DISTRIBUTION OF FULL CYLINDERS AND THE DIOPHANTINE PROPERTIES OF THE ORBITS IN β-EXPANSIONS

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ABSTRACT. Let $\beta > 1$ be a real number. Let T_β denote the β-transformation on $[0, 1]$. A cylinder of order n is a set of real numbers in $[0, 1]$ having the same first n digits in their β-expansion. A cylinder is called full if it has maximal length, i.e., if its length is equal to β^{-n}. In this paper, we show that full cylinders are well distributed in $[0, 1]$ in a suitable sense. As an application to the metrical theory of β-expansions, we determine the Hausdorff dimension of the set

$$\Big\{x \in [0, 1] : |T_\beta^n x - z_n| < e^{-S_n f(x)} \ \text{ for infinitely many } \ n \in \mathbb{N}\Big\},$$

where $\{z_n\}_{n\geq 1}$ is a sequence of real numbers in $[0, 1]$, the function $f : [0, 1] \to \mathbb{R}^+$ is continuous, and $S_n f(x)$ denotes the ergodic sum $f(x) + \ldots + f(T_\beta^{n-1} x)$.

1. INTRODUCTION

1.1. β-expansions. Let $\beta > 1$ be a real number. Let $T_\beta : [0, 1] \to [0, 1]$ be the *β-transformation* defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor,$$

where $\lfloor\cdot\rfloor$ denote the integer part of a real number. It is well-known [16] that every real number $x \in [0, 1]$ can be uniquely expanded into a finite or an infinite series:

$$x = \frac{\varepsilon_1(x,\beta)}{\beta} + \ldots + \frac{\varepsilon_n(x,\beta) + T_\beta^n x}{\beta^n} = \sum_{n=1}^{\infty} \frac{\varepsilon_n(x,\beta)}{\beta^n}, \tag{1.1}$$

where, for $n \geq 1$,

$$\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1} x \rfloor$$

is called the nth digit of x. Sometimes we identify x with its *digit sequence*

$$\varepsilon(x,\beta) := (\varepsilon_1(x,\beta), \ldots, \varepsilon_n(x,\beta), \ldots)$$

and call also the digit sequence $\varepsilon(x,\beta)$ the *β-expansion* of x. We call the system $([0, 1], T_\beta)$ the β-dynamical system.

It is well-known that the β-dynamical system is, in general, not a subshift of finite type with mixing properties. This causes difficulties in studying metrical questions related to β-expansions.

For an admissible sequence $(\varepsilon_1, \ldots, \varepsilon_n)$, i.e. a prefix of the digit sequence of some $x \in [0, 1)$, we define the cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ of order n by

$$I_n(\varepsilon_1, \ldots, \varepsilon_n) := \{x \in [0, 1) : \varepsilon_j(x,\beta) = \varepsilon_j, \ \text{ for } \ j = 1, \ldots, n\}.$$

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We write $I_n(x)$ for the cylinder of order n containing x and $|I_n(x)|$ for the length of $I_n(x)$. It follows from the definition of the β-expansion that the length of a cylinder always satisfies

$$|I_n(\varepsilon_1, \ldots, \varepsilon_n)| \le \beta^{-n}. \tag{1.2}$$

We stress that there is no nontrivial universal lower bound for the length of a cylinder, which can be much smaller than β^{-n}.

From the ergodic theorem, it is well known [13] that for almost all x,

$$\lim_{n\to\infty} \frac{-\log_\beta |I_n(x)|}{n} = 1,$$

where $\log_\beta$ denotes the logarithm with respect to the base β. This means that, in some sense, almost all cylinders are of almost maximal length.

Cylinders with maximal length have very good properties; see for example Lemma 3.2 below. Thus, we would like to know whether there exist cylinders with maximal length, which cylinders have maximal length and how they are distributed in the unit interval $[0, 1]$.

Definition 1.1 (Full cylinder)**.** *A cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ is called* full *if it has maximal length, i.e., if*

$$|I_n(\varepsilon_1, \ldots, \varepsilon_n)| = \beta^{-n}.$$

The properties of full cylinders were firstly investigated in [6]. In the present paper, we give a full characterization of full cylinders and investigate the distribution of full cylinders in the unit interval.

Theorem 1.2. *For $n \ge 1$, among every $(n + 1)$ consecutive cylinders of order n, there exists at least one full cylinder.*

Theorem 1.2 enables us to prove a modified mass distribution principle to study the Hausdorff dimension of sets defined in terms of β-expansions. The reader is referred to Falconer's book [5] for the definition of Hausdorff dimension and the "Mass distribution principle", which is a classical tool to obtain a lower bound for the Hausdorff dimension of a set.

Proposition 1.3 (Modified mass distribution principle)**.** *Let E be a Borel measurable set in $[0, 1]$ and μ be a Borel measure with $\mu(E) > 0$. Assume that there exist a positive constant $c > 0$ and an integer n_0 such that, for any $n \ge n_0$ the measure of any cylinder I_n of order n satisfies $\mu(I_n) \le c|I_n|^s$. Then, $\dim_H E \ge s$.*

In the classical form of the mass distribution principle [5, Proposition 4.2], one needs to estimate the measure of an arbitrary ball, while the above proposition tells us that, for β-expansions, it is sufficient to consider only the measure of cylinders. This will simplify the argument in determining the Hausdorff dimension of sets defined in terms of β-expansions.

To give an application of Proposition 1.3 to the metrical theory of β-expansions, we determine the Hausdorff dimension of the following shrinking target set:

$$\mathfrak{S}(f) := \Big\{ x \in [0, 1] : |T_\beta^n x - z_n| < e^{-S_n f(x)} \ \text{ for infinitely many } \ n \in \mathbb{N} \Big\},$$

where $\{z_n\}_{n\ge1}$ is a sequence of real numbers in $[0, 1]$, the function $f : [0, 1] \to \mathbb{R}^+$ is continuous, and $S_n f(x)$ denotes the ergodic sum $f(x) + \ldots + f(T_\beta^{n-1} x)$. For the background and more results on shrinking target problems, the reader is referred to [1, 2, 3, 4, 8, 9, 10, 11, 12, 20, 21, 24, 25] and the references quoted therein.

The special case when $z_n = z$ is a constant function and $f(x) = b$ for all $x \in [0, 1]$ was investigated in [14, 19]. The following result is much more general.

Theorem 1.4. *Let $\{z_n\}_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$ and $f : [0, 1] \to \mathbb{R}^+$ be a continuous function. The Hausdorff dimension of the set $\mathfrak{S}(f)$ is the unique solution s to the pressure equation*

$$P\left(T_\beta, -s(f + \log\beta)\right) = 0,$$

where $P(T_\beta, \phi)$ denotes the pressure function related to the potential ϕ.

We display a particular instance of Theorem 1.4. Throughout this paper, $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension.

Corollary 1.5. *Let $\{z_n\}_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$ and b be a positive real number. Then,*

$$\dim_H \left\{x \in [0, 1] : |T_\beta^n x - z_n| < \beta^{-bn} \textit{ for infinitely many } n \in \mathbb{N}\right\} = \frac{1}{1 + b}.$$

Actually, very similar ideas allow us to extend a result of [19] as follows.

Theorem 1.6. *Let ψ be a positive function defined on the set of positive integers. Let $\{z_n\}_{n\geq 1}$ be a sequence of real numbers in $[0, 1]$. Then,*

$$\dim_H \left\{x \in [0, 1] : |T_\beta^n x - z_n| < \psi(n) \textit{ for infinitely many } n \in \mathbb{N}\right\} = \frac{1}{1 + b},$$

where

$$b = \liminf_{n\to+\infty} \frac{-\log_\beta \psi(n)}{n}.$$

We omit the proof of Theorem 1.6.

Remark 1. We stress that the Hausdorff dimensions determined in the above theorem and corollary do not depend on the choice of the sequence $\{z_n\}_{n\geq 1}$. This is not always the case for general systems; see Reeve [17] for an example of a conformal iterated function system showing that sometimes the dimension depends on the centers of the targets.

For more dimensional results related to β-expansions, the reader is referred to the papers of C.-E Pfister and W. G. Sullivan [15], J. Schmeling [18], D. Thompson [22], D. Färm, T. Persson and J. Schmeling [7] and the references therein.

The paper is organized as follows. The next section is devoted to recalling some elementary properties of β-expansions. The distribution of full cylinders is studied in Section 3. Since no further new ideas are needed to prove Theorem 1.4, only an outline of the proof is presented in the last section.

Throughout, we use the symbol $\sharp$ to denote the cardinality of a finite set.

2. Preliminaries

In this section we give a brief account on β-expansions.

From the definition of T_β, it is clear that, for $n \geq 1$, the nth digit $\varepsilon_n(x, \beta)$ of x belongs to the alphabet $\mathcal{A} = \{0, \ldots, \lceil\beta - 1\rceil\}$, where $\lceil y\rceil$ denotes the smallest integer greater than or equal to y. We stress that not all sequences $\varepsilon \in \mathcal{A}^{\mathbb{N}}$ are the β-expansion of some $x \in [0, 1]$. This leads to the notion of *β-admissible sequence*.

Definition 2.1. *A finite or an infinite sequence* $(\varepsilon_1, \ldots, \varepsilon_n, \ldots)$ *is called* β*-admissible, if there exists an* $x \in [0, 1)$ *such that the* β*-expansion of* x *begins with* $\varepsilon_1, \ldots, \varepsilon_n, \ldots$

Denote by Σ_β^n the set of all β-admissible sequences of length n and by Σ_β the set of all infinite β-admissible sequences:

$$\Sigma_\beta = \Big\{\varepsilon \in \mathcal{A}^{\mathbb{N}} : \varepsilon \text{ is the } \beta\text{-expansion of some } x \in [0, 1)\Big\}.$$

When there is no possible confusion, we simply write *admissible* instead of β-*admissible*.

In order to characterize the admissible sequences, let us first define *the infinite expansion of 1*. Let $\beta > 1$ be given. If the β-expansion of 1 terminates, i.e. if there exists $m \geq 1$ such that $\varepsilon_m(1,\beta) \geq 1$ but $\varepsilon_n(1,\beta) = 0$ for $n > m$, then β is called *a simple Parry number*. Whence, we put

$$(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots) = (\varepsilon_1(1,\beta), \ldots, \varepsilon_{m-1}(1,\beta), \varepsilon_m(1,\beta) - 1)^\infty,$$

where $(\varepsilon)^\infty$ denotes the periodic sequence $(\varepsilon, \varepsilon, \varepsilon, \ldots)$. If β is not a simple Parry number, we use $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots)$ to denote the β-expansion of 1. In both cases, we say that the sequence

$$\varepsilon^*(\beta) := (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots)$$

is the infinite β-expansion of 1 (or of unity).

The lexicographical order $\prec$ on $\mathcal{A}^{\mathbb{N}}$ is defined as follows: we write

$$(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n, \ldots) \prec (\varepsilon_1', \varepsilon_2', \ldots, \varepsilon_n', \ldots)$$

if there exists $k \geq 1$ such that $\varepsilon_j = \varepsilon_j'$ for $1 \leq j < k$, while $\varepsilon_k < \varepsilon_k'$. This order can be extended to finite blocks by identifying a finite block $(\varepsilon_1, \ldots, \varepsilon_n)$ with the sequence $(\varepsilon_1, \ldots, \varepsilon_n, 0, 0, \ldots)$.

The admissible sequences and the topological entropy are characterized in the following two theorems.

Theorem 2.2 (Parry [13])**.** *(1). Let* $\beta > 1$ *be given. A sequence* $(\varepsilon_1, \varepsilon_2, \ldots)$ *of non-negative integers is* β*-admissible if and only if, for any* $k \geq 1$*,*

$$(\varepsilon_k, \varepsilon_{k+1}, \ldots) \prec (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots),$$

where $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots)$ *is the infinite* β*-expansion of unity.*

(2). If $1 < \beta_1 < \beta_2$*, then* $\Sigma_{\beta_1} \subset \Sigma_{\beta_2}$*.*

Theorem 2.3 (Rényi [16])**.** *For any* $\beta > 1$*, we have*

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \beta^{n+1}/(\beta - 1), \quad \lim_{n\to\infty} \frac{\log \sharp\Sigma_\beta^n}{n} = \log\beta.$$

In particular, the topological entropy of the dynamical system $([0, 1], T_\beta)$ *is equal to* $\log\beta$*.*

We end this section by a definition of the pressure function. In the β-dynamical system, following [26], the pressure function P associated to a continuous potential g can be defined by the formula

$$P(T_\beta, g) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{(\varepsilon_1,\ldots,\varepsilon_n)\in\Sigma_\beta^n} \sup_{y\in I_n(\varepsilon_1,\ldots,\varepsilon_n)} e^{S_n g(y)}, \tag{2.1}$$

where $S_n g(y)$ denotes the ergodic sum $\sum_{j=0}^{n-1} g(T_\beta^j y)$.

This definition of the pressure function looks different from the one given in P. Walters' book [27]; however, both of them fulfill the same variational principle, namely

$$P(T_\beta, g) = \sup\Big\{h_\mu + \int g d\mu : \ \mu \in \mathcal{M}_1(T_\beta)\Big\},$$

where h_μ is the measure-theoretic entropy of μ and $\mathcal{M}_1(T_\beta)$ denotes the collection of T_β-invariant Borel probability measures. Thus, the two definitions coincide.

3. Distribution of full cylinders

In this section, we consider the distribution of cylinders with maximal lengths. We start with an auxiliary lemma.

Lemma 3.1. *Assume that the infinite β-expansion of 1 is purely periodic with minimal period ℓ, denoted by*

$$\varepsilon^*(\beta) = \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big)^\infty.$$

Then

$$\Big(\varepsilon_{i+1}^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big) \prec \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_{\ell-i}^*(\beta)\Big), \text{ for } i = 1, \dots, \ell - 1. \tag{3.1}$$

Proof. Let i be an integer with $1 \le i < \ell$. It immediately follows from the minimality of ℓ that

$$\Big(\varepsilon_{i+1}^*(\beta), \dots, \varepsilon_{\ell+i}^*(\beta)\Big) \neq \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big). \tag{3.2}$$

Furthermore, the admissibility of $\varepsilon^*(\beta)$ implies

$$\Big(\varepsilon_{i+1}^*(\beta), \dots, \varepsilon_{i+\ell}^*(\beta)\Big) \preceq \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\Big(\varepsilon_{i+1}^*(\beta), \dots, \varepsilon_{i+\ell}^*(\beta)\Big) \prec \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big). \tag{3.4}$$

We are led to compare the tails of the words in (3.4). The left one ends with $(\varepsilon_{\ell+1}^*(\beta), \dots, \varepsilon_{i+\ell}^*(\beta))$, while the right one ends with $(\varepsilon_{\ell-i+1}^*(\beta), \dots, \varepsilon_\ell^*(\beta))$. Using the periodicity and admissibility of $\varepsilon^*(\beta)$ again, we get

$$\Big(\varepsilon_{\ell+1}^*(\beta), \dots, \varepsilon_{i+\ell}^*(\beta)\Big) = \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_i^*(\beta)\Big) \succeq \Big(\varepsilon_{\ell-i+1}^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big).$$

In other words, the smaller word in (3.4) has larger tails, thus we conclude

$$\Big(\varepsilon_{i+1}^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big) \prec \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_{\ell-i}^*(\beta)\Big),$$

as asserted. □

Fan and Wang [6] gave several criteria and properties of full cylinders.

Lemma 3.2. [6] (1). *The cylinder $I_n(w_1, \dots, w_n)$ is full if and only if for any $m \ge 1$ and $(u_1, \dots, u_m) \in \Sigma_\beta^m$, the sequence $(w_1, \dots, w_n, u_1, \dots, u_m)$ is still admissible.*

(2). *Let $(w_1, \dots, w_{n-1}, w_n')$ be an admissible sequence with $w_n' \neq 0$. Then, for any integer w_n with $0 \le w_n < w_n'$, the cylinder*

$$I_n(w_1, \dots, w_{n-1}, w_n) \textit{ is full.}$$

(3). *If $I_n(w_1, \dots, w_n)$ is full, then for any $m \ge 1$ and any $(u_1, \dots, u_m) \in \Sigma_\beta^m$,*

$$\begin{aligned} &\Big|I_{n+m}(w_1, \dots, w_n, u_1, \dots, u_m)\Big| \\ &= \Big|I_n(w_1, \dots, w_n)\Big| \cdot \Big|I_m(u_1, \dots, u_m)\Big| = \beta^{-n}\Big|I_m(u_1, \dots, u_m)\Big|. \end{aligned}$$

Thus, the concatenation $I_{n+m}(w_1,\dots,w_n,u_1,\dots,u_m)$ of two full cylinders $I_n(w_1,\dots,w_n)$ and $I_m(u_1,\dots,u_m)$ is still full.

Proof. Items (1) and (3) are immediate. For the sake of completeness, we establish item (2). In view of item (1), it is sufficient to check that for any $(w_{n+1},\dots,w_{n+m})\in\Sigma_\beta^m$ with $m\ge 1$, the word

$$(w_1,\dots,w_n,w_{n+1},\dots,w_{n+m})$$

is admissible. This follows from a direct application of the criterion of admissibility of a sequence (Theorem 2.2). When $k<n$, since $w_n<w_n'$ and $(w_1,\dots,w_{n-1},w_n')$ is admissible, we have

$$\Big(w_{k+1},\dots,w_n,w_{n+1},\dots,w_{n+m}\Big)\prec\Big(w_{k+1},\dots,w_n'\Big)\preceq\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k}^*(\beta)\Big).$$

When $k\ge n$, by the admissibility of $(w_{n+1},\dots,w_{n+m})$, it is clear that

$$\Big(w_{k+1},\dots,w_{n+m}\Big)\preceq\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n+m-k}^*(\beta)\Big).$$

□

We give below a new necessary and sufficient condition ensuring that a cylinder is full.

Proposition 3.3. *Let $(w_1,\dots,w_n)$ be in Σ_β^n.*

(i) If $\varepsilon^*(\beta)$, the infinite β-expansion of 1, is not purely periodic, then the cylinder $I_n(w_1,\dots,w_n)$ is full if and only if for $k=0,\dots,n-1$,

$$\Big(w_{k+1},\dots,w_n\Big)\prec\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k}^*(\beta)\Big);$$

(ii) if $\varepsilon^*(\beta)$, the infinite β-expansion of 1, is purely periodic, then the cylinder $I_n(w_1,\dots,w_n)$ is full if and only if

$$\Big(w_{k+1},\dots,w_n\Big)\prec\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k}^*(\beta)\Big),\ \ \textit{for } k=0,\dots,n-1,\tag{3.5}$$

or $(w_1,\dots,w_n)$ ends with a period of $\varepsilon^*(\beta)$.

Proof. (i) We prove the sufficient part first. With almost the same argument as in the proof of item (2) of Lemma 3.2, we check that, for any $m\ge 1$ and $(u_1,\dots,u_m)\in\Sigma_\beta^m$, the sequence

$$(w_1,\dots,w_n,u_1,\dots,u_m)$$

is β-admissible. Then, the sufficient part follows by applying item (1) of Lemma 3.2. The necessary part is proved by contraposition. Assume that for some integer $k=0,\dots,n-1$,

$$\Big(w_{k+1},\dots,w_n\Big)=\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k}^*(\beta)\Big).\tag{3.6}$$

For any $m\ge n-k$, consider the admissible sequence $(\varepsilon_1^*(\beta),\dots,\varepsilon_m^*(\beta))$. Since the interval $I_n(w_1,\dots,w_n)$ is full, by item (1) of Lemma 3.2, we get an admissible sequence

$$\Big(w_1,\dots,w_n,\varepsilon_1^*(\beta),\dots,\varepsilon_m^*(\beta)\Big).$$

By the criterion of admissibility of a sequence, we have

$$\Big(w_{k+1},\dots,w_n,\varepsilon_1^*(\beta),\dots,\varepsilon_m^*(\beta)\Big)\preceq\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{m+n-k}^*(\beta)\Big).$$

By (3.6), it then follows that

$$\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k}^*(\beta),\varepsilon_1^*(\beta),\dots,\varepsilon_m^*(\beta)\Big)\preceq\Big(\varepsilon_1^*(\beta),\dots,\varepsilon_{m+n-k}^*(\beta)\Big).\tag{3.7}$$

Cutting the common prefix $(\varepsilon_1^*(\beta), \dots, \varepsilon_{n-k}^*(\beta))$ of the two sequences in (3.7), we get

$$\Big(\varepsilon_1^*(\beta), \dots, \varepsilon_m^*(\beta)\Big) \preceq \Big(\varepsilon_{n-k+1}^*(\beta), \dots, \varepsilon_{m+n-k}^*(\beta)\Big). \tag{3.8}$$

By applying the criterion of admissibility again, we get an equality in (3.8), thus (3.7) is also an equality. Consequently, $(\varepsilon_1^*(\beta), \dots, \varepsilon_m^*(\beta))$ is periodic of period $n-k$. Since m is arbitrary, we deduce that $\varepsilon^*(\beta)$ is periodic. This is a contradiction.

(ii). Let

$$\varepsilon^*(\beta) = \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big)^\infty$$

be purely periodic with ℓ being the minimal period.

For the sufficient part, it is clear that $I_n(w_1, \dots, w_n)$ is full if (3.5) holds. Thus, we show that $I_n(w_1, \dots, w_n)$ is also full if the admissible sequence $(w_1, \dots, w_n)$ ends with $(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta))$.

Let $t \geq 1$ be the largest integer such that $(w_1, \dots, w_n)$ can be written as

$$\Big(w_1, \dots, w_k, (\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta))^t\Big).$$

First we claim that

$$\Big(w_1, \dots, w_k\Big) \prec \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_k^*(\beta)\Big). \tag{3.9}$$

If this is not the case, then, by the admissibility of $(w_1, \dots, w_k)$, we have

$$\Big(w_1, \dots, w_k\Big) = \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_k^*(\beta)\Big). \tag{3.10}$$

We show that (3.10) contradicts the admissibility of $(w_1, \dots, w_n)$. Indeed, by the maximality of t, the admissible word $(w_1, \dots, w_n)$ can be written as

$$\Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big)^{t_1}, \varepsilon_1^*(\beta), \dots, \varepsilon_i^*(\beta), \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big)^t$$

for some integers $t_1 \geq 0$ and $1 \leq i < \ell$. Consider the subword $\widetilde{w}$ of $(w_1, \dots, w_n)$ defined by

$$\widetilde{w} := \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_i^*(\beta), \varepsilon_1^*(\beta), \dots, \varepsilon_{\ell-i}^*(\beta)\Big).$$

Lemma 3.1 implies that

$$\Big(\varepsilon_1^*(\beta), \dots, \varepsilon_i^*(\beta), \varepsilon_1^*(\beta), \dots, \varepsilon_{\ell-i}^*(\beta)\Big) \succ \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta)\Big).$$

This means that $\widetilde{w}$ is not admissible. From the criterion of admissibility, it is clear that any subword of an admissible word is admissible. Applying this fact to $(w_1, \dots, w_n)$, the non-admissibility of $\widetilde{w}$ contradicts the admissibility of $(w_1, \dots, w_n)$. Thus, (3.9) holds. In the same way, we can show that

$$\Big(w_{i+1}, \dots, w_k\Big) \prec \Big(\varepsilon_1^*(\beta), \dots, \varepsilon_{k-i}^*(\beta)\Big), \text{ for } i = 0, \dots, k-1. \tag{3.11}$$

For any admissible word v, it can be checked directly using (3.11) and the criterion of admissibility that

$$\Big(w_1, \dots, w_k, (\varepsilon_1^*(\beta), \dots, \varepsilon_\ell^*(\beta))^t, v\Big)$$

is admissible. This implies that the cylinder $I_n(w_1, \dots, w_n)$ is full.

Now, we show the necessary part. Assume that (3.5) does not hold. Let then $k < n$ be the largest integer such that

$$(w_{k+1}, \dots, w_n) = (\varepsilon_1^*(\beta), \dots, \varepsilon_{n-k}^*(\beta)).$$

To reach the desired conclusion, it is sufficient to show that $n-k=\ell$. Since $\varepsilon^*(\beta)$ is periodic with period ℓ, by the maximality of k, it follows that $n-k\le\ell$. If $n-k\ne\ell$, consider the word

$$w' := \Big(w_{k+1},\dots,w_n,\varepsilon_1^*(\beta),\dots,\varepsilon_\ell^*(\beta)\Big),$$

which is admissible since $I_n(w_1,\dots,w_n)$ is full. Combined with (3.10), this leads to a contradiction with the admissibility of w'. Thus we get $n-k=\ell$. □

Proposition 3.3 gives us an easily checkable criterion for full cylinders, which will be used frequently later.

Corollary 3.4. *Let $(w_1,\dots,w_n)$ be in Σ_β^n. If $I_n(w_1,\dots,w_n)$ is not full, then there exists an integer $k=0,\dots,n-1$ such that $I_k(w_1,\dots,w_k)$ is full and $(w_{k+1},\dots,w_n)$ is a prefix of the infinite β-expansion of 1.*

We are now in position to prove Theorem 1.2 indicating that the full cylinders are well distributed, in a suitable sense.

Proof of Theorem 1.2. Let $w^{(0)},w^{(1)},\dots,w^{(n)}$ be $n+1$ consecutive words in Σ_β^n in the lexicographic order. Then $I_n(w^{(0)}),\dots,I_n(w^{(n)})$ are $n+1$ consecutive cylinders of order n in $[0,1]$.

Step 1. Assume that $I_n(w^{(n)})$ is not full. Then, by Corollary 3.4, there exists an integer $0\le k_0<n$ such that

$$I_{k_0}(w_1^{(n)},\dots,w_{k_0}^{(n)}) \text{ is full and } \quad w_{k_0+1}^{(n)}=\varepsilon_1^*(\beta),\dots,w_n^{(n)}=\varepsilon_{n-k_0}^*(\beta).$$

Step 2. Assume that $I_n(w^{(n-1)})$ is as well not full.

Firstly we claim that $w^{(n-1)}$ and $w^{(n)}$ have a common prefix up to at least the k_0-th digit. In fact, since $(w_1^{(n)},\dots,w_{k_0}^{(n)},\varepsilon_1^*(\beta))$ is admissible and $\varepsilon_1^*(\beta)\ne0$, we know that $(w_1^{(n)},\dots,w_{k_0}^{(n)},0)$ is another admissible sequence smaller than $(w_1^{(n)},\dots,w_{k_0}^{(n)},\varepsilon_1^*(\beta))$. Thus,

$$\Big(w_1^{(n)},\dots,w_{k_0}^{(n)},0\Big)\le w^{(n-1)}\prec\Big(w_1^{(n)},\dots,w_{k_0}^{(n)},\varepsilon_1^*(\beta),\varepsilon_2^*(\beta),\dots,\varepsilon_{n-k_0}^*(\beta)\Big).$$

This shows that $w^{(n-1)}$ also begins with $(w_1^{(n)},\dots,w_{k_0}^{(n)})$.

Since $w^{(n-1)}\prec w^{(n)}$ and they have a common prefix at least up to the k_0-th position, there exists $\overline{k}_1>k_0$ such that

$$w_1^{(n-1)}=w_1^{(n)},\dots,w_{\overline{k}_1-1}^{(n-1)}=w_{\overline{k}_1-1}^{(n)}, \text{ but } w_{\overline{k}_1}^{(n-1)}<w_{\overline{k}_1}^{(n)}.$$

By item (2) of Lemma 3.2, we know that the cylinder

$$I_{\overline{k}_1}\Big(w_1^{(n-1)},\dots,w_{\overline{k}_1}^{(n-1)}\Big)$$

is full. Then, by item (3) of Lemma 3.2, the cylinder

$$I_{n-\overline{k}_1}\Big(w_{\overline{k}_1+1}^{(n-1)},\dots,w_n^{(n-1)}\Big)$$

is not full, since otherwise $I_n(w_1^{(n-1)},\dots,w_n^{(n-1)})$ would be full. Applying Corollary 3.4 to the cylinder $I_{n-\overline{k}_1}(w_{\overline{k}_1+1}^{(n-1)},\dots,w_n^{(n-1)})$ and using then item (3) of Lemma 3.2, we deduce that there exists $k_1\ge\overline{k}_1$ such that

$$I_{k_1}\Big(w_1^{(n-1)},\dots,w_{k_1}^{(n-1)}\Big) \text{ is full and } \Big(w_{k_1+1}^{(n-1)},\dots,w_n^{(n-1)}\Big)=(\varepsilon_1^*(\beta),\dots,\varepsilon_{n-k_1}^*(\beta)). \tag{3.12}$$

To sum up, if $I_n(w^{(n)})$ and $I_n(w^{(n-1)})$ are both not full, then there exists $k_1 > k_0$ such that (3.12) is satisfied.

STEP 3. We repeat the argument in STEP 2 to show that, if $I_n(w^{(n-2)})$ is not full, then there exists $k_2 > k_1$ such that

$$I_{k_2}\Big(w_1^{(n-2)}, \dots, w_{k_2}^{(n-2)}\Big) \text{ is full and } \Big(w_{k_2+1}^{(n-2)}, \dots, w_n^{(n-2)}\Big) = (\varepsilon_1^*(\beta), \dots, \varepsilon_{n-k_2}^*(\beta)).$$

We then continue this procedure. If there exists some k_i ($0 \le i \le n$) such that $k_i = n$, this procedure ends since $I_n(w^{(i)})$ is full. Otherwise, assuming that every $I_n(w^{(i)})$ ($0 \le i \le n$) is not full, we would have a sequence of integers

$$k_n > k_{n-1} > \dots > k_1 > k_0 \ge 0 \tag{3.13}$$

such that

$$I_{k_n}\Big(w_1^{(0)}, \dots, w_{k_n}^{(0)}\Big) \text{ is full and } \Big(w_{k_n+1}^{(0)}, \dots, w_n^{(0)}\Big) = (\varepsilon_1^*, \dots, \varepsilon_{n-k_n}^*). \tag{3.14}$$

Since $k_n \le n$, we must have $k_n = n$ by (3.13). Thus the first part in (3.14) implies that $I_n(w^{(0)})$ is full which leads to a contradiction.

Therefore, there must be at least one full cylinder among $I_n(w^{(i)})$ ($i = 0, 1, \dots, n$). □

4. DIMENSIONAL THEORY FOR β-EXPANSIONS

In this section, we prove Proposition 1.3. We begin with two propositions concerning the relationship between balls and cylinders.

Proposition 4.1 (Covering properties)**.** *Let $\beta > 1$. For any $y \in [0, 1]$ and any positive integer ℓ, the ball $B(y, \beta^{-\ell})$ can be covered by at most $4(\ell + 1)$ cylinders of order ℓ.*

Proof. By Theorem 1.2, among any $4(\ell + 1)$ consecutive cylinders of order ℓ, there are at least 4 full cylinders. So the total length of these intervals is larger than $4\beta^{-\ell}$. Thus $B(y, \beta^{-\ell})$ can be covered by at most $4(\ell + 1)$ cylinders of order ℓ. □

Proposition 4.2 (Packing properties)**.** *Let $\delta > 0$. Let $n_0 \ge 3$ be an integer such that $(\beta n_0)^{1+\delta} < \beta^{n_0\delta}$. Then, for any real number r with $0 < r < n_0\beta^{-n_0}$ and for any $x_0 \in [0, 1]$, there exists a cylinder I_n satisfying the following three conditions:*

(1). The cylinder I_n is a full cylinder.

(2). The cylinder I_n is contained in the ball $B(x_0, r)$.

(3). The length of I_n is comparable with r, in the sense that $r^{1+\delta} < |I_n| < r$.

Proposition 4.2 was shown for the first time in [19], by means of a constructive method. Here, we apply Theorem 1.2 to give a simpler proof.

Proof. Let $n \ge n_0$ be the integer defined by

$$n\beta^{-n} \le r < (n-1)\beta^{-n+1}.$$

Since the length of every cylinder of order n is at most equal to β^{-n}, the ball $B(x, r)$ contains at least $2n - 2 \ge n + 1$ consecutive cylinders of order n. Thus, by Theorem 1.2, it contains a full cylinder of order n. Denote by I_n such a full cylinder. By the choice of n and n_0, we have

$$r^{1+\delta} < \Big((n-1)\beta^{-n+1}\Big)^{1+\delta} \le \beta^{-n} = |I_n|.$$

This completes the proof of the proposition. □

We apply Proposition 4.1 to prove Proposition 1.3.

Proof of Proposition 1.3. Let $\eta > 0$. Let n_0 be the smallest integer such that $\beta^{n\eta} \geq 8n$. For any interval U with length $|U| \leq \beta^{-n_0}$, let $n \geq n_0$ be the integer defined by $\beta^{-n-1} < |U| \leq \beta^{-n}$. It follows from Proposition 4.1 that U can be covered by at most $8n$ cylinders of order n. Denoting by Γ the collection of these cylinders of order n, we get

$$\mu(U) \leq \sum_{I_n \in \Gamma} \mu(I_n) \leq \sum_{I_n \in \Gamma} c|I_n|^s \leq c \cdot 8n\beta^{-ns} \leq c\beta|U|^{s-\eta}.$$

Since η can be chosen arbitrarily small, we conclude by the classical form of the mass distribution principle [5, Proposition 4.2]. □

5. Proof of Theorem 1.4

As usual, the proof of Theorem 1.4 is divided into two parts: upper bound and lower bound. In the following, unless otherwise specified, when we need to take a point y in a cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$, we always take for y the left endpoint of $I_n(\varepsilon_1, \ldots, \varepsilon_n)$, i.e.

$$y = \frac{\varepsilon_1}{\beta} + \ldots + \frac{\varepsilon_n}{\beta^n}.$$

Instead of $\mathfrak{S}(f)$, we consider the following set

$$\overline{\mathfrak{S}}(f) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_n)\in\Sigma_\beta^n} J(\varepsilon_1, \ldots, \varepsilon_n),$$

where

$$J(\varepsilon_1, \ldots, \varepsilon_n) = \left\{x \in I_n(\varepsilon_1, \ldots, \varepsilon_n), |T_\beta^n x - z_n| < e^{-S_n f(y)}\right\},$$

with y being the left endpoint of $I_n(\varepsilon_1, \ldots, \varepsilon_n)$.

It follows from the continuity of f that, for any $\delta > 0$ and n large enough,

$$\left|S_n f(x) - S_n f(y)\right| < n\delta, \text{ with } x, y \in I_n(\varepsilon_1, \ldots, \varepsilon_n).$$

Thus we have

$$\overline{\mathfrak{S}}(f + \delta) \subset \mathfrak{S}(f) \subset \overline{\mathfrak{S}}(f - \delta).$$

Therefore, it is sufficient to determine the dimension of $\overline{\mathfrak{S}}(f)$.

Let $s(\beta)$ be the solution to the pressure equation

$$P(T_\beta, -s(f + \log\beta)) = 0.$$

5.1. **Upper bound.**

The upper bound can be obtained by considering the obvious covering system of $\overline{\mathfrak{S}}(f)$ given by

$$\left\{J(\varepsilon_1, \ldots, \varepsilon_n) : (\varepsilon_1, \ldots, \varepsilon_n) \in \Sigma_\beta^n, n \geq N\right\}, \text{ for } N \geq 1.$$

The length of $J(\varepsilon_1, \ldots, \varepsilon_n)$ satisfies

$$|J(\varepsilon_1, \ldots, \varepsilon_n)| \leq 2\beta^{-n}e^{-S_n f(y)},$$

since, for every x in $J(\varepsilon_1, \ldots, \varepsilon_n)$, we have

$$\left|x - \left(\frac{\varepsilon_1}{\beta} + \ldots + \frac{\varepsilon_n + z_n}{\beta^n}\right)\right| = \left|\frac{T_\beta^n x - z_n}{\beta^n}\right| < \beta^{-n}e^{-S_n f(y)}.$$

Thus, for $s \in (0,1]$, we get

$$\begin{aligned}\mathcal{H}^s(\overline{\mathfrak{S}}(f)) &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(\varepsilon_1,\dots,\varepsilon_n)\in\Sigma_\beta^n} \left|J(\varepsilon_1,\dots,\varepsilon_n)\right|^s \\ &\le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{(\varepsilon_1,\dots,\varepsilon_n)\in\Sigma_\beta^n} \left(2\beta^{-n} e^{-S_n f(y)}\right)^s.\end{aligned}$$

By the definitions of the pressure function and of $s(\beta)$, we have, for any $s > s(\beta)$,

$$\mathcal{H}^s(\overline{\mathfrak{S}}(f)) < \infty.$$

This implies that the Hausdorff dimension of $\overline{\mathfrak{S}}(f)$ satisfies

$$\dim_{\mathrm{H}} \overline{\mathfrak{S}}(f) \le s(\beta).$$

5.2. **Lower bound.** We apply the modified mass distribution principle (Proposition 1.3) to give a lower bound for $\dim_{\mathrm{H}} \overline{\mathfrak{S}}(f)$. We first construct a large Cantor set $\mathbb{F}_\infty$ inside $\overline{\mathfrak{S}}(f)$ and then we define a suitable probability measure μ supported on $\mathbb{F}_\infty$ and estimate the Hölder exponent of μ on cylinders.

5.2.1. *Construction of a Cantor subset of $\overline{\mathfrak{S}}(f)$.*

At first, we give concisely a family of full cylinders. Recall that the sequence $\varepsilon^*(\beta) = (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots)$ is the infinite β-expansion of unity. When $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots)$ is periodic, let $\beta_N = \beta$ for $N \ge 1$. Otherwise, for every N with $\varepsilon_N^*(\beta) \ge 1$, define β_N to be the unique positive solution to the equation

$$1 = \frac{\varepsilon_1^*(\beta)}{\beta_N^1} + \frac{\varepsilon_2^*(\beta)}{\beta_N^2} + \dots + \frac{\varepsilon_N^*(\beta)}{\beta_N^N}. \tag{5.1}$$

In the latter case, it is easy to see that β_N increases to β as $N \to \infty$, and thus $\Sigma_{\beta_N}^n \subset \Sigma_\beta^n$ for $n \ge 1$. Moreover, the infinite β_N-expansion of unity is given by

$$(\varepsilon_1^*(\beta), \dots, \varepsilon_{N-1}^*(\beta), \varepsilon_N^*(\beta) - 1)^\infty.$$

The following fact, which will be used several times, is a consequence of Proposition 3.3 on the criterion of full cylinders. It should be reminded that all the cylinders appearing below are cylinders in β-expansion, but not in β_N-expansion.

Corollary 5.1. *(1). When $\varepsilon^*(\beta) = (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots, \varepsilon_N^*(\beta))^\infty$ is a periodic sequence with period length N, for any β-admissible sequence $(\varepsilon_1, \dots, \varepsilon_n)$ with $n \ge 1$, the cylinder $I_{n+N}(\varepsilon_1, \dots, \varepsilon_n, 0^N)$ is full.*

(2) When $\varepsilon^*(\beta)$ is not periodic, for any β_N-admissible sequence $(\varepsilon_1, \dots, \varepsilon_n)$ with $n \ge 1$, the cylinder $I_{n+N}(\varepsilon_1, \dots, \varepsilon_n, 0^N)$ is full.

Proof. We claim that for every β_N-admissible sequence $(\varepsilon_1, \dots, \varepsilon_n)$,

$$(\varepsilon_1, \dots, \varepsilon_n, 0^N) \prec (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots, \varepsilon_{n+N}^*(\beta)).$$

Then, by applying the above claim repeatedly and by the first item of Proposition 3.3, we get the desired result.

To check the claim, we distinguish two cases, according as whether $\varepsilon^*(\beta)$ is periodic or not.

(i). Assume that $\varepsilon^*(\beta) = (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots, \varepsilon_N^*(\beta))^\infty$ is a periodic sequence with period length N. Firstly, the admissibility of $(\varepsilon_1, \dots, \varepsilon_n, 0^N)$ implies that

$$(\varepsilon_1, \dots, \varepsilon_n, 0^N) \preceq (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \dots, \varepsilon_{n+N}^*(\beta)). \tag{5.2}$$

Secondly, the periodicity of $\varepsilon^*(\beta)$ implies that any subword of length N in $\varepsilon^*(\beta)$ cannot be 0^N. Thus the inequality $\leq$ in (5.2) is strict.

(ii). Assume that $\varepsilon^*(\beta)$ is not periodic. Recall that $(\varepsilon_1, \ldots, \varepsilon_n)$ is β_N-admissible and that the infinite β_N-expansion of 1. When $n \geq N$, by the criterion of admissibility, we have

$$(\varepsilon_1, \ldots, \varepsilon_N) \leq (\varepsilon_1^*(\beta), , \ldots, \varepsilon_{N-1}^*(\beta), \varepsilon_N^*(\beta) - 1) \prec (\varepsilon_1^*(\beta), \ldots, \varepsilon_N^*(\beta)).$$

When $n < N$, with the same argument as in (i), we have

$$(\varepsilon_1, \ldots, \varepsilon_n, 0^N) \prec (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots, \varepsilon_{n+N}^*(\beta)).$$

This proves the claim. □

Now, we are in the position to construct a Cantor subset $\mathbb{F}_\infty$ of $\overline{\mathfrak{S}}(f)$. Fix $\delta > 0$ and choose a very rapidly increasing subsequence $\{m_k\}_{k\geq 1}$ of positive integers with m_1 large enough.

Generation 1 *of the Cantor set.* Let $n_1 = m_1$. For every $(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}) \in \Sigma_{\beta_N}^{n_1}$ ending with 0^N, consider the set

$$\Big\{x \in I_{n_1}(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}) : |T_\beta^{n_1} x - z_{n_1}| < e^{-S_{n_1} f(y_1)}\Big\}, \tag{5.3}$$

where $y_1 \in I_{n_1}(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)})$.

Applying Proposition 4.2 to the ball $B(z_{n_1}, e^{-S_{n_1} f(y_1)})$, we get a full cylinder $I_{\ell_1}(w_1)$ such that

$$I_{\ell_1}(w_1) \subset B(z_{n_1}, e^{-S_{n_1} f(y_1)}), \text{ and } \Big|I_{\ell_1}(w_1)\Big| \geq \Big(e^{-S_{n_1} f(y_1)}\Big)^{1+\delta}.$$

Then, we get a subset of (5.3), namely the cylinder

$$I_{n_1+\ell_1}\Big(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}, w_1\Big).$$

We point out that the cylinder $I_{n_1+\ell_1}$ given above is a full cylinder, since $I_{n_1}(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)})$ (by Corollary 5.1) and $I_{\ell_1}(w_1)$ are both full.

Now the first generation of the Cantor set is defined as

$$\mathbb{F}_1 = \Big\{I_{n_1+\ell_1}\Big(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}, w_1\Big) : (\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}) \in \Sigma_{\beta_N}^{n_1} \text{ ending with } 0^N\Big\}.$$

From the construction of $\mathbb{F}_1$, it is clear that for any $x \in I_{n_1+\ell_1}(u_1) \in \mathbb{F}_1$,

$$T_\beta^{n_1} x \in I_{\ell_1}(w_1) \subset B(z_{n_1}, e^{-S_{n_1} f(y_1)}). \tag{5.4}$$

It should be noted that ℓ_1 and w_1 depend on $(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)})$. This dependence will not play a role in the following argument, thus will not be indicated explicitly. We abbreviate by u_1 the word $(\varepsilon_1^{(1)}, \ldots, \varepsilon_{n_1}^{(1)}, w_1)$.

Generation 2 *of the Cantor set.*

Choose a large integer m_2 such that

$$\frac{\delta}{1+\delta} \cdot m_2 \log \beta \geq \Big(n_1 + \sup\{\ell_1 : I_{n_1+\ell_1}(u_1) \in \mathbb{F}_1\}\Big)\|f\|, \tag{5.5}$$

where $\|f\| = \sup\{|f(x)| : x \in [0, 1]\}$.

Fix an element $I_{n_1+\ell_1}(u_1) \in \mathbb{F}_1$ and set $n_2 = n_1 + \ell_1 + m_2$. For every $\varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)} \in \Sigma_{\beta_N}^{m_2}$, ending with 0^N, consider the set

$$\Big\{x \in I_{n_2}(u_1, \varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)}) : |T_\beta^{n_2} x - z_{n_2}| < e^{-S_{n_2} f(y_2)}\Big\} \tag{5.6}$$

with $y_2 \in I_{n_2}(u_1, \varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)})$.

Applying Proposition 4.2 to the ball $B(z_{n_2}, e^{-S_{n_2} f(y_2)})$, we get a full cylinder $I_{\ell_2}(w_2)$ such that

$$I_{\ell_2}(w_2) \subset B(z_{n_2}, e^{-S_{n_2} f(y_2)}), \text{ and } \left|I_{\ell_2}(w_2)\right| \geq \left(e^{-S_{n_2} f(y_2)}\right)^{1+\delta}.$$

Then, we get a subset of (5.6), namely the cylinder

$$I_{n_2+\ell_2}\Big(u_1, \varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)}, w_2\Big).$$

As above, we note that $I_{n_2+\ell_2}$ is a full cylinder.

A subfamily of the second generation of the Cantor set is defined as

$$\mathbb{F}_2(I_{n_1+\ell_1}(u_1)) := \Big\{I_{n_2+w_2}\Big(u_1, \varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)}, w_2\Big) : (\varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)}) \in \Sigma_{\beta_N}^{m_2} \text{ ending with } 0^N\Big\}.$$

Then, abbreviating $(u_1, \varepsilon_1^{(2)}, \ldots, \varepsilon_{m_2}^{(2)}, w_2)$ by u_2, the second generation of the Cantor set is defined as

$$\mathbb{F}_2 = \Big\{I_{n_2+\ell_2}(u_2) \in \mathbb{F}_2(I_{n_1+\ell_1}(u_1)) : I_{n_1+\ell_1}(u_1) \in \mathbb{F}_1\Big\}.$$

From the construction of $\mathbb{F}_2$, it is clear that for any $x \in I_{n_2+\ell_2}(u_2) \in \mathbb{F}_2$,

$$T_\beta^{n_2} x \in I_{\ell_2}(w_2) \subset B(z_{n_2}, e^{-S_{n_2} f(y_2)}). \tag{5.7}$$

The Cantor set. Continuing the process, we obtain a nested sequence $\{\mathbb{F}_k\}_{k\geq 1}$ composed of full cylinders, called *basic cylinders.* And then the desired Cantor set is

$$\mathbb{F}_\infty = \bigcap_{k=1}^{\infty} \bigcup_{I_{n_k+\ell_k}(u_k)\in\mathbb{F}_k} I_{n_k+\ell_k}(u_k).$$

By (5.4) and (5.7), it is clear that

$$\mathbb{F}_\infty \subset \overline{\mathfrak{S}}(f). \tag{5.8}$$

To apply the modified mass distribution principle, we will construct a probability measure μ supported on $\mathbb{F}_\infty$ and then estimate the Hölder exponent of the measure μ on cylinders. Since the construction and the estimation are quite analogous to those in [23], we do not give all the details.

5.2.2. *Supporting measure.*

Now we construct a probability measure μ supported on $\mathbb{F}_\infty$, which is defined by distributing masses among the cylinders with non-empty intersection with $\mathbb{F}_\infty$.

Recall that $s(\beta)$ is the solution to the equation

$$P(T_\beta, -s(\log\beta + f)) = 0.$$

Fix an integer N. For $k \geq 1$, we define a sequence of real numbers connected to the Hausdorff dimension of $\mathbb{F}_\infty$: let s_k be the solution to the equation

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_{m_k})\in\Sigma_{\beta_N}^{m_k}\text{: ending with } 0^N} \left(\beta^{-m_k} e^{-S_{m_k} f(y_k')}\right)^s = 1,$$

where $y_k' \in I_{m_k}(\varepsilon_1, \ldots, \varepsilon_{m_k})$. By the continuity of the pressure function $P(T_\beta, f)$ with respect to β [23, Theorem 4.1], it can be shown that

$$\lim_{N\to\infty} \lim_{k\to\infty} s_k = s(\beta).$$

At first, we define the measure μ on the *basic cylinders*.

(1). Define the measure μ on the basic cylinders in $\mathbb{F}_1$. For every basic interval $I_{n_1+\ell_1}(\varepsilon_1^{(1)}, \ldots, \varepsilon_{m_1}^{(1)}, w_1) = I_{n_1+\ell_1}(u_1) \in \mathbb{F}_1$, take

$$\mu(I_{n_1+\ell_1}(u_1)) = \left(\frac{1}{\beta^{m_1}} e^{-S_{m_1} f(y_1')}\right)^{s_1},$$

where $y_1' \in I_{m_1}(\varepsilon_1^{(1)}, \ldots, \varepsilon_{m_1}^{(1)})$.

(2). Define the measure μ inductively on the basic cylinders in $\mathbb{F}_k$. For every $I_{n_k+\ell_k}(u_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}, w_k) = I_{n_k+\ell_k}(u_k) \in \mathbb{F}_k$, define

$$\begin{aligned} \mu\Big(I_{n_k+\ell_k}(u_k)\Big) &= \mu\Big(I_{n_{k-1}+\ell_{k-1}}(u_{k-1})\Big)\left(\frac{1}{\beta^{m_k}} e^{-S_{m_k} f(y_k')}\right)^{s_k} \\ &= \prod_{j=1}^{k}\left(\frac{1}{\beta^{m_j}} e^{-S_{m_j} f(y_j')}\right)^{s_j}, \end{aligned} \tag{5.9}$$

where $y_j' \in I_{m_j}(\varepsilon_1^{(j)}, \ldots, \varepsilon_{m_j}^{(j)})$ for $j = 1, \ldots, k$.

We emphasize that there are differences between the definitions of y' in (5.9) and y in (5.6).

To ensure that μ is indeed a measure, the measure of every cylinder which is not a basic cylinder is defined to be the total measure of basic cylinders contained in it.

5.2.3. *The lengths of cylinders.* In this subsection, we estimate the lengths of the cylinders $\{I_n(x) : n \geq 1\}$ for all $x \in \mathbb{F}_\infty$. As in the previous subsection, let

$$x = (u_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}, w_k, \ldots)$$

be the β-expansion of x.

(1) When $n = n_k + \ell_k$, since $I_{n_{k-1}+\ell_{k-1}}(u_{k-1})$, $I_{m_k}(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)})$ and $I_{\ell_k}(w_k)$ are all full cylinders, we have

$$\begin{aligned} \Big|I_n(x)\Big| &= \Big|I_{n_{k-1}+\ell_{k-1}}(u_{k-1})\Big| \cdot |I_{m_k}(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)})| \cdot \Big|I_{\ell_k}(w_k)\Big| \\ &\geq \Big|I_{n_{k-1}+\ell_{k-1}}(u_{k-1})\Big| \cdot \beta^{-m_k} \cdot \Big(e^{-S_{n_k} f(y_k)}\Big)^{1+\delta}, \end{aligned}$$

where $y_k \in I_{n_k}(u_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)})$. Thus, by induction, we get

$$\Big|I_n(x)\Big| \geq \prod_{j=1}^{k}\Big(\beta^{-m_j} \cdot \Big(e^{-S_{n_j} f(y_j)}\Big)^{1+\delta}\Big), \tag{5.10}$$

where $y_j \in I_{n_j}(u_{j-1}, \varepsilon_1^{(j)}, \ldots, \varepsilon_{m_j}^{(j)})$ for $j = 1, \ldots, k$.

Now, we compare $S_{n_k} f(y_k)$ and $S_{m_k} f(y_k')$: by (5.5), we have

$$\Big|S_{n_k} f(y_k) - S_{m_k} f(y_k')\Big| = \Big|S_{n_{k-1}+\ell_{k-1}} f(y_k)\Big| \leq (n_{k-1} + \ell_{k-1})\|f\|_\infty \leq \frac{\delta}{1+\delta} m_k \log\beta.$$

Combining this with (5.10), we get

$$\Big|I_n(x)\Big| \geq \prod_{j=1}^{k}\Big(\beta^{-m_j} \cdot e^{-S_{m_j} f(y_j')}\Big)^{1+\delta}, \tag{5.11}$$

where $y_j' \in I_{m_j}(\varepsilon_1^{(j)}, \ldots, \varepsilon_{m_j}^{(j)})$ for $j = 1, \ldots, k$.

(2). When $n_k \le n < n_k + \ell_k$,

$$\left|I_n(x)\right| \ge \left|I_{n_k+\ell_k}(x)\right| \ge \prod_{j=1}^{k} \left(\beta^{-m_j} \cdot e^{-S_{m_j} f(y'_j)}\right)^{1+\delta}. \tag{5.12}$$

(3). When $n_{k-1} + \ell_{k-1} < n \le n_k$, write

$$I_n(x) = I_n(u_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_\ell^{(k)}).$$

Since $I_{n_{k-1}+\ell_{k-1}}(u_{k-1})$ is full and $(\varepsilon_1^{(k)}, \ldots, \varepsilon_\ell^{(k)}) \in \Sigma_{\beta_N}^\ell$, we have

$$\left|I_n(x)\right| \ge \left|I_{n_{k-1}+\ell_{k-1}}(u_{k-1})\right| \cdot \left|I_\ell(\varepsilon_1^{(k)}, \ldots, \varepsilon_\ell^{(k)})\right| \ge \beta^{-n_{k-1}-\ell_{k-1}} \beta^{-\ell-N} = \beta^{-n-N}. \tag{5.13}$$

5.2.4. *Hölder exponent of the measure μ.*

Once the μ-measure of a cylinder and the length of a cylinder are given, it only remains for us to check that $\mu(I_n(x)) \le |I_n(x)|^s$ holds for some suitably chosen s. Here, we omit the argument and the reader is referred to [23] for a detailed calculation.

Lemma 5.2. *For any $s < s(\beta)$, there exist an integer n_1, a measure μ supported on $\mathbb{F}_\infty$ and a constant $C_0 = C_0(s)$ such that for all $x \in [0, 1]$ and $n \ge n_1$,*

$$\mu\Big(I_n(x)\Big) \le C_0 \cdot \left|I_n(x)\right|^s.$$

We then conclude using the modified mass distribution principle (Proposition 1.3) that

$$\dim_{\mathsf{H}} \overline{\mathfrak{S}}(f) \ge s.$$

This ends the proof of the theorem.

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