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Badly approximable numbers and Littlewood-type problems

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Abstract

We establish that the set of pairs (α, β) of real numbers such that

$$\liminf_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| > 0,$$

where $\| \cdot \|$ denotes the distance to the nearest integer, has full Hausdorff dimension in $\mathbf{R}^2$. Our proof rests on a method introduced by Peres and Schlag, that we further apply to various Littlewood-type problems.

1. *Introduction*

A famous open problem in simultaneous Diophantine approximation, called the Littlewood conjecture [**15**], claims that, for any given pair (α, β) of real numbers, we have

$$\inf_{q \geqslant 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{1·1}$$

where $\| \cdot \|$ denotes the distance to the nearest integer. Throughout the present paper, we denote by $\boldsymbol{Bad}$ the set of badly approximable numbers, that is,

$$\boldsymbol{Bad} = \{\alpha \in \mathbf{R} : \inf_{q \geqslant 1} q \cdot \|q\alpha\| > 0\},$$

and we recall that $\boldsymbol{Bad}$ has Lebesgue measure zero and full Hausdorff dimension [**13**]. Consequently, (1·1) holds for almost every pair (α, β) of real numbers. Recently, this result was considerably improved by Einsiedler, Katok and Lindenstrauss [**9**], who established that the set of pairs (α, β) for which (1·1) do not hold has Hausdorff dimension zero; see also [**25**] for a weaker statement, and [**5**, section 10·1] for a survey of related results.

Another metrical statement connected to the Littlewood conjecture was established by Gallagher [**12**] in 1962 and can be formulated as follows (see e.g. [**3**]).

THEOREM G. *Let n be a positive integer. Let $\Psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a non-increasing function. The set of points $(x_1, \ldots, x_n)$ in $\mathbf{R}^n$ such that there are infinitely many positive integers* *q satisfying*

$$\prod_{i=1}^{n} \|qx_i\| < \Psi(q)$$

has full Lebesgue measure if the sum

$$\sum_{h \geqslant 1} \Psi(h)^n (\log h)^{n-1}$$

diverges, and has zero Lebesgue measure otherwise.

In particular, it follows from Gallagher's theorem that

$$\liminf_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| = 0 \tag{1·2}$$

for almost every pair (α, β) of real numbers. The main purposes of the present note are to establish the existence of exceptional pairs (α, β) which do not satisfy (1·2) – a result first proved in [**22**] –, and to prove that the set of these pairs has full Hausdorff dimension in $\mathbf{R}^2$. We further consider various questions closely related to the Littlewood conjecture.

Our main results are stated in Section 2 and proved in Sections 4 and 5, with the help of auxiliary lemmas gathered in Section 3. Several additional results are given without proofs in Section 6.

Throughout this paper, $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the greatest integer less than or equal to x and the smallest integer greater than or equal to x, respectively.

2. *Main results*

Our first result shows that there are many pairs (α, β) of real numbers that are not well multiplicatively approximable.

THEOREM 1. *For every real number α in **Bad**, the set of real numbers β such that*

$$\liminf_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| > 0 \tag{2·1}$$

has full Hausdorff dimension.

The proof of Theorem 1 uses a method introduced by Peres and Schlag [**24**], which was subsequently applied in [**18–22**].

Since the set ***Bad*** has full Hausdorff dimension, the next result follows from Theorem 1 by an immediate application of [**11**, corollary 7·12].

THEOREM 2. *The set of pairs (α, β) of real numbers satisfying*

$$\liminf_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| > 0$$

has full Hausdorff dimension in $\mathbf{R}^2$.

Theorem 1 can be viewed as a complement to the following result of Pollington and Velani [**25**].

THEOREM PV. *For every real number α in **Bad**, there exists a subset $G(\alpha)$ of **Bad** with full Hausdorff dimension such that, for any β in $G(\alpha)$, there exist arbitrarily large integers q satisfying*

$$q \cdot (\log q) \cdot \|q\alpha\| \cdot \|q\beta\| \leqslant 1.$$

In [**1**], the authors constructed explicitly for every α in $\boldsymbol{Bad}$ uncountably many β in $\boldsymbol{Bad}$ such that the pair (α, β) satisfies (1·1), and even a strong form of this inequality. It would be very interesting to construct explicit examples of pairs of real numbers that satisfy (2·1).

A modification of an auxiliary lemma yields a slight improvement on Theorem 1.

THEOREM 3. *Let a be a real number with $0 < a < 1$. For every real number α in $\boldsymbol{Bad}$, the set of real numbers β such that*

$$\liminf_{q\to+\infty} q \cdot (\log q)^{2-a} \cdot (\log 1/\|q\alpha\|)^a \cdot \|q\alpha\| \cdot \|q\beta\| > 0$$

has full Hausdorff dimension.

Theorem 3 is stronger than Theorem 1 since, for every α in $\boldsymbol{Bad}$, there exists a positive real number δ such that $\log(1/\|q\alpha\|) \leqslant \delta \log q$ holds for every integer $q \geqslant 2$.

Cassels and Swinnerton–Dyer [**8**] proved that (1·1) is equivalent to the equality

$$\inf_{(x,y)\in\mathbf{Z}\times\mathbf{Z}\setminus\{(0,0)\}} \max\{|x|, 1\} \cdot \max\{|y|, 1\} \cdot \|x\alpha + y\beta\| = 0,$$

and used it to show that (1·1) holds if α and β belong to the same cubic number field (see also [**23**]). In this context, we have the following metrical result, extracted from [**4**, page 455]. For integers $q_1, \dots, q_n$, set

$$\Pi(q_1, \dots, q_n) = \prod_{i=1}^{n} \max\{1, |q_i|\}.$$

THEOREM BKM. *Let n be a positive integer. Let $\Psi: \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a non-increasing function. The set of points $(x_1, \dots, x_n)$ in $\mathbf{R}^n$ such that there are infinitely many integers $q_1, \dots, q_n$ satisfying*

$$||q_1x_1 + \cdots + q_nx_n|| < \Psi\big(\Pi(q_1, \dots, q_n)\big) \tag{2·2}$$

has full Lebesgue measure if the sum

$$\sum_{h\geqslant 1} \Psi(h)\,(\log h)^{n-1} \tag{2·3}$$

diverges, and has zero Lebesgue measure otherwise.

For $n \geqslant 2$, there is no known example of points $(x_1, \dots, x_n)$ in $\mathbf{R}^n$ and of a function Ψ as in Theorem BKM such that the sum (2·3) diverges and (2·2) has only finitely many solutions. The Peres–Schlag method allows us to show that such examples do exist.

THEOREM 4. *The set of pairs (α, β) of real numbers satisfying*

$$\liminf_{x,y\geqslant 0} \max\{2, |xy|\} \cdot \|x\alpha + y\beta\| \cdot (\log \max\{2, |xy|\})^2 > 0$$

has full Hausdorff dimension in $\mathbf{R}^2$.

The proof of Theorem 4 is briefly outlined in Section 5. Note that Theorem 4 (resp. Theorem 1) does not follow from Theorem 1 (resp. Theorem 4) by some transference principle.

In analogy with the Littlewood conjecture, de Mathan and Teulié [**17**] proposed recently a 'mixed Littlewood conjecture'. For any prime number p, the usual p-adic absolute value $|\cdot|_p$ is normalized in such a way that $|p|_p = p^{-1}$.

De Mathan–Teulié conjecture. For every real number α and every prime number p, we have

$$\inf_{q \geqslant 1} q \cdot \|q\alpha\| \cdot |q|_p = 0.$$

Despite several recent results [**6**, **10**], this conjecture is still unsolved. The following metrical statement, established in [**7**], should be compared with Theorem G.

THEOREM BHV. *Let k be a positive integer. Let $p_1, \ldots, p_k$ be distinct prime numbers. Let $\Psi\colon \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a non-increasing function. The set of real numbers α such that there are infinitely many positive integers q satisfying*

$$\|q\alpha\| \cdot |q|_{p_1} \cdots |q|_{p_k} < \Psi(q)$$

has full Lebesgue measure if the sum

$$\sum_{h \geqslant 1} \Psi(h) \, (\log h)^k$$

diverges, and has zero Lebesgue measure otherwise.

As an immediate consequence of Theorem BHV, we get that, for every prime number p, almost every real number α satisfies

$$\inf_{q \geqslant 2} q \cdot (\log q)^2 \cdot (\log \log q) \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{2·4}$$

The method of proof of Theorem 1 allows us to confirm the existence of real numbers for which (2·4) does not hold.

THEOREM 5. *Let a be a real number with $0 \leqslant a < 1$. For every prime number p, the set of real numbers α such that*

$$\liminf_{q \to +\infty} q \cdot (\log q)^{2-a} \cdot \|q\alpha\| \cdot |q|_p \cdot (\log 2/|q|_p)^a > 0$$

has full Hausdorff dimension.

We display an immediate consequence of Theorem 5.

COROLLARY 1. *For every prime number p, the set of real numbers α such that*

$$\liminf_{q \to +\infty} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot |q|_p > 0$$

has full Hausdorff dimension.

In the present note, we have restricted our attention to 2-dimensional questions. However, our method can be successfully applied to prove that, given an integer $n \geqslant 2$, there are real numbers $\alpha_1, \ldots, \alpha_n$ such that

$$\liminf_{q \to +\infty} q \cdot (\log q)^n \cdot \|q\alpha_1\| \cdots \|q\alpha_n\| > 0,$$

as well as real numbers $\beta_1, \ldots, \beta_n$ such that

$$\liminf_{x_1, \ldots, x_n \geqslant 0} \max\{2, |x_1 \ldots x_n|\} \cdot \|x_1\beta_1 + \ldots + x_n\beta_n\| \cdot (\log \max\{2, |x_1 \ldots x_n|\})^n > 0$$

This will be the subject of subsequent work by E. Ivanova.

3. *Auxiliary results*

The original method of Peres and Schlag is a construction of nested intervals. A useful tool for estimating from below the Hausdorff measure of a Cantor set is the mass distribution principle, which we recall now.

We consider a set $\mathcal{K}$ included in a bounded interval E, and defined as follows. Set $\mathcal{E}_0 = E$ and assume that, for any positive integer k, there exists a finite family $\mathcal{E}_k$ of disjoint compact intervals in E such that any interval U belonging to $\mathcal{E}_k$ is contained in exactly one of the intervals of $\mathcal{E}_{k-1}$ and contains at least two intervals belonging to $\mathcal{E}_{k+1}$. Suppose also that the maximum of the lengths of the intervals in $\mathcal{E}_k$ tends to 0 when k tends to infinity. For $k \geqslant 0$, denote by E_k the union of the intervals belonging to the family $\mathcal{E}_k$, and set

$$\mathcal{K} := \bigcap_{k=1}^{+\infty} E_k.$$

LEMMA 1. *Keep the same notation as above. Assume further that there exists a positive integer k_0 such that, for any $k \geqslant k_0$, each interval of E_{k-1} contains at least $m_k \geqslant 2$ intervals of E_k, these being separated by at least ε_k, where $0 < \varepsilon_{k+1} < \varepsilon_k$. We then have*

$$\dim \mathcal{K} \geqslant \liminf_{k\to+\infty} \frac{\log(m_1 \dots m_{k-1})}{-\log(m_k\varepsilon_k)}.$$

Proof. This is [**11**, example 4·6], see also [**5**, proposition 5·2].

LEMMA 2. *Let α be in **Bad**. There exists a positive constant $C(\alpha)$ such that, for every integer $q \geqslant 2$, we have*

$$\sum_{x=q}^{q^3} \frac{1}{||\alpha x||\, x \log_2^2 x} \leqslant C(\alpha).$$

Proof. This is a straightforward consequence of [**14**, example 3·2, page 124], where it is established that there exists a positive constant $C_1(\alpha)$ such that

$$\sum_{x=1}^{m} \frac{1}{||\alpha x||\, x} \leqslant C_1(\alpha)(\log m)^2,$$

for all positive integers m.

Theorem 3 depends on the following refinement of Lemma 2.

LEMMA 3. *Let α be in **Bad**. Let a be a real number with $0 < a < 1$. There exists a positive constant $C(\alpha)$ such that, for every integer $q \geqslant 2$, we have*

$$\sum_{x=q}^{q^3} \frac{1}{||\alpha x||\, x\, (\log 1/\|x\alpha\|)^a \cdot (\log x)^{2-a}} \leqslant C(\alpha).$$

Proof. Let $(p_j/q_j)_{j\geqslant 0}$ denote the sequence of convergents to α. Let m (resp. n) be the largest (resp. the smallest) integer j such that $q_j \leqslant q$ (resp. $q_j \geqslant q^3$). As the sequence $(q_j)_{j\geqslant 0}$ grows exponentially fast, we have

$$\log q \ll n \ll m \ll \log q,$$

where, as throughout this proof, the numerical constants implied by $\ll$ depend only on α.

Let j be an integer satisfying $m \leqslant j < n$ and consider

$$S_j := \sum_{x=q_j}^{q_{j+1}} \frac{1}{||\alpha x||\, x\, (\log 1/\|x\alpha\|)^a}.$$

Denote by $y_1 < y_2 < \cdots < y_{q_{j+1}}$ the increasing sequence of the q_{j+1} points $\{\alpha x\}$, $x = 1, \ldots, q_{j+1}$. It follows from the Three Distance Theorem (see, e.g. [**2**, section 3]) and the fact that α is badly approximable that

$$\frac{x}{q_{j+1}} \ll y_x \ll \frac{x}{q_{j+1}}, \quad \text{for } x = 1, \ldots, q_{j+1}.$$

Consequently,

$$\begin{aligned} S_j &\ll \frac{1}{q_j} \sum_{x=1}^{q_{j+1}} \frac{1}{||\alpha x||\, (\log 1/\|x\alpha\|)^a} \ll \frac{1}{q_j} \sum_{x=1}^{q_{j+1}/2} \frac{(q_{j+1}/x)}{(\log (q_{j+1}/x))^a} \\ &\ll \frac{q_{j+1}}{q_j} \int_2^{q_{j+1}} \frac{du}{u(\log u)^{1-a}} \ll (\log q_j)^{1-a}, \end{aligned}$$

since q_{j+1}/q_j is bounded from above by an absolute constant depending only on α. Now,

$$\sum_{j=m}^{n} S_j \ll (\log q)^{2-a},$$

which proves the lemma.

The key tool for the proof of Theorem 5 is Lemma 4 below.

LEMMA 4. *Let p be a prime number. Let a be a real number with $0 \leqslant a < 1$. There exists a positive constant $C(a, p)$ such that, for every integer $q \geqslant 2$, we have*

$$\sum_{x=q}^{q^3} \frac{1}{x \cdot |x|_p\, (\log (2/|x|_p))^a \cdot (\log x)^{2-a}} \leqslant C(a, p).$$

Proof. Observe that

$$\sum_{x=q}^{q^3} \frac{1}{x \cdot |x|_p \cdot (\log (2/|x|_p))^a} \ll \sum_{j=0}^{3\log q} \sum_{x=\lceil q/p^j \rceil}^{\lfloor q^3/p^j \rfloor} \frac{1}{x(j+1)^a}.$$

Consequently,

$$\sum_{x=q}^{q^3} \frac{1}{x \cdot |x|_p \cdot (\log (2/|x|_p))^a} \ll \sum_{j=0}^{3\log q} \frac{\log q}{(j+1)^a} \ll (\log q)^{2-a},$$

and the lemma is proved.

4. *Proof of Theorem* 1

Let α be in ***Bad*** and δ be a positive real number satisfying

$$q \cdot ||q\alpha|| \geqslant \delta, \quad \text{for every } q \geqslant 1. \tag{4·1}$$

Let ε be such that

$$0 < \varepsilon < \left(2^{10} C(\alpha)\right)^{-1}, \tag{4·2}$$

where $C(\alpha)$ is given by Lemma 2.

We follow a method introduced by Peres and Schlag [**24**]. First, we construct 'dangerous' sets of real numbers. These sets depend on α, but, to simplify the notation, we choose not to indicate this dependence. For integers x and y with $x \geqslant 2$ and $0 \leqslant y \leqslant x$, define

$$E(x, y) = \left[\frac{y}{x} - \frac{\varepsilon}{||\alpha x|| x^2 \log_2^2 x}, \frac{y}{x} + \frac{\varepsilon}{||\alpha x|| x^2 \log_2^2 x}\right] \tag{4·3}$$

and

$$E(x) = \bigcup_{y=0}^{x} \big(E(x, y) \cap [0, 1] \big). \tag{4·4}$$

Set also

$$l_0 = 0, \quad l_x = \lfloor \log_2(||\alpha x|| x^2 \log_2^2 x/(2\varepsilon)) \rfloor, \quad \text{for } x \in \mathbf{Z}_{\geqslant 1}. \tag{4·5}$$

Each interval from the union $E(x)$ defined in (4·4) can be covered by an open dyadic interval of the form

$$\left(\frac{b}{2^{l_x}}, \frac{b+2}{2^{l_x}}\right), \quad b \in \mathbf{Z}_{\geqslant 0}.$$

Let $A(x)$ be the smallest union of all such dyadic intervals which covers the whole set $E(x)$ and put

$$A^c(x) = [0, 1] \setminus A(x).$$

Observe that $A^c(x)$ is a union of closed intervals of the form

$$\left[\frac{a}{2^{l_x}}, \frac{a+1}{2^{l_x}}\right], \quad a \in \mathbf{Z}_{\geqslant 0}.$$

Let q_0 be an integer such that

$$q_0 \geqslant (100\varepsilon)^3 \quad \text{and} \quad ||q_0 \alpha|| \geqslant 1/4. \tag{4·6}$$

For $q \geqslant q_0$, define

$$B_q = \bigcap_{x=q_0}^{q} A^c(x).$$

The sets B_q, $q \geqslant q_0$, are closed and nested. Our aim is to show inductively that they are non-empty. Set $L_0 = l_0$ and

$$q_k := q_0^{3^k}, \quad L_k = \lfloor \log_2(q_k^2 \log_2^2 q_k/(4\varepsilon)) \rfloor, \quad k \geqslant 1. \tag{4·7}$$

Observe that $l_x \leqslant L_k$ when $x \leqslant q_k$.

For every integer $k \geqslant 0$ we construct inductively subsets C_{q_k} and D_{q_k} of B_{q_k} with the following property (P_k):

The set C_{q_k} is the union of $2^{-5k-3+L_k}$ intervals of length 2^{-L_k}, separated by at least 2^{-L_k}, and such that at least $2^{-5k-5+L_k}$ among them include at least $2^{L_{k+1}-L_k-3}$ intervals composing $B_{q_{k+1}}$, which are also separated by at least $2^{-L_{k+1}}$. Let denote by $C_{q_{k+1}}$ (resp. by D_{q_k}) the *union of* $2^{-5(k+1)-3+L_{k+1}}$ *of these intervals (resp. of the corresponding* $2^{-5k-5+L_k}$ *intervals from* C_{q_k}*). In particular, we have* $\mathrm{mes}(C_{q_k}) = 4\mathrm{mes}(D_{q_k}) = 2^5\mathrm{mes}(C_{q_{k+1}})$.

We deduce from (4·2), (4·3) and Lemma 2 that

$$\mathrm{mes}(B_{q_1}) \geqslant 1 - \sum_{x=q_0}^{q_1} \mathrm{mes}\big(A(x)\big) \geqslant 31/32.$$

Consequently, B_{q_1} is the union of at least 2^{L_1-1} intervals of length 2^{-L_1}. By (4·6), the set B_{q_0} is the union of at least 2^{L_0-1} intervals of length 2^{-L_0}. This allows us to define the sets C_{q_0}, D_{q_0} and C_{q_1}. This proves (P_0).

Let k be a non-negative integer such that (P_k) holds, and consider the set $B'_{q_{k+2}} := C_{q_{k+1}} \cap B_{q_{k+2}}$. Observe that

$$B'_{q_{k+2}} = C_{q_{k+1}} \setminus \Bigg(\bigcup_{x=q_{k+1}+1}^{q_{k+2}} A(x) \Bigg),$$

hence

$$\mathrm{mes}(B'_{q_{k+2}}) \geqslant \mathrm{mes}(C_{q_{k+1}}) - \sum_{x=q_{k+1}+1}^{q_{k+2}} \mathrm{mes}\big(C_{q_{k+1}} \cap A(x)\big). \tag{4·8}$$

By construction, the set C_{q_k} can be written as a union, say

$$C_{q_k} = \bigcup_{\nu=1}^{T_{q_k}} J_\nu,$$

of T_{q_k} dyadic intervals J_ν of the form

$$\left[\frac{a}{2^{L_k}}, \frac{a+1}{2^{L_k}}\right], \quad a \in \mathbf{Z}_{\geqslant 0},$$

where L_k is given by (4·7). Let $x \geqslant q_k^3$ be an integer. Since, by (4·6),

$$2^{L_k} \leqslant \frac{q_k^2 \log_2^2 q_k}{4\varepsilon} \leqslant \frac{q_k^3}{2} \leqslant \frac{x}{2},$$

each interval J_ν contains at least the rationals y/x, $(y+1)/x$ for some integer y, and we infer from (4·3) that

$$\mathrm{mes}(J_\nu \cap A(x)) \leqslant \frac{2^4 \varepsilon}{||\alpha x||\, x \log_2^2 x} \times \mathrm{mes}(J_\nu). \tag{4·9}$$

Summing (4·9) from $\nu = 1$ to $\nu = T_{q_k}$, we get

$$\mathrm{mes}\big(C_{q_k} \cap A(x)\big) \leqslant \frac{2^4 \varepsilon}{||\alpha x||\, x \log_2^2 x} \times \mathrm{mes}(C_{q_k}). \tag{4·10}$$

It then follows from (4·10) that

$$\begin{aligned} \mathrm{mes}(C_{q_{k+1}} \cap A(x)) &\leqslant \mathrm{mes}(C_{q_k} \cap A(x)) \\ &\leqslant \frac{2^4 \varepsilon}{||\alpha x||\, x \log_2^2 x} \times \mathrm{mes}(C_{q_k}) \leqslant \frac{2^9 \varepsilon}{||\alpha x||\, x \log_2^2 x} \times \mathrm{mes}(C_{q_{k+1}}). \end{aligned}$$

Combined with (4·8) and Lemma 2, this gives

$$\operatorname{mes}(B'_{q_{k+2}}) \geqslant \big(\operatorname{mes}(C_{q_{k+1}})\big)\left(1 - \sum_{x=q_{k+1}+1}^{q_{k+2}} \frac{2^9 \varepsilon}{||\alpha x||\, x \log_2^2 x}\right) \geqslant \frac{\operatorname{mes}(C_{q_{k+1}})}{2}.$$

Thus, at least one quarter of the intervals composing $C_{q_{k+1}}$ contains at least $2^{L_{k+2}-L_{k+1}-2}$ intervals composing $B'_{q_{k+2}}$, thus at least $2^{L_{k+2}-L_{k+1}-3}$ intervals composing $B'_{q_{k+2}}$, if we impose that these intervals are mutually distant by at least $2^{-L_{k+2}}$. This allows us to define the sets $C_{q_{k+2}}$ and $D_{q_{k+1}}$ with the required properties. This proves (P_{k+1}).

It then follows that the set

$$\mathcal{K} := \bigcap_{k \geqslant 0} D_{q_k}$$

is non-empty. By construction, every point β in this set avoids all the intervals $E(x, y)$ with $x \geqslant q_0$, thus, the pair (α, β) satisfies (2·1).

To establish that the set $\mathcal{K}$ has full Hausdorff dimension, we apply Lemma 1 with

$$m_k = 2^{L_{k+1}-L_k-5} \quad \text{and} \quad \varepsilon_k := 2^{-L_{k+1}}.$$

Note that

$$\frac{\log (m_1 \dots m_{k-1})}{-\log (m_k \varepsilon_k)} \geqslant \frac{\log (32^{-k} 2^{L_k})}{-\log (2^{-L_k+5})}$$

We infer from (4·1), (4·5) and (4·7) that

$$2^{L_k} \geqslant \delta q_0^{3^k}.$$

Consequently,

$$\lim_{k \to +\infty} \frac{\log (m_1 \dots m_{k-1})}{-\log (m_k \varepsilon_k)} = 1,$$

and it follows from Lemma 1 that the set $\mathcal{K}$ has full Hausdorff dimension. This completes the proof of our theorem.

5. *Proofs of Theorems* 3, 4 *and* 5

The proofs of Theorems 3 and 5 follow exactly the same steps as that of Theorem 1. Instead of the intervals

$$E(x, y) = \left[\frac{y}{x} - \frac{\varepsilon}{||\alpha x|| x^2 \log_2^2 x}, \frac{y}{x} + \frac{\varepsilon}{||\alpha x|| x^2 \log_2^2 x}\right],$$

we use respectively the intervals

$$\left[\frac{y}{x} - \frac{\varepsilon}{||\alpha x|| x^2 (\log_2 x)^{2-a} (\log 1/||\alpha x||)^a}, \frac{y}{x} + \frac{\varepsilon}{||\alpha x|| x^2 (\log_2 x)^{2-a} (\log 1/||\alpha x||)^a}\right]$$

and

$$\left[\frac{y}{x} - \frac{\varepsilon}{|x|_p x^2 (\log_2 x)^{2-a} (\log 2/|x|_p)^a}, \frac{y}{x} + \frac{\varepsilon}{|x|_p x^2 (\log_2 x)^{2-a} (\log 2/|x|_p)^a}\right].$$

Furthermore, we apply Lemmas 3 and 4 in place of Lemma 2.

For the proof of Theorem 4, we work directly in the plane. The idea is the following. For a triple (x, y, z) of integers and a positive ε, the inequality $|xX + yY + z| \leqslant \varepsilon$ defines a strip composed of points (X, Y) close to the line $xX + yY + z = 0$. Since we are working in the unit square, to a given pair (x, y) of integers corresponds a unique z, and the length of the intersection of the line with the unit square is at most equal to $\sqrt{2}$. Setting

$$\varepsilon_{x,y} = \frac{\varepsilon}{|xy| \log^2 |xy|},$$

for a given (very small) positive ε, the strips $|xX + yY + z| \leqslant \varepsilon_{x,y}$ play the same role as the intervals (4·3) in the proof of Theorem 1.

Since, for every large integer q, we have

$$\begin{aligned}\sum_{q \leqslant xy \leqslant q^3} \varepsilon_{x,y} &\ll \sum_{x=1}^{q^3} \sum_{y=\lfloor q/x \rfloor}^{\lfloor q^3/x \rfloor} \frac{\varepsilon}{|xy| \log^2 q} \\ &\ll \sum_{x=1}^{q^3} \frac{\varepsilon}{x \log q} \ll \varepsilon,\end{aligned}$$

the Peres–Schlag method can be applied as in the proof of Theorem 1. We omit the details.

6. *Further results*

We gather in the present section several results that can be obtained with the same method as in the proof of Theorem 1, that is, by combining the Peres–Schlag method with the mass distribution principle.

A result on lacunary sequences.

THEOREM 6. *Let M be a positive real number and $(t_j)_{j \geqslant 1}$ be a sequence such that $t_{j+1}/t_j > 1 + 1/M$ for $j \geqslant 1$. Let c be a real number with $0 < c < 1/10$. Let ε be a positive real number. Then, the Hausdorff dimension of the set*

$$\{\xi \in [0, 1] : \forall n \geqslant 1, ||\xi t_n|| \geqslant c/(M \log M)\}$$

is at least $1 - \varepsilon$ if M is sufficiently large.

Theorem 6 complements the results from [**18**, **24**].

A result on sequences with polynomial growth.

THEOREM 7. *Let C_1, C_2 and γ be positive real numbers. Let $(t_n)_{n \geqslant 1}$ be a sequence of real numbers such that*

$$C_1 n^\gamma \leqslant t_n \leqslant C_2 n^\gamma, \quad \text{for } n \geqslant 1.$$

Then, there exist a positive C and an integer n_0 such that the set

$$\bigcap_{n \geqslant n_0} \left\{ \xi \in \mathbf{R} : ||\xi t_n|| > \frac{C}{n \log n} \right\} \tag{6·1}$$

has full Hausdorff dimension.

This improves [**20**, theorem 1], where it is established that the Hausdorff dimension of the set (6·1) is at least $\gamma/(\gamma+1)$.

As an immediate application, we get that the set of real numbers ξ for which

$$\liminf_{n\to+\infty}\ n(\log n)||\xi n^2|| > 0$$

has full Hausdorff dimension.

We have stated homogeneous statements, but the method is flexible enough to allow us to deal with inhomogeneous approximation.

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