

$$\{\xi \in \mathbf{R} : \hat{v}_b(\xi) \ge \hat{v}\} \cap \{\xi \in \mathbf{R} : v_b(\xi) = \theta\hat{v}\}$$

is empty. Otherwise, we have

$$\dim(\{\xi \in \mathbf{R} : \hat{v}_b(\xi) = \hat{v}\} \cap \{\xi \in \mathbf{R} : v_b(\xi) = \theta\hat{v}\}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}. \tag{1.7}$$

Furthermore,

$$\dim\{\xi \in \mathbf{R} : \hat{v}_b(\xi) = 1\} = 0. \tag{1.8}$$

A key observation in the proof of Theorem 1.2 is the fact that the right hand side inequality of (1.2) can be considerably improved. Namely, we show in Subsection 2.1 that

$$v_b(\xi) \text{ is infinite when } \hat{v}_b(\xi) = 1 \tag{1.9}$$

and

$$v_b(\xi) \geq \hat{v}_b(\xi)/(1 - \hat{v}_b(\xi)) \text{ when } \hat{v}_b(\xi) < 1.$$

The latter inequality immediately implies the first statement of Theorem 1.2. Furthermore, the combination of (1.3) and (1.9) gives (1.8).

A rapid calculation shows that the right hand side of (1.7) is a continuous function of the parameter θ on the interval $[1/(1-\hat{v}), +\infty)$, reaching its maximum at the point $\theta_0 := 2/(1-\hat{v})$ and only at that point. This maximum is precisely equal to $(1-\hat{v})^2/(1+\hat{v})^2$, namely the right hand side of (1.6). This essentially shows that Theorem 1.2 implies Theorem 1.1 (a complete argument is given at the end of Subsection 2.1).

We remark that Theorem 1.2 allows us to reprove (1.5). To see this, write $v = \theta\hat{v}$, then $\hat{v} = v/\theta$ and (1.7) becomes

$$\begin{aligned}\dim(\{\xi \in \mathbf{R} : \hat{v}_b(\xi) = v/\theta\} \cap \{\xi \in \mathbf{R} : v_b(\xi) = v\}) &= \frac{\theta - 1 - v}{(1+v)(\theta - 1)} \\ &= \frac{1}{1+v}\Big(1 - \frac{v}{\theta - 1}\Big).\end{aligned}$$

Letting θ tend to infinity, we see that $\dim \mathcal{V}_b(v) \geq 1/(1+v)$. The reverse inequality can easily be obtained by using the natural covering.

Beside b-ary expansions, we can as well consider β-expansions. For $\beta > 1$, let T_β be the β-transformation defined on $[0,1]$ by

$$T_\beta(x) := \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor \cdot \rfloor$ denotes the integer part function. We assume that the reader is familiar with the classical results on β-expansions. Some useful facts are recalled in Section 3.

For a real number $\beta > 1$, we define in a similar way the functions v_β and $\hat{v}_\beta$.

Definition 1.3. *Let $\beta > 1$ be a real number. Let $x \in [0,1]$. We denote by $v_\beta(x)$ the supremum of the real numbers v for which the equation*

$$T_\beta^n x < (\beta^n)^{-v}$$

has infinitely many solutions in positive integers n. We denote by $\hat{v}_\beta(x)$ the supremum of the real numbers $\hat{v}$ for which, for every sufficiently large integer N, the equation

$$T_\beta^n x < (\beta^N)^{-\hat{v}}$$

has a solution n with $1 \leq n \leq N$.

Observe that Definitions 1.1 and 1.3 do not coincide when β is an integer at least equal to 2. However, this should not cause any trouble and we may consider that the following result, established by Shen and Wang [15], extends (1.4) to β-expansions.

Theorem SW. *Let $\beta > 1$ be a real number and v be a positive real number. Then,*

$$\dim\{x \in [0,1] : v_\beta(x) \geq v\} = \frac{1}{1+v}.$$

We establish the following analogues of Theorems 1.1 and 1.2 for β-expansions.

Theorem 1.4. *Let $\beta > 1$ be a real number. Let θ and $\hat{v}$ be positive real numbers with $\hat{v} < 1$. If $\theta < 1/(1-\hat{v})$, then the set*

$$\{x \in [0,1] : \hat{v}_\beta(x) \geq \hat{v}\} \cap \{x \in [0,1] : v_\beta(x) = \theta\hat{v}\}$$

is empty. Otherwise, we have

$$\dim(\{x \in [0,1] : \hat{v}_\beta(x) = \hat{v}\} \cap \{x \in [0,1] : v_\beta(x) = \theta\hat{v}\}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}.$$

Furthermore,

$$\dim(\{x \in [0,1] : \hat{v}_\beta(x) = 1\}) = 0.$$

In the same way as Theorem 1.2 implies Theorem 1.1, the next statement follows from Theorem 1.4.

Theorem 1.5. *Let $\beta > 1$ be a real number and $\hat{v}$ be a real number in $[0,1]$. Then we have*

$$\dim\{x \in [0,1] : \hat{v}_\beta(x) \geq \hat{v}\} = \dim\{x \in [0,1] : \hat{v}_\beta(x) = \hat{v}\} = \Big(\frac{1-\hat{v}}{1+\hat{v}}\Big)^2.$$

Persson and Schmeling [13] have adopted another point of view, by considering the β-expansions of 1 and letting β vary.

Theorem PS. *Let β_0, β_1 and v be real numbers with $1 < \beta_0 < \beta_1 < 2$ and $v > 0$. Then,*

$$\dim\{\beta \in (\beta_0, \beta_1) : v_\beta(1) \geq v\} = \frac{1}{1+v}.$$

The assumption $\beta_1 < 2$ in Theorem PS can easily be removed; see [11]. Applying some of the ideas from [13], we obtain the following theorem.

Theorem 1.6. *Let θ and $\hat{v}$ be positive real numbers with $\hat{v} < 1$. If $\theta < 1/(1-\hat{v})$, then the set*

$$\{\beta > 1 : \hat{v}_\beta(1) \geq \hat{v}\} \cap \{\beta > 1 : v_\beta(1) = \theta\hat{v}\}$$

is empty. Otherwise, we have

$$\dim(\{\beta > 1 : \hat{v}_\beta(1) = \hat{v}\} \cap \{\beta > 1 : v_\beta(1) = \theta\hat{v}\}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}.$$

Furthermore,

$$\dim\{\beta > 1 : \hat{v}_\beta(1) \geq \hat{v}\} = \Big(\frac{1-\hat{v}}{1+\hat{v}}\Big)^2,$$

and
$$\dim(\{\beta > 1 : \hat{v}_\beta(1) = 1\}) = 0.$$

Our paper is organized as follows. Theorem 1.2 is proved in Section 2. We recall classical results from the theory of β-expansion in Section 3 and establish Theorems 1.4 and 1.6 in Sections 4 and 5, respectively. Diophantine approximation on Cantor sets is briefly discussed in Section 6.

Throughout this text, we denote by $|I|$ the length of the interval I and we use $\sharp$ to denote the cardinality of a finite set.

2. Proof of Theorem 1.2

2.1. Upper bound.

Let $\hat{v}$ be a real number with $0 < \hat{v} \leq 1$. We wish to bound from above the dimension of
$$\{\xi \in \mathbf{R} : \hat{v}_b(\xi) \geq \hat{v}\}.$$
By (1.3), it is sufficient to consider the set
$$\{\xi \in \mathbf{R} : \hat{v}_b(\xi) \geq \hat{v}\} \cap \{\xi \in \mathbf{R} : v_b(\xi) < +\infty\}.$$
Let ξ be an irrational real number. Throughout this section, in view of the preceding observations, we assume that
$$0 < \hat{v}_b(\xi) \leq v_b(\xi) < +\infty.$$

Let
$$\xi := \lfloor \xi \rfloor + \sum_{j \geq 1} \frac{a_j}{b^j}$$
denote the b-ary expansion of ξ. It is understood that the digits $a_1, a_2, \ldots$ all belong to the set $\{0, 1, \ldots, b-1\}$.

Define the increasing sequences $(n'_k)_{k\geq 1}$ and $(m'_k)_{k\geq 1}$ as follows: for $k \geq 1$, we have either
$$a_{n'_k} > 0, a_{n'_k+1} = \ldots = a_{m'_k-1} = 0, a_{m'_k} > 0$$
or
$$a_{n'_k} < b-1, a_{n'_k+1} = \ldots = a_{m'_k-1} = b-1, a_{m'_k} < b-1.$$
Furthermore, for every j such that $a_j = 0$ or $b-1$, there exists an index k satisfying $n'_k < j < m'_k$. Since $v_b(\xi)$ is positive, we get
$$\limsup_{k \to +\infty} (m'_k - n'_k) = +\infty. \tag{2.1}$$

Now, we take the maximal subsequences $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ of $(n'_k)_{k\geq 1}$ and $(m'_k)_{k\geq 1}$, respectively, in such a way that the sequence $(m_k - n_k)_{k\geq 1}$ is non-decreasing. More precisely, take $n_1 = n'_1$ and $m_1 = m'_1$. Let $k \geq 1$ be such that $n_k = n'_{j_k}$ and $m_k = m'_{j_k}$ have been defined. Set

$$j_{k+1} := \min\{j > j_k : m'_j - n'_j \geq m_k - n_k\}.$$

Then, define

$$n_{k+1} = n'_{j_{k+1}} \quad \text{and} \quad m_{k+1} = m'_{j_{k+1}}.$$

Observe that, by (2.1), the sequence $(j_k)_{k\geq 1}$ is well defined. Furthermore, $m_k - n_k$ tends to infinity as k tends to infinity.

Note that

$$b^{n_k - m_k} < \|b^{n_k}\xi\| < b^{n_k - m_k + 1}.$$

By construction, we have

$$v_b(\xi) = \limsup_{k\to+\infty} \frac{m_k - n_k}{n_k} = \limsup_{k\to+\infty} \frac{m_k}{n_k} - 1 \tag{2.2}$$

and

$$\hat{v}_b(\xi) = \liminf_{k\to+\infty} \frac{m_k - n_k}{n_{k+1}} \leq \liminf_{k\to+\infty} \frac{m_k - n_k}{m_k} = 1 - \limsup_{k\to+\infty} \frac{n_k}{m_k}. \tag{2.3}$$

Since

$$\left(\limsup_{k\to+\infty} \frac{n_k}{m_k}\right) \cdot \left(\limsup_{k\to+\infty} \frac{m_k}{n_k}\right) \geq 1,$$

we derive from (2.2) and (2.3) that

$$\hat{v}_b(\xi) \leq 1 - \frac{1}{1 + v_b(\xi)} = \frac{v_b(\xi)}{1 + v_b(\xi)}.$$

Noticing that $\hat{v}_b(\xi) < 1$ since $v_b(\xi)$ is assumed to be finite, we have proved that

$$v_b(\xi) \geq \frac{\hat{v}_b(\xi)}{1 - \hat{v}_b(\xi)}. \tag{2.4}$$

Let ξ be a real number with $\hat{v}_b(\xi) \geq \hat{v}$. Take a subsequence $(k_j)_{j\geq 1}$ along which the supremum of (2.2) is obtained. For simplicity, we still write $(n_k)_{k\geq 1}, (m_k)_{k\geq 1}$ for the subsequences $(n_{k_j})_{j\geq 1}, (m_{k_j})_{j\geq 1}$. We remark that when passing to the subsequence, the first equality in (2.3) becomes an inequality.

Let ε be a real number with $0 < \varepsilon < v_b(\xi)/2$. Observe that for k large enough we have

$$(v_b(\xi) - \varepsilon)n_k \leq m_k - n_k \leq (v_b(\xi) + \varepsilon)n_k \tag{2.5}$$

and

$$m_k - n_k \geq n_{k+1}(\hat{v} - \varepsilon). \tag{2.6}$$

The last inequality means that the length of the block of 0 (or $b - 1$) starting at index $n_k + 1$ is at least equal to $n_{k+1}(\hat{v} - \varepsilon)$.

The combination of the second inequality of (2.5) and (2.6) gives

$$(v_b(\xi)+\varepsilon)n_k \geq (\hat{v}-\varepsilon)n_{k+1}.$$

Consequently, there exist an integer n' and a positive real number ε' such that the sum of all the lengths of the blocks of 0 or $b-1$ in the prefix of length n_k of the infinite sequence $a_1a_2\ldots$ is, for k large enough, at least equal to

$$\begin{aligned}(\hat{v}-\varepsilon)n_k\Big(1+\frac{\hat{v}-\varepsilon}{v_b(\xi)+\varepsilon}+\Big(\frac{\hat{v}-\varepsilon}{v_b(\xi)+\varepsilon}\Big)^2+\ldots\Big)-n' &= n_k\,\frac{(\hat{v}-\varepsilon)(v_b(\xi)+\varepsilon)}{v_b(\xi)-\hat{v}+2\varepsilon}-n' \\ &\geq n_k\Big(\frac{\hat{v}v_b(\xi)}{v_b(\xi)-\hat{v}}-\varepsilon'\Big) \\ &= n_k\Big(\frac{\theta\hat{v}}{\theta-1}-\varepsilon'\Big),\end{aligned} \tag{2.7}$$

where the parameter θ is defined by

$$v_b(\xi)=\theta\hat{v}.$$

Note that, by (2.4), we must have

$$\theta \geq \frac{1}{1-\hat{v}}. \tag{2.8}$$

By the first inequality of (2.5), we have

$$m_k \geq (1+v_b(\xi)-\varepsilon)n_k \geq (1+v_b(\xi)-\varepsilon)m_{k-1},$$

for k large enough. Thus $(m_k)_{k\geq 1}$ increases at least exponentially. Since $n_k \geq m_{k-1}$ for $k \geq 2$, the sequence $(n_k)_{k\geq 1}$ also increases at least exponentially. Consequently, there exists a positive real number C such that $k \leq C\log n_k$, for k large enough.

Now, let us construct a covering. Remind that all the integers n_k, m_k defined above depend on ξ. Fix $\hat{v}$ and θ with $\theta \geq \frac{1}{1-\hat{v}}$. Let $(n_k)_{k\geq 1}$ and $(m_k)_{k\geq 1}$ be sequences such that

$$\lim_{k\to\infty}\frac{m_k-n_k}{n_k}=\theta\hat{v}, \quad \text{and} \quad \liminf_{k\to\infty}\frac{m_k-n_k}{n_{k+1}}\geq \hat{v}.$$

For fixed k, we collect all those ξ with $\hat{v}_b(\xi)=\hat{v}$ and $v_b(\xi)=\theta\hat{v}$ whose b-ary expansions have blocks of 0 (or of $b-1$) between n_k and m_k.

By the precedent analysis, there are at most $C\log n_k$ blocks of 0 (or of $b-1$) in the prefix of length n_k of the infinite sequence $a_1a_2\ldots$. Since there are, obviously, at most n_k possible choices for their first index, we have in total at most

$$2^{C\log n_k}\,n_k^{C\log n_k}=(2n_k)^{C\log n_k}$$

possible choices. For each of these choices, it follows from (2.7) that at least $n_k(\theta\hat{v}/(\theta-1)-\varepsilon')$ digits are prescribed (and are equal to 0 or $b-1$). Consequently, defining the positive real number ε'' by the next equality, at most

$$n_k-n_k\Big(\frac{\theta\hat{v}}{\theta-1}-\varepsilon'\Big)+1=n_k(1+\varepsilon'')\cdot\frac{\theta-1-\theta\hat{v}}{\theta-1}$$

digits in the prefix $a_1a_2\ldots a_{n_k}$, and thus in $a_1a_2\ldots a_{m_k}$, are free. The set of real numbers whose b-ary expansion starts with $a_1a_2\ldots a_{m_k}$ defines an interval of length b^{-m_k}. By (2.5), we have

$$b^{-m_k} \le b^{-(1+\theta\hat{v})(1-\varepsilon'')n_k},$$

for k large enough. We have shown that the set of those ξ corresponding to $(n_k)_{k\ge1}$, $(m_k)_{k\ge1}$ is covered by

$$(2n_k)^{C\log n_k}\, b^{n_k(1+\varepsilon'')(\theta-1-\theta\hat{v})/(\theta-1)}.$$

intervals of length at most $b^{-(1+\theta\hat{v})(1-\varepsilon'')n_k}$.

Then, a standard covering argument shows that we have to consider the series

$$\sum_{N\ge1}(2N)^{C\log N}\, b^{N(1+\varepsilon'')(\theta-1-\theta\hat{v})/(\theta-1)}\, b^{-(1+\theta\hat{v})(1-\varepsilon'')Ns}. \tag{2.9}$$

The critical exponent s_0 such that (2.9) converges if $s>s_0$ and diverges if $s<s_0$ is given by

$$s_0=\frac{1+\varepsilon''}{1-\varepsilon''}\cdot\frac{\theta-1-\theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}.$$

It then follows that

$$\dim(\{\xi\in\mathbf{R}:\hat{v}_b(\xi)\ge\hat{v}\}\cap\{\xi\in\mathbf{R}:v_b(\xi)=\theta\hat{v}\})\le\frac{\theta-1-\theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}. \tag{2.10}$$

Actually, we have proved that, for every $\theta\ge(1-\hat{v})^{-1}$ and for every sufficiently small positive number δ, we have

$$\dim(\{\xi\in\mathbf{R}:\hat{v}_b(\xi)\ge\hat{v}\}\cap\{\xi\in\mathbf{R}:\theta\le v_b(\xi)\le\theta+\delta\})\le\frac{\theta-1-\theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}+5\delta\hat{v}.$$

Regarding the right-hand side of (2.10) as a function of θ and taking (2.8) into account, a short calculation shows that the maximum is attained for $\theta=2/(1-\hat{v})$, giving, for any positive ε, that

$$\dim\{\xi\in\mathbf{R}:\hat{v}_b(\xi)\ge\hat{v}\}\le\frac{1+\varepsilon}{1-\varepsilon}\,\frac{(1-\hat{v})^2}{(1+\hat{v})^2}.$$

We have established the required upper bound.

2.2. Lower bound.

To obtain the lower bound, we construct a suitable Cantor type set. Let $\hat{v}$ be in $(0,1)$. Let θ be a real number with $\theta\ge\frac{1}{1-\hat{v}}$. Choose two sequences $(m_k)_{k\ge1}$ and $(n_k)_{k\ge1}$ such that $n_k<m_k<n_{k+1}$ for $k\ge1$, and such that $(m_k-n_k)_{k\ge1}$ is non-decreasing. Furthermore, we assume that

$$\lim_{k\to+\infty}\frac{m_k-n_k}{n_{k+1}}=\hat{v}, \tag{2.11}$$

and

$$\lim_{k\to+\infty}\frac{m_k-n_k}{n_k}=\theta\hat{v}. \tag{2.12}$$

An easy way to construct such sequences is to start with

$$n'_k = \lfloor \theta^k \rfloor, \quad m'_k = \lfloor (\theta \hat{v} + 1) n'_k \rfloor,$$

and then to make a small adjustment to guarantee that $(m_k - n_k)_{k \geq 1}$ is non-decreasing.

We consider the set of real numbers ξ in $(0,1)$ whose b-ary expansion $\xi = \sum_{j \geq 1} \frac{a_j}{b^j}$ satisfies, for $k \geq 1$,

$$a_{n_k} = 1, \; a_{n_k+1} = \ldots = a_{m_k-1} = 0, \; a_{m_k} = 1,$$

and

$$a_{m_k+(m_k-n_k)} = a_{m_k+2(m_k-n_k)} = \; \cdots \; = a_{m_k+t_k(m_k-n_k)} = 1,$$

where t_k is the largest integer such that $m_k + t_k(m_k - n_k) < n_{k+1}$. Observe that, since

$$t_k < \frac{n_{k+1} - m_k}{m_k - n_k} \leq \frac{2}{\hat{v}},$$

for k large enough, the sequence $(t_k)_{k \geq 1}$ is bounded.

We check that the maximal length of blocks of zeros in the prefix of length n_{k+1} of the infinite sequence $a_1 a_2 \ldots$ is equal to $m_k - n_k - 1$. Thus, we deduce that

$$\hat{v}_b(\xi) = \hat{v} \quad \text{and} \quad v_b(\xi) = \theta \hat{v}.$$

Actually, the above two equalities might not be true if $b = 2$. However, they will be valid if we take the block 10 in place of 1 in the definition of $a_{m_k+t(m_k-n_k)}$ for $t = 1, \ldots, t_k$. The following proof will be almost the same. So, for simplicity, we assume that $b \geq 3$ and leave to the reader the slight change to deal with the case $b = 2$ (or the reader may look at Subsection 4.2).

Our Cantor type subset $E_{\theta,\hat{v}}$ consists precisely of the real numbers in $(0,1)$ whose b-ary expansion has the above property. We will now estimate the Hausdorff dimension of $E_{\theta,\hat{v}}$ from below.

Let n be a large positive integer. For $a_1, \ldots, a_n$ in $\{0, 1, \ldots, b-1\}$, denote by $I_n(a_1, \ldots, a_n)$ the interval composed of the real numbers in $(0,1)$ whose b-ary expansion starts with $a_1 \ldots a_n$. Define a Bernoulli measure μ on $E_{\theta,\hat{v}}$ as follows. We distribute the mass uniformly. For $k \geq 1$, set

$$\delta_k := m_k - n_k - 1 \quad \text{and} \quad u_k := m_k + t_k(m_k - n_k).$$

If there exists $k \geq 2$ such that $n_k \leq n \leq m_k$, then define

$$\mu(I_n(a_1, \ldots, a_n)) = b^{-(n_1 - 1 + \sum_{j=1}^{k-1}(t_j \delta_j + n_{j+1} - u_j - 1))} = b^{-(n_k - 1 - \sum_{j=1}^{k-1}(m_j - n_j + 1 + t_j))}.$$

Observe that we have $\mu(I_{n_k}) = \mu(I_{n_k+1}) = \ldots = \mu(I_{m_k})$.

If there exists $k \geq 2$ such that $m_k < n < n_{k+1}$, then define

$$\begin{aligned}\mu(I_n(a_1,\ldots,a_n)) &= b^{-(n_1-1+\sum_{j=1}^{k-1}(t_j\delta_j+n_{j+1}-u_j-1)+t\delta_k+n-(m_k+t\delta_k))} \\ &= b^{-n+\sum_{j=1}^{k-1}(m_j-n_j+1+t_j)+m_k-n_k+1+t},\end{aligned}$$

where t is the largest integer such that $m_k + t(m_k - n_k) \leq n$. It is routine to check that μ is well defined on $E_{\theta,\hat{v}}$.

Now we calculate the *local dimension* of μ at $x \in E_{\theta,\hat{v}}$, i.e.,

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r},$$

where $B(x,r)$ stands for the ball centered at x with radius r. To this end, we first calculate the same lower limit for the basic intervals, and we prove that

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} = \frac{\theta - 1 - \theta\hat{v}}{(\theta-1)(\theta\hat{v}+1)}. \tag{2.13}$$

Since the lengths of basic intervals decrease 'regularly', the above limit (2.13) is the same as the local dimension of μ at $x \in E_{\theta,\hat{v}}$. The details are left to the reader. Finally, the required lower bound follows from the mass distribution principle; see [7], page 26.

Let us finish the proof by showing (2.13). First, we check that if $n = m_k$, then

$$\begin{aligned}\liminf_{k\to\infty} \frac{\log \mu(I_{m_k})}{\log |I_{m_k}|} &= \liminf_{k\to\infty} \frac{n_k - 1 - \sum_{j=1}^{k-1}(m_j - n_j + 1 + t_j)}{m_k} \\ &= \liminf_{k\to\infty} \frac{n_1 - 1 + \sum_{j=1}^{k-1}(n_{j+1} - m_j + 1 + t_j)}{m_k}.\end{aligned}$$

Recalling that $(t_k)_{k\geq 1}$ is bounded and that $(m_k)_{k\geq 1}$ grows exponentially fast in terms of k, we have

$$\liminf_{k\to\infty} \frac{\log \mu(I_{m_k})}{\log |I_{m_k}|} = \liminf_{k\to\infty} \frac{\sum_{j=1}^{k-1}(n_{j+1} - m_j)}{m_k}.$$

By (2.11) and (2.12), we see that

$$\lim_{k\to\infty} \frac{m_k}{n_k} = \theta\hat{v} + 1, \quad \lim_{k\to\infty} \frac{m_{k+1}}{m_k} = \theta, \quad \text{and} \quad \lim_{k\to\infty} \frac{n_{k+1}}{m_k} = \frac{\theta}{\theta\hat{v}+1}.$$

Thus, by the Stolz–Cesàro Theorem,

$$\begin{aligned}\lim_{k\to\infty} \frac{\sum_{j=1}^{k-1}(n_{j+1} - m_j)}{m_k} &= \lim_{k\to\infty} \frac{n_{k+1} - m_k}{m_{k+1} - m_k} \\ &= \lim_{k\to\infty} \frac{\frac{n_{k+1}}{m_k} - 1}{\frac{m_{k+1}}{m_k} - 1} = \frac{\theta - 1 - \theta\hat{v}}{(\theta-1)(\theta\hat{v}+1)}.\end{aligned}$$

Hence,

$$\liminf_{k\to\infty} \frac{\log \mu(I_{m_k})}{\log |I_{m_k}|} = \frac{\theta - 1 - \theta\hat{v}}{(\theta-1)(\theta\hat{v}+1)}.$$

Let n be a large positive integer. If there exists $k \geq 2$ such that $n_k \leq n \leq m_k$, then

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\log \mu(I_n)}{\log |I_{m_k}|} = \frac{\log \mu(I_{m_k})}{\log |I_{m_k}|}.$$

If there exists $k \geq 2$ such that $m_k < n < n_{k+1}$, write $n = m_k + t(m_k - n_k) + \ell$, where t, ℓ are integers with $0 \leq t \leq t_k$ and $0 \leq \ell < m_k - n_k$. Then we have

$$\mu(I_n) = \mu(I_{m_k}) \cdot b^{-(t\delta_k + \ell)} \quad \text{and} \quad |I_n| = |I_{m_k}| \cdot b^{-(t(m_k-n_k)+\ell)}.$$

Since $0 \leq t \leq t_k$ and $(t_k)_{k\geq 1}$ is bounded, for n large enough,

$$\frac{-\log \mu(I_n)}{-\log |I_n|} = \frac{-\log \mu(I_{m_k}) + t\delta_k + \ell}{-\log |I_{m_k}| + t(m_k - n_k) + \ell} \geq \frac{-\log \mu(I_{m_k})}{-\log |I_{m_k}|},$$

where we have used the fact that

$$\frac{a+x}{b+x} \geq \frac{a}{b}, \quad \text{for all } 0 < a \leq b, x \geq 0.$$

We have established (2.13).

3. Classical results on β-expansions

Throughout this section, β denotes a real number greater than 1 and $\lfloor \beta \rceil$ is equal to $\beta - 1$ if β is an integer and to $\lfloor \beta \rfloor$ otherwise. The notion of β-expansion was introduced by Rényi [14] in 1957. We denote by T_β the transformation defined on $[0,1]$ by $T_\beta(x) = \{\beta x\}$, where $\{\cdot\}$ denotes the fractional part function.

Definition 3.1. *The expansion of a number x in $[0,1]$ to base β, also called the β-expansion of x, is the sequence $(\varepsilon_n)_{n\geq 1} = (\varepsilon_n(x,\beta))_{n\geq 1}$ of integers from $\{0, 1, \ldots, \lfloor \beta \rceil\}$ such that*

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \ldots + \frac{\varepsilon_n}{\beta^n} + \ldots,$$

and, unless $x = 1$ and β is an integer, defined by one of the following equivalent properties:

$$\sum_{k>n} \frac{\varepsilon_k}{\beta^k} < \frac{1}{\beta^n}, \quad \textit{for all } n \geq 0;$$

$$\varepsilon_1 = \lfloor \beta x \rfloor,\ \varepsilon_2 = \lfloor \beta\{\beta x\} \rfloor,\ \varepsilon_3 = \lfloor \beta\{\beta\{\beta x\}\} \rfloor, \ldots$$
$$\varepsilon_n = \lfloor \beta T_\beta^{n-1}(x) \rfloor, \quad \textit{for all } n \geq 1.$$

We then write

$$d_\beta(x) = \varepsilon_1\varepsilon_2\ldots\varepsilon_n\ldots.$$

For $x < 1$, the β-expansion coincides with the representation of x computed by the 'greedy algorithm'. If β is an integer, then the digits ε_i of x lie in the set $\{0, 1, \ldots, \beta - 1\}$ and $d_\beta(x)$ corresponds, for $x \neq 1$, to the usual β-ary expansion of x. If the β-expansion

$$d_\beta(1) = \varepsilon_1(1,\beta)\varepsilon_2(1,\beta)\ldots\varepsilon_n(1,\beta)\ldots$$

of 1 is finite, i.e., if there exists $m \geq 1$ such that $\varepsilon_m(1,\beta) \neq 0$ and $\varepsilon_n(1,\beta) = 0$ for all $n > m$, then β is called a *simple Parry number*. In this case, we define the *infinite β-expansion of* 1 by

$$(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots, \varepsilon_n^*(\beta), \ldots) := (\varepsilon_1^*(1,\beta), \varepsilon_2^*(1,\beta), \ldots, (\varepsilon_m(1,\beta) - 1))^\infty,$$

where $(w)^\infty$ stands for the periodic sequence $(w, w, w, \ldots)$.

We endow the set $\{0, 1, \ldots, \lfloor\beta\rceil\}^{\mathbf{Z}_{\geq 1}}$ with the product topology and the one-sided shift operator σ defined by $\sigma((s_n)_{n\geq 1}) = (s_{n+1})_{n\geq 1}$, for any infinite sequence $(s_n)_{n\geq 1}$ in $\{0, 1, \ldots, \lfloor\beta\rceil\}^{\mathbf{Z}_{\geq 1}}$.

The lexicographic order on $\{0, 1, \ldots, \lfloor\beta\rceil\}^{\mathbf{Z}_{\geq 1}}$, denoted by $<_{\text{lex}}$, is defined as follows: we write

$$w = (w_1, w_2, \ldots) <_{\text{lex}} w' = (w_1', w_2', \ldots)$$

if there exists $k \geq 1$ such that for all $j < k$ we have $w_j = w_j'$, but $w_k < w_k'$. We use the notation $w \leq_{\text{lex}} w'$ if $w <_{\text{lex}} w'$ or $w = w'$.

Definition 3.2. *The closure of the set of all β-expansions of x in $[0, 1]$ is called the β-shift and denoted by Σ_β.*

Parry [12] proved that the β-shift Σ_β is fully determined by $d_\beta(1)$.

Theorem 3.3. *If $d_\beta(1) = \varepsilon_1\ldots\varepsilon_m 00\ldots 0\ldots$ is finite with $\varepsilon_m \neq 0$, then $\mathbf{s} = (s_n)_{n\geq 1}$ belongs to Σ_β if, and only if,*

$$\sigma^k(\mathbf{s}) \leq_{\text{lex}} (\varepsilon_1, \ldots, \varepsilon_{m-1}, (\varepsilon_m - 1))^\infty, \quad \text{for } k \geq 1.$$

If $d_\beta(1)$ does not terminate with zeros only, then $\mathbf{s} = (s_n)_{n\geq 1}$ belongs to Σ_β if, and only if,

$$\sigma^k(\mathbf{s}) \leq_{\text{lex}} d_\beta(1), \quad \text{for } k \geq 1.$$

It follows from Theorem 3.3 that Σ_β is contained in $\Sigma_{\beta'}$ if, and only if, $\beta \leq \beta'$.

Definition 3.4. *A block $\varepsilon_1\ldots\varepsilon_m$, respectively, an infinite sequence $\varepsilon_1\varepsilon_2\ldots$, on the alphabet $\{0, 1, \ldots, \lfloor\beta\rceil\}$ is β-admissible (or, simply, admissible) if*

$$\sigma^k(\varepsilon_1\ldots\varepsilon_m) \leq_{\text{lex}} d_\beta(1), \quad \text{for } k = 0, 1, \ldots, m-1,$$

respectively, if

$$\sigma^k(\varepsilon_1\varepsilon_2\ldots) \leq_{\text{lex}} d_\beta(1), \quad \text{for } k \geq 0.$$

An infinite sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ *is self-admissible if*

$$\sigma^k(\varepsilon_1\varepsilon_2\ldots) \leq_{\text{lex}} (\varepsilon_1\varepsilon_2\ldots), \quad \textit{for } k \geq 0.$$

Denote by Σ_β^n the set of all β-admissible blocks of length n. Then, its cardinality satisfies (Rényi [14], formula 4.9)

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1}. \tag{3.1}$$

For any $(\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n$, call

$$I_n(\varepsilon_1, \ldots, \varepsilon_n) := \{x \in [0,1] : d_\beta(x) \text{ starts with } \varepsilon_1 \ldots \varepsilon_n\}$$

an *n-th order basic interval* (with respect to the base β). Denote by $I_n(x)$ the n-th order basic interval containing x. We remark that the basic intervals are also called cylinders by some authors.

The next theorem was proved by Parry [12].

Theorem 3.5. *A sequence of digits* $(\varepsilon_1, \varepsilon_2, \ldots)$ *is the* β*-expansion of* 1 *for some* $\beta > 1$ *if and only if it is self-admissible.*

Now, we estimate the length of the basic intervals. We will use the notion of "full cylinder" introduced by Fan and Wang [8].

Definition 3.6. *Let* $(\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n$ *be a* β*-admissible block in* Σ_β*. A basic interval* $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ *is called full if it is of length* β^{-n}.

Proposition 3.7. *([8], Lemma 3.1) A basic interval* $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ *is full if and only if, for any admissible block* $(\varepsilon'_1, \ldots, \varepsilon'_m) \in \Sigma_\beta^m$*, the concatenation* $(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon'_1, \ldots, \varepsilon'_m)$ *is also admissible, i.e., in* Σ_β^{n+m}.

Proposition 3.8. *([15], Corollary 2.6) For any* $w \in \Sigma_\beta^n$*, if* $I_n(w)$ *is full, then for any* $w' \in \Sigma_\beta^m$*, we have*

$$|I_{n+m}(w, w')| = |I_n(w)| \cdot |I_m(w')| = \beta^{-n}|I_m(w')|.$$

The following approximation of β-shift is very useful. For $\beta > 1$, let $(\varepsilon_k^*(\beta))_{k\geq 1}$ denote the infinite β-expansion of 1. Let β_N be the unique real number which satisfies the equation

$$1 = \frac{\varepsilon_1^*(\beta)}{z} + \cdots + \frac{\varepsilon_N^*(\beta)}{z^N}.$$

Then we have $\beta_N < \beta$ and the sequence $(\beta_N)_{N\geq 1}$ increases and converges to β when N tends to infinity. Furthermore, the subshift of finite type Σ_{β_N} is a subset of the β-shift Σ_β. The subsets Σ_{β_N} are increasing and converge to Σ_β.

Proposition 3.9. *([15], Lemma 2.7) For any $w \in \Sigma^n_{\beta_N}$ viewed as an element of Σ^n_β, we have*

$$\beta^{-(n+N)} \le |I_n(w)| \le \beta^{-n}.$$

4. Proof of Theorem 1.4

4.1. Upper bound.

The proof is essentially similar to that in Subsection 2.1. Let $x \in [0,1]$ be a real number and let

$$x = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \dots$$

denote its β-expansion. Assume that $v_\beta(x)$ is positive.

Define the increasing sequences $(n'_k)_{k\ge1}$ and $(m'_k)_{k\ge1}$ as follows: for $k \ge 1$, we have

$$a_{n'_k} > 0, a_{n'_k+1} = \dots = a_{m'_k-1} = 0, a_{m'_k} > 0,$$

and, furthermore, for every j such that $a_j = 0$, there exists an index k satisfying $n'_k < j < m'_k$. Then take the maximal subsequences $(n_k)_{k\ge1}$ and $(m_k)_{k\ge1}$ of $(n'_k)_{k\ge1}$ and $(m'_k)_{k\ge1}$, respectively, in such a way that the sequence $(m_k - n_k)_{k\ge1}$ is non-decreasing. Observe that, since $v_\beta(x) > 0$, this sequence tends to infinity with k. Similarly, notice that

$$\beta^{n_k-m_k} < T^{n_k}_\beta x < \beta^{n_k-m_k+1}.$$

We also have

$$v_\beta(x) = \limsup_{k\to+\infty} \frac{m_k - n_k}{n_k}, \quad \hat{v}_\beta(x) \le \liminf_{k\to+\infty} \frac{m_k - n_k}{n_{k+1}},$$

and the relations

$$\hat{v}_\beta(x) \le \frac{v_\beta(x)}{1 + v_\beta(x)} \quad \text{and} \quad v_\beta(x) \ge \frac{\hat{v}_\beta(x)}{1 - \hat{v}_\beta(x)}.$$

The last inequality means that $v_\beta(x)$ is infinite when $\hat{v}_\beta(x) = 1$. Combined with Theorem SW, it implies the last assertion of Theorem 1.4.

Let $\hat{v}$ be in $(0,1)$. Let θ be the real number defined by

$$v_\beta(x) = \theta\hat{v}.$$

Then

$$\theta \ge \frac{1}{1-\hat{v}}.$$

Arguing as in the proof of Theorem 1.2, we see that there exists a positive real number ε such that the sum of all the lengths of the blocks of zeros in the prefix of length n_k of the infinite sequence $a_1a_2\dots$ is at least equal to

$$n_k\Big(\frac{\theta\hat{v}}{\theta-1} - \varepsilon\Big),$$

for $k \geq 1$.

Now we study the possible choices of digits among $a_1, \ldots, a_{m_k}$ as we did in Subsection 2.1. Note that there are k blocks of digits which are 'free'. Denote their lengths by $\ell_1, \ldots, \ell_k$. Then we have $\ell_j = n_j - m_{j-1}$ and

$$\sum_{j=1}^{k} \ell_j \leq n_k - n_k \left(\frac{\theta \hat{v}}{\theta - 1} - \varepsilon \right) = n_k (1 + \varepsilon')(\theta - 1 - \theta \hat{v})/(\theta - 1),$$

with $\varepsilon' > 0$ being a small real number. By (3.1), there are at most

$$\frac{\beta}{\beta - 1} \cdot \beta^{\ell_j}$$

ways to choose the block of length ℓ_j. Thus, for the k blocks, we have in total

$$\left(\frac{\beta}{\beta - 1} \right)^k \cdot \beta^{\sum_{j=1}^{k} \ell_j} \leq \left(\frac{\beta}{\beta - 1} \right)^k \cdot \beta^{n_k (1+\varepsilon')(\theta - 1 - \theta \hat{v})/(\theta - 1)}$$

choices.

As in Subsection 2.1, there exists a real number $C > 1$ such that k is less than $C \log n_k$, and there are at most n_k possible choices for the first index of the k blocks. Thus, we have at most

$$n_k^{C \log n_k}$$

possible choices for the k blocks. We get that, for k sufficiently large, the set of real numbers x with the above properties is contained in a union of no more than

$$\left(\frac{\beta n_k}{\beta - 1} \right)^{C \log n_k} \beta^{n_k (1+\varepsilon')(\theta - 1 - \theta \hat{v})/(\theta - 1)}$$

basic intervals of order m_k, each of them being of length at most β^{-m_k}.

Furthermore, there are infinitely many indices k such that

$$\beta^{-m_k} \leq \beta^{-(1+\theta \hat{v})(1-\varepsilon') n_k}.$$

Arguing as in Subsection 2.1, we end up with the series

$$\sum_{N \geq 1} N^{C \log N} \beta^{N(1+\varepsilon')(\theta - 1 - \theta \hat{v})/(\theta - 1)} \beta^{-(1+\theta \hat{v})(1-\varepsilon') N s}. \tag{4.1}$$

The critical exponent s_0 such that (4.1) converges if $s > s_0$ and diverges if $s < s_0$ is given by

$$s_0 = \frac{1 + \varepsilon'}{1 - \varepsilon'} \cdot \frac{\theta - 1 - \theta \hat{v}}{(1 + \theta \hat{v})(\theta - 1)}.$$

We then get that

$$\dim(\{x \in (0,1) : \hat{v}_\beta(x) \geq \hat{v}\} \cap \{x \in (0,1) : v_\beta(x) = \theta\hat{v}\}) \leq \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}.$$

For the rest of the proof, we argue exactly as at the end of Subsection 2.1. We omit the details.

4.2. Lower bound.

Let $\hat{v}$ be in $(0,1)$. Let θ be a real number with $\theta \geq \frac{1}{1-\hat{v}}$. We construct a subset $E_{\theta,\hat{v}}$ whose elements x satisfy

$$\hat{v}_\beta(x) = \hat{v} \quad \text{and} \quad v_\beta(x) = \theta\hat{v}.$$

We will suitably modify the construction we performed in Subsection 2.2 when dealing with b-ary expansions.

Let N be a positive integer. Let β_N be the real number defined from the infinite β-expansion of 1 as explained at the end of Section 3. As in Subsection 2.2, let $(m_k)_{k\geq 1}$ and $(n_k)_{k\geq 1}$ be sequences of positive integers with $n_k < m_k < n_{k+1}$ for $k \geq 1$, such that $(m_k - n_k)_{k\geq 1}$ is non-decreasing and (2.11) and (2.12) are satisfied. Start with the construction performed in Subsection 2.2 and replace the digit 1 for a_{n_k}, a_{m_k} and $a_{m_k + j(m_k - n_k)}$, $1 \leq j \leq t_k$, by the block $0^N 1 0^N$. Fill other places by blocks in Σ_{β_N}. Thus, we have completed the modifications and have constructed the subset $E_{\theta,\hat{v}}$.

Since N is fixed and $(t_k)_{k\geq 1}$ is bounded, we check that for every x in $E_{\theta,\hat{v}}$,

$$\hat{v}_\beta(x) = \lim_{k\to+\infty} \frac{m_k - n_k - 1 + 2N}{n_{k+1} + (4k+2)N + \sum_{j=1}^{k} 2Nt_j} = \lim_{k\to+\infty} \frac{m_k - n_k}{n_{k+1}} = \hat{v},$$

and

$$v_\beta(x) = \lim_{k\to+\infty} \frac{m_k - n_k - 1 + 2N}{n_k + (4k-2)N + \sum_{j=1}^{k-1} 2Nt_j} = \lim_{k\to+\infty} \frac{m_k - n_k}{n_k} = \theta\hat{v}.$$

Moreover, the sequence $d_\beta(x)$ (in Σ_β) is, by construction, also in Σ_{β_N}.

Let n be a large positive integer. Denote by $I_n(a_1, \ldots, a_n)$ the interval composed of the real numbers in $(0,1)$ whose β-expansion starts with $a_1 \ldots a_n$. We will define a Bernoulli measure μ on $E_{\theta,\hat{v}}$. We distribute the mass uniformly when we meet a block in Σ_{β_N} and keep the mass when we go through the positions where the digits are determined by the construction. Precisely, we can write down the first levels as follows.

If $n < n_1$, define

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^n}.$$

If $n_1 \leq n \leq m_1 + 4N$, then take

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1 - 1}}.$$

If there exists an integer t such that $0 \le t \le t_1$ and

$$m_1+4N+t(m_1-n_1)+2Nt < n \le \min\{n_2+4N+2Nt_1, m_1+4N+(t+1)(m_1-n_1)+2Nt\},$$

then set

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{(\sharp\Sigma_{\beta_N}^{m_1-n_1-1})^t} \cdot \frac{1}{\sharp\Sigma_{\beta_N}^{n-(m_1+4N+t(m_1-n_1)+2Nt)}}.$$

If there exists an integer t such that $0 \le t \le t_1 - 1$ and

$$m_1 + 4N + (t+1)(m_1-n_1) + 2Nt < n \le m_1 + 4N + (t+1)(m_1-n_1) + 2N(t+1),$$

then set

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{(\sharp\Sigma_{\beta_N}^{m_1-n_1-1})^{t+1}}.$$

More generally, for $k \ge 2$, set

$$l_k := n_k + (4k-4)N + \sum_{j=1}^{k-1} 2Nt_j, \quad h_k := m_k + 4kN + \sum_{j=1}^{k-1} 2Nt_j.$$

and

$$\delta_k := m_k - n_k - 1, \quad u_k := h_k + t_k(m_k-n_k) + 2Nt_k.$$

If $l_k \le n \le h_k$, define

$$\mu(I_n) = \frac{1}{\sharp\Sigma_{\beta_N}^{n_1-1}} \cdot \frac{1}{\prod_{j=1}^{k-1}(\sharp\Sigma_{\beta_N}^{\delta_j})^{t_j}\sharp\Sigma_{\beta_N}^{l_{j+1}-u_j-1}} (= \mu(I_{l_k}) = \mu(I_{h_k})).$$

If there exists an integer t such that $0 \le t \le t_k - 1$ and

$$h_k + t(m_k-n_k) + 2Nt < n \le \min\{l_{k+1}, h_k + (t+1)(m_k-n_k) + 2Nt\},$$

then define

$$\mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{(\sharp\Sigma_{\beta_N}^{\delta_k})^t} \cdot \frac{1}{\sharp\Sigma_{\beta_N}^{n-(h_k+t(m_k-n_k)+2Nt)}}.$$

If there exists an integer t such that $0 \le t \le t_k - 1$ and

$$h_k + (t+1)(m_k-n_k) + 2Nt < n \le h_k + (t+1)(m_k-n_k) + 2N(t+1),$$

then define

$$\mu(I_n) = \mu(I_{h_k}) \cdot \frac{1}{(\sharp\Sigma_{\beta_N}^{\delta_k})^{t+1}}.$$

By the construction and Proposition 3.7, we deduce that I_{h_k} is full and thus has length β^{-h_k}. Then, by recalling that N is fixed and $(t_k)_{k\geq 1}$ is bounded, we deduce from (3.1) that

$$\begin{aligned}
\liminf_{k\to\infty} \frac{\log \mu(I_{h_k})}{\log |I_{h_k}|} &= \liminf_{k\to\infty} \frac{\log \sharp\Sigma_{\beta_N}^{n_1-1} + \sum_{j=1}^{k-1} \left(t_j \log \sharp\Sigma_{\beta_N}^{\delta_j} + \log \sharp\Sigma_{\beta_N}^{l_{j+1}-u_j-1}\right)}{h_k \log \beta} \\
&= \lim_{k\to\infty} \frac{n_1 - 1 + \sum_{j=1}^{k-1}(t_j\delta_j + l_{j+1} - u_j - 1)}{h_k} \cdot \frac{\log \beta_N}{\log \beta} \\
&= \lim_{k\to\infty} \frac{n_1 - 1 + \sum_{j=1}^{k-1}(l_{j+1} - h_j - 2Nt_j - 1)}{h_k} \cdot \frac{\log \beta_N}{\log \beta} \\
&= \lim_{k\to\infty} \frac{\sum_{j=1}^{k-1}(n_{j+1} - m_j)}{m_k} \cdot \frac{\log \beta_N}{\log \beta} \\
&= \frac{\theta - 1 - \theta\hat{v}}{(\theta-1)(\theta\hat{v}+1)} \cdot \frac{\log \beta_N}{\log \beta}.
\end{aligned}$$

Let n be a large positive integer. If there exists $k \geq 2$ such that $l_k \leq n \leq h_k$, then

$$\frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\log \mu(I_n)}{\log |I_{h_k}|} = \frac{\log \mu(I_{h_k})}{\log |I_{h_k}|}.$$

If there exist integers $k \geq 2$ and t such that $0 \leq t \leq t_k - 1$ and

$$h_k + t(m_k - n_k) + 2Nt < n \leq \min\{l_{k+1}, h_k + (t+1)(m_k - n_k) + 2Nt\},$$

then, letting $\ell = n - (h_k + t(m_k - n_k) + 2Nt)$, we have

$$\mu(I_n) \leq \mu(I_{h_k}) \cdot \beta_N^{-\delta_k t - \ell}.$$

Since I_{h_k} is full, then

$$|I_n| = |I_{h_k}| \cdot |I_{n-h_k}(w')|,$$

where w' is an admissible block in Σ_{β_N} of length $n - h_k$. By Proposition 3.9,

$$|I_{n-h_k}(w')| \geq \beta^{-(n-h_k+N)}.$$

Hence,

$$|I_n| \geq |I_{h_k}| \cdot \beta^{-(n-h_k+N)}.$$

Notice that

$$n - h_k + N = t(m_k - n_k) + 2Nt + \ell + N = \delta_k t + \ell + t + N(2t+1).$$

Since N is fixed and t is bounded, we argue as in Subsection 2.2 to show that, for n large enough, we have

$$\begin{aligned}
\frac{-\log \mu(I_n)}{-\log |I_n|} &\geq \frac{-\log \mu(I_{h_k}) + (t\delta_k + \ell)\log \beta_N}{-\log |I_{h_k}| + (t\delta_k + \ell + t + N(2t+1))\log \beta} \\
&\geq \frac{-\log \mu(I_{h_k})}{-\log |I_{h_k}|} \cdot \eta(N),
\end{aligned}$$

where $\eta(N) < 1$ and $\eta(N)$ tends to 1 as N tends to infinity.

Similarly, if there exist integers $k \geq 2$ and t such that $0 \leq t \leq t_k - 1$ and

$$h_k + (t+1)(m_k - n_k) + 2Nt < n \leq h_k + (t+1)(m_k - n_k) + 2N(t+1),$$

we also have for n large enough,

$$\frac{-\log \mu(I_n)}{-\log |I_n|} \geq \frac{-\log \mu(I_{h_k})}{-\log |I_{h_k}|} \cdot \eta(N).$$

So, in all cases, we have

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\theta - 1 - \theta\hat{v}}{(\theta-1)(\theta\hat{v}+1)} \cdot \frac{\log \beta_N}{\log \beta} \cdot \eta(N).$$

Now, we consider a general ball $B(x,r)$ with x a point in the Cantor-type subset $E_{\theta,\hat{v}}$ and r satisfying

$$|I_{n+1}(x)| \leq r < |I_n(x)|.$$

By the construction and Proposition 3.9, any n-th order basic interval I_n satisfies

$$|I_n| \geq \beta^{-(n+N)}.$$

Thus, the ball $B(x,r)$ intersects at most $\lceil 2\beta^N \rceil + 2$ basic intervals of order n. Noting that all n-th order basic intervals have the same measure, we deduce that

$$\mu(B(x,r)) \leq (\lceil 2\beta^N \rceil + 2) \cdot \mu(I_n(x)).$$

On the other hand, by Proposition 3.9,

$$r \geq |I_{n+1}(x)| \geq \beta^{-(n+1+N)} = \beta^{-(N+1)} \cdot \beta^{-n} \geq \beta^{-(N+1)} \cdot |I_n(x)|$$

Since N is fixed, we have

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \liminf_{n\to\infty} \frac{\log \mu(I_n(x))}{\log |I_n(x)|}.$$

Finally, the lower bound follows by letting N tend to infinity.

To end this section, we remark that the last step of the proof of the lower bound, i.e., the fact that the two lower limits concerning the basic intervals and the general balls are the same, always holds in the setting of β-transformation. This conclusion was proved very recently by Bugeaud and Wang in [6], who called it a *modified mass distribution principle.*

5. Proof of Theorem 1.6

We follow the approach of Persson and Schmeling [13]. The main idea is to take a correspondence between the β-shift and the parameter space. Then the results in the shift space can be translated to the parameter space.

5.1. Upper bound.

First we reduce the question to a small interval (β_0, β_1), where $1 < \beta_0 < \beta_1$. For $\hat{v} \in [0,1]$ and $\theta \geq 1$, set

$$D_{\theta,\hat{v}} := \{\beta > 1 : \hat{v}_\beta(1) = \hat{v}\} \cap \{\beta > 1 : v_\beta(1) = \theta\hat{v}\},$$

and

$$D_{\theta,\hat{v}}(\beta_0,\beta_1) := \{\beta \in (\beta_0,\beta_1) : \hat{v}_\beta(1) = \hat{v}\} \cap \{\beta \in (\beta_0,\beta_1) : v_\beta(1) = \theta\hat{v}\}.$$

By Theorem 3.5, to every self-admissible sequence corresponds a real number $\beta > 1$. Let K_{β_1} be the set of all self-admissible sequences in Σ_{β_1}. Let π_β be the natural projection from the β-shift to the unit interval $[0,1]$. Then there is a one-to-one map $\varrho_{\beta_1} : \pi_{\beta_1}(K_{\beta_1}) \to (1, \beta_1)$.

Let $B_{\theta,\hat{v}}$ be the subset of Σ_{β_1} defined by

$$B_{\theta,\hat{v}} := \pi_{\beta_1}^{-1}\big(\{x \in [0,1] : \hat{v}_{\beta_1}(x) = \hat{v}\} \cap \{x \in [0,1] : v_{\beta_1}(x) = \theta\hat{v}\}\big).$$

The Hölder exponent of the restriction of the map ϱ_{β_1} to the subset $\pi_{\beta_1}(K_{\beta_1} \cap B_{\theta,\hat{v}})$ is equal to $\log\beta_0/\log\beta_1$. Since $D_{\theta,\hat{v}}(\beta_0,\beta_1) \subset \varrho_{\beta_1}(\pi_{\beta_1}(K_{\beta_1} \cap B_{\theta,\hat{v}}))$ we have

$$\dim D_{\theta,\hat{v}}(\beta_0,\beta_1) \leq \dim \varrho_{\beta_1}(\pi_{\beta_1}(K_{\beta_1} \cap B_{\theta,\hat{v}})) \leq \frac{\log\beta_1}{\log\beta_0} \dim \pi_{\beta_1}(B_{\theta,\hat{v}}),$$

while, by Theorem 1.5,

$$\dim \pi_{\beta_1}(B_{\theta,\hat{v}}) = \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}.$$

Letting β_1 tend to β_0, we obtain the requested upper bound.

5.2. Lower bound.

Take β_2 such that $1 < \beta_0 < \beta_1 < \beta_2$ and that the β_2-expansion of 1 ends with zeros, i.e., such that the β-shift Σ_{β_2} is a subshift of finite type. We establish the following lemma.

Lemma 5.1. *For real numbers $\beta_0, \beta_1, \beta_2, \hat{v}$ and θ such that $1 < \beta_0 < \beta_1 < \beta_2$, $\hat{v} \in [0,1]$ and $\theta \geq 1$, we have*

$$\dim(\varrho_{\beta_2}^{-1}(D_{\theta,\hat{v}}(\beta_0,\beta_1))) \geq \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)} \cdot \frac{\log\beta_1}{\log\beta_2}.$$

It follows from Lemma 5.1 and the proof of Theorem 14 of Persson and Schmeling [13] that

$$\dim(D_{\theta,\hat{v}}(\beta_0,\beta_1)) \geq \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)} \cdot \frac{\log\beta_1}{\log\beta_2}.$$

Finally, letting β_2 tend to β_1, we complete the proof of the lower bound.

Proof of Lemma 5.1.

Denote by $(\varepsilon_k^*)_{k\geq 1} := (\varepsilon_k^*(\beta_1))_{k\geq 1}$ the infinite β_1-expansion of 1. Take an integer N sufficiently large such that $\varepsilon_N^* \neq 0$ and the β_0-expansion of 1 is smaller than $\varepsilon_1^* \ldots \varepsilon_N^*$ in lexicographical order. Let $\tilde{\beta}_N$ be the unique real number which satisfies the equation

$$1 = \frac{\varepsilon_1^*}{z} + \cdots + \frac{\varepsilon_N^*}{z^N}.$$

Then we have $\beta_0 < \tilde{\beta}_N < \beta_1$, and $\tilde{\beta}_N$ tends to β_1 as N tends to infinity. Moreover, the infinite $\tilde{\beta}_N$-expansion of 1 is given by

$$(\varepsilon_1^*, \ldots, (\varepsilon_N^* - 1))^\infty.$$

We construct a subset $\pi_{\beta_2}(K_N)$ of $\varrho_{\beta_2}^{-1}(D_{\theta,\hat{v}}(\beta_0, \beta_1))$ by using the same construction as in Subsection 4.2. Take any sequence $\underline{a} \in \Sigma_{\tilde{\beta}_N}$ constructed in Subsection 4.2 with $\beta = \tilde{\beta}_N$. Make the concatenation $\varepsilon_1^* \ldots \varepsilon_N^* 0^N \underline{a}$. By Lemma 5.2 from [11], the sequence $\varepsilon_1^* \ldots \varepsilon_N^* 0^N \underline{a}$ is self-admissible and thus, by Theorem 3.5, is the β-expansion of 1 for some β. By checking the lexicographic ordering, we see that $\beta_0 < \beta < \beta_1$. We define the subset K_N to be the collection of these sequences. Notice that K_N is also a subset of Σ_{β_2}.

Now we can define a measure $\tilde{\mu}$ on the set $\pi_{\beta_2}(K_N)$. Consider a basic interval $I_{2N+m}(\varepsilon_1^*, \ldots, \varepsilon_N^*, 0^N, a_1, \ldots, a_m)$. Let μ be the measure defined in Section 4.2 by replacing β_N there by $\tilde{\beta}_N$. Define

$$\tilde{\mu}(I_{2N+m}(\varepsilon_1^*, \ldots, \varepsilon_N^*, 0^N, a_1, \ldots, a_m)) := \mu(I_m(a_1, \ldots, a_m)).$$

By Proposition 3.7, the basic interval $I_{2N}(\varepsilon_1^*, \ldots, \varepsilon_N^*, 0^N)$ is full. Then, it follows from Proposition 3.8 that

$$\begin{aligned} |I_{2N+m}(\varepsilon_1^*, \ldots, \varepsilon_N^*, 0^N, a_1, \ldots, a_m)| &= |I_{2N}(\varepsilon_1^*, \ldots, \varepsilon_N^*, 0^N)| \cdot |I_m(a_1, \ldots, a_m)|, \\ &= {\beta_2}^{-2N} |I_m(a_1, \ldots, a_m)|. \end{aligned}$$

Since Σ_{β_2} is of finite type, there exists a positive real number C such that

$$C^{-1}\beta_2^{-m} \leq |I_m(a_1, \ldots, a_m)| \leq C\beta_2^{-m}.$$

Thus, noting that N is fixed, we can deduce, as in Section 4.2, that

$$\liminf_{n\to\infty} \frac{\log \mu(I_n)}{\log |I_n|} \geq \frac{\theta - 1 - \theta\hat{v}}{(\theta - 1)(\theta\hat{v} + 1)} \cdot \frac{\log \tilde{\beta}_N}{\log \beta_2} \cdot \eta(N).$$

Similarly as in the previous section, we have the same inequality for the general ball $B(x, r)$, i.e., for any x in the Cantor-type set $\pi_{\beta_2}(K_N)$, we have

$$\liminf_{r\to 0} \frac{\log \mu(B(x, r))}{\log r} \geq \frac{\theta - 1 - \theta\hat{v}}{(\theta - 1)(\theta\hat{v} + 1)} \cdot \frac{\log \tilde{\beta}_N}{\log \beta_2} \cdot \eta(N).$$

Hence

$$\dim(\varrho_{\beta_2}^{-1}(D_{\theta,\hat{v}}(\beta_0, \beta_1))) \geq \dim(\pi_{\beta_2}(K_N)) \geq \frac{\theta - 1 - \theta\hat{v}}{(\theta - 1)(\theta\hat{v} + 1)} \cdot \frac{\log \tilde{\beta}_N}{\log \beta_2} \cdot \eta(N).$$

Letting N tend to infinity, this proves the lemma. □

6. Diophantine approximation and Cantor sets

Throughout this section, b denotes an integer at least equal to 3 and S is a subset of $\{0, 1, \ldots, b-1\}$ of cardinality at least two and containing at least one of the digits 0 and $b-1$. Let $K_{b,S}$ denote the set of real numbers ξ in $[0, 1]$ which can be expressed as

$$\xi = \sum_{j \geq 1} \frac{a_j}{b^j},$$

with $a_j \in K_{b,S}$ for $j \geq 1$. Note that for $b = 3$ and $S = \{0, 2\}$ the set $K_{b,S}$ is the middle third Cantor set, which we simply denote by K. Furthermore, recall that the Hausdorff dimension of $K_{b,S}$ is given by

$$\dim K_{b,S} = \frac{\log \sharp S}{\log b}.$$

As a corollary of a more general result, Levesley, Salp and Velani [10] established that, for $v \geq 0$,

$$\dim\{\xi \in K : v_3(\xi) \geq v\} = \dim\{\xi \in K : v_3(\xi) = v\} = \frac{\log 2}{\log 3} \cdot \frac{1}{1+v}. \tag{6.1}$$

A suitable modification of the proofs of Theorems 1.1 and 1.2 allows us to extend (6.1) as follows.

Theorem 6.1. *Let $b \geq 3$ be an integer and S a subset of $\{0, 1, \ldots, b-1\}$ of cardinality at least two and containing at least one of the digits 0 and $b-1$. Let θ and $\hat{v}$ be positive real numbers with $\hat{v} < 1$. If $\theta < 1/(1-\hat{v})$, then the set*

$$\{\xi \in K_{b,S} : \hat{v}_b(\xi) \geq \hat{v}\} \cap \{\xi \in K_{b,S} : v_b(\xi) = \theta\hat{v}\}$$

is empty. Otherwise, we have

$$\dim(\{\xi \in K_{b,S} : \hat{v}_b(\xi) = \hat{v}\} \cap \{\xi \in K_{b,S} : v_b(\xi) = \theta\hat{v}\}) = \frac{\log \sharp S}{\log b} \cdot \frac{\theta - 1 - \theta\hat{v}}{(1+\theta\hat{v})(\theta-1)}. \tag{6.2}$$

Furthermore,

$$\dim\{\xi \in K_{b,S} : \hat{v}_b(\xi) = 1\} = 0.$$

and

$$\dim\{\xi \in K_{b,S} : \hat{v}_b(\xi) \geq \hat{v}\} = \dim\{\xi \in K_{b,S} : \hat{v}_b(\xi) = \hat{v}\} = \frac{\log \sharp S}{\log b} \cdot \Big(\frac{1-\hat{v}}{1+\hat{v}}\Big)^2.$$

The assumption that S contains at least one of the digits 0 and $b-1$ is necessary, since, otherwise, we trivially have $v_b(\xi) = 0$ for every ξ in $K_{b,S}$.

For additional results on Diophantine approximation on Cantor sets, the reader may consult Chapter 7 of [4] and the references quoted therein.

Proof of Theorem 6.1. We follow step by step the proof of Theorems 1.1 and 1.2. To prove that the right hand side of (6.2) is an upper bound for the dimension, we use the same covering argument, but, instead of (2.9), we have to consider the series

$$\sum_{N\geq 1} (2N)^{C\log N}\,(\sharp S)^{N(1+\varepsilon')(\theta-1-\theta\hat{v})/(\theta-1)}\,b^{-(1+\theta\hat{v})(1-\varepsilon')Ns}.$$

As for the lower bound, we again consider a Bernoulli measure and we distribute the mass uniformly among the elements of S. Also, if 1 does not belong to S, we cannot take $a_{n_k}, a_{m_k}, \ldots$ equal to 1 and we then choose them equal to a non-zero element of S. We omit the details. □

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