Fractional parts of powers and Sturmian words

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Abstract. *Let $b \geq 2$ be an integer. In terms of combinatorics on words we describe all irrational numbers $\xi > 0$ with the property that the fractional parts $\{\xi b^n\}$, $n \geq 0$, all belong to a semi-open or an open interval of length $1/b$. The length of such an interval cannot be smaller, that is, for irrational ξ, the fractional parts $\{\xi b^n\}$, $n \geq 0$, cannot all belong to an interval of length smaller than $1/b$.*

Parties fractionnaires de puissances et mots sturmiens

Résumé. *Soit $b \geq 2$ un entier. Au moyen de résultats de la combinatoire des mots, nous caractérisons l'ensemble des nombres réels $\xi > 0$ tels que les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$, appartiennent toutes à un intervalle semi-ouvert ou ouvert de longueur $1/b$. La longueur d'un tel intervalle ne peut pas être plus petite, c'est-à-dire, quel que soit le nombre irrationnel ξ, aucun intervalle de longueur strictement inférieure à $1/b$ ne contient toutes les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$.*

Version française abrégée

Dans tout ce qui suit, $\{\cdot\}$ désigne la fonction partie fractionnaire. Suivant la définition énoncée en 1968 par Mahler [11], un Z-nombre est un nombre réel positif ξ vérifiant $0 \leq \{\xi(3/2)^n\} < 1/2$ pour tout entier $n \geq 0$. L'ensemble des Z-nombres est au plus dénombrable [11] et même vraisemblablement vide, mais ce problème difficile n'est à ce jour pas résolu. Plus généralement, étant donnés un nombre réel $\alpha > 1$ et un sous-intervalle $[s, t[$ de $[0, 1[$, on souhaiterait savoir s'il existe un nombre réel $\xi > 0$ vérifiant $s \leq \{\xi\alpha^n\} < t$ pour tout entier $n \geq 0$, ou bien, plus modestement, on aimerait déterminer la plus petite longueur $t - s$ pour laquelle un tel ξ existe.

Cette note répond à deux objectifs : nous annonçons des résultats nouveaux obtenus pour α algébrique par le second auteur et nous apportons une réponse complète aux questions *supra* lorsque $\alpha \geq 2$ est un entier.

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Soient p et q des entiers vérifiant $p > q \geq 2$. Flatto, Lagarias & Pollington [9] établirent que, pour tout intervalle I de longueur strictement inférieure à $1/p$, il n'existe aucun nombre réel $\xi > 0$ vérifiant $\{\xi(p/q)^n\} \in I$ pour tout entier $n \geq 0$ (*cf.* également [3]). Une nouvelle démonstration, plus simple, de ce résultat, ainsi que sa généralisation aux nombres algébriques réels > 1 qui ne sont ni de Pisot, ni de Salem, se trouvent dans deux travaux récents [4, 6] du second auteur.

Théorème 1 ([4, 6]). *Soit $\alpha > 1$ un nombre réel algébrique, et soit $P(X) \in \mathbf{Z}[X]$ son polynôme minimal. Soit $F(X)$ un polynôme à coefficients réels, de degré $r \geq 0$, et dont le coefficient dominant est positif. Supposons en outre que $F(X) \notin \mathbf{Q}(\alpha)[X]$ si ou bien α est un nombre de Pisot, ou bien $r = 0$ et α est un nombre de Salem. Alors, les parties fractionnaires $\{F(n)\alpha^n\}$, $n \geq 0$, ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à $1/\ell(P^{r+1})$.*

Ici, $\ell(P^{r+1})$ désigne la *longueur réduite* du polynôme $P(X)^{r+1}$, définie *infra*. Les hypothèses sur le polynôme $F(X)$ sont nécessaires [14].

Nous supposons désormais que α est un entier > 1, et choisissons de le noter b. Il découle du Théorème 1 que, pour tout *irrationnel* ξ, la longueur de tout intervalle I contenant toutes les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$, est au moins égale à $1/b$. Dans cette note, nous caractérisons complètement les paires (ξ, I), formées d'un nombre réel irrationnel $\xi > 0$ et d'un intervalle I, pour lesquelles $\{\xi b^n\}$ appartient à I pour tout $n \geq 0$. Nous employons la terminologie de la combinatoire des mots [2, 10], et notamment la notion de suite sturmienne.

Théorème 2. *Soient $b \geq 2$ un entier et ξ un nombre réel irrationnel. Les parties fractionnaires $\{\xi b^n\}$, $n \geq 0$, ne peuvent pas toutes se trouver dans un intervalle de longueur strictement inférieure à $1/b$. En outre, les nombres $\{\xi b^n\}$, $n \geq 0$, sont tous dans un intervalle fermé I de longueur $1/b$ si, et seulement si, $\xi = g + k/(b-1) + t_b(\mathbf{w})$, où g est un entier quelconque, k appartient à $\{0, 1, \ldots, b-2\}$ et $\mathbf{w}$ est un mot sturmien sur $\{0, 1\}$. Si tel est le cas, alors ξ est transcendant et l'intervalle I est semi-ouvert. De plus, I est ouvert sauf s'il existe un entier $j \geq 0$ et un mot sturmien caractéristique $\mathbf{u}$ tels que $T^j(\mathbf{w}) = \mathbf{u}$.*

En particulier, puisqu'il existe une infinité non dénombrable de suites sturmiennes sur $\{0, 1\}$, le Théorème 2 montre qu'il existe une infinité non dénombrable de paires (ξ, s), où ξ est irrationnel et $s \in]0, 1-1/b[$, telles que $s < \{\xi b^n\} < s + 1/b$ pour tout $n \geq 0$.

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1. Introduction

In 1968, Mahler [11] introduced the notion of Z-numbers. These are precisely the positive real numbers ξ such that $0 \leq \{\xi(3/2)^n\} < 1/2$ for all integers $n \geq 0$. Here and below, $\{\cdot\}$ denotes the fractional part. The set of Z-numbers is at most countable [11], and it is widely believed it is even empty. This raises the following more general questions. Given a real number $\alpha > 1$ and an interval $[s, t)$ included in $[0, 1)$, are there any positive numbers ξ such that $s \le \{\xi\alpha^n\} < t$ for all integers $n \ge 0$? What is the smallest possible difference $t - s$ for which such positive numbers ξ do exist?

The purpose of this note is twofold. Firstly, we announce several new results obtained by the second named author for algebraic numbers α. Secondly, we give a complete answer to the above questions for rational integers $\alpha = b \ge 2$.

Let p and q be coprime integers with $p > q \ge 2$. Flatto, Lagarias & Pollington [9] showed that, for any interval I of length strictly smaller than $1/p$, there are no $\xi > 0$ such that $\{\xi(p/q)^n\} \in I$ for all integers $n \ge 0$. Presumably, this also holds for any interval $I = [s, s+1/p)$, with $s \in [0, 1-1/p]$. Actually, it was proved in [9] that this is the case for any s lying in a dense subset of $[0, 1-1/p]$, and, later, the first named author established [3] that this is also the case for any s lying in a subset of full Lebesgue measure of $[0, 1-1/p]$.

A new, simpler proof of the result of Flatto, Lagarias & Pollington and its generalization from rational non-integer numbers $\alpha = p/q$ to arbitrary real algebraic numbers α which are neither PV-numbers nor Salem numbers has been recently given by the second named author [4, 6]. Recall that an algebraic integer $\alpha > 1$ is called a *PV-number* (resp. *Salem number*) if its remaining conjugates (if any) are all inside the unit disc $|z| < 1$ (resp. in $|z| \le 1$ with at least one conjugate lying on $|z| = 1$). To state these results, we define the *reduced length* of a polynomial $P(X) \in \mathbf{R}[X]$, denoted by $\ell(P)$, to be the infimum of the lengths (that is, the sums of the absolute values of the coefficients) of the polynomials $P(X) \cdot G(X)$, taken over every polynomial $G(X) \in \mathbf{R}[X]$ with either leading coefficient or constant coefficient equal to 1. It is easy to prove that $\ell(qX - p) = p$ for all integers $p > q \ge 1$ (see [4] or [13], where the reduced length of a polynomial was studied in detail).

Theorem 1 ([4, 6]). *Let $\alpha > 1$ be a real algebraic number with minimal defining polynomial $P(X) \in \mathbf{Z}[X]$, and let $F(X)$ be a degree $r \ge 0$ real polynomial with positive leading coefficient. Suppose, in addition, that $F(X) \notin \mathbf{Q}(\alpha)[X]$ if either α is a PV-number or $r = 0$ and α is a Salem number. Then the fractional parts $\{F(n)\alpha^n\}$, $n \ge 0$, cannot all lie in an interval of length smaller than $1/\ell(P^{r+1})$.*

The extra conditions on $F(X)$ in Theorem 1 that concern PV and Salem numbers α are necessary. This is clear for PV-numbers α, whereas for Salem numbers α the necessity of the condition $\xi = F(X) \notin \mathbf{Q}(\alpha)$, where $r = \deg F = 0$, is shown in [14]. Other results related to Theorem 1 have been obtained in [1] and [5].

From now on, suppose that $\alpha > 1$ is an integer, say $\alpha = b \ge 2$. It is a PV-number, so Theorem 1 implies that, for any interval I of length strictly smaller than $1/b$, there are no *irrational* numbers ξ for which $\{\xi b^n\} \in I$ for every integer $n \ge 0$. In particular, it follows from this that $1/b$ is the smallest possible length of an interval to which all the fractional parts $\{\xi b^n\}$, $n \ge 0$, with fixed irrational ξ, can belong. This also raises the question whether, for an interval I of length $1/b$, there exists an irrational number ξ such that $\{\xi b^n\} \in I$ for all integers $n \ge 0$. In the present note, we show that there are uncountably many pairs (ξ, I) with this property and describe all of them.

Note that, writing the b-adic expansion of $\{\xi\}$, namely, $\xi = g + x_1 b^{-1} + x_2 b^{-2} + \ldots$, where $g = [\xi]$ and $x_1, x_2, \ldots \in \{0, 1, \ldots, b-1\}$, we have

$$\{\xi b^n\} = x_{n+1} b^{-1} + x_{n+2} b^{-2} + x_{n+2} b^{-3} + \ldots := 0.x_{n+1} x_{n+2} x_{n+3} \ldots$$

for any $n \geq 0$. So, in other words, we are interested in the following question: determine the smallest possible interval I to which belong all the *tails* of an irrational number $\xi = g + 0.x_1x_2x_3\ldots$ (in its b-adic expansion), namely, the numbers $0.x_{n+1}x_{n+2}x_{n+2}\ldots$, where $n \geq 0$.

2. Main result

We will use the terminology from combinatorics on words (see, for instance, [2] or [10]). For an infinite word $\mathbf{w}$, let us denote by $p(\mathbf{w}, m)$ the number of distinct blocks of length m occurring in $\mathbf{w}$. Morse and Hedlund [12] proved that the function $m \mapsto p(\mathbf{w}, m)$ is either bounded, or strictly increasing. Consequently, $\mathbf{w}$ is not ultimately periodic (in this context usually called *aperiodic*) if, and only if, $p(\mathbf{w}, m) \geq m + 1$ holds for every positive integer m. By definition, an infinite word $\mathbf{w}$ is called *Sturmian* if we have $p(\mathbf{w}, m) = m+1$ for any positive integer m. (In particular, since then $p(\mathbf{w}, 1) = 2$, this implies that $\mathbf{w}$ is a word on an alphabet of two letters.) There are many equivalent definitions for Sturmian words, and we refer the reader to Chapter 2 from [10] or to Chapter 6 from [2]. We just recall that $\mathbf{w} := w_1w_2\ldots$ is a *characteristic Sturmian word* if, and only if, there exists an irrational number β (the *slope*) in $(0, 1)$ such that $w_n = [\beta(n+1)] - [\beta n]$ for every positive integer n.

Suppose that T^j maps the word $\mathbf{w} = w_1 \ldots w_j w_{j+1} \ldots$ to the word $w_{j+1}w_{j+2}\ldots$ and set $t_b(\mathbf{w}) := 0.w_1w_2\ldots = \sum_{j=1}^{\infty} w_j b^{-j}$. With this notation, we can state our main result.

Theorem 2. *Let $b \geq 2$ be an integer and ξ be an irrational real number. Then the numbers $\{\xi b^n\}$, $n \geq 0$, cannot all lie in an interval of length strictly smaller than $1/b$. On the other hand, the real numbers $\{\xi b^n\}$, $n \geq 0$, are all lying in a closed interval I of length $1/b$ if, and only if, $\xi = g + k/(b-1) + t_b(\mathbf{w})$, where g is an arbitrary integer, k is in $\{0, 1, \ldots, b-2\}$, and $\mathbf{w}$ is a Sturmian word on $\{0, 1\}$. If this is the case, then ξ is transcendental and the interval I is semi-open. Moreover, it is open, unless there exists an integer $j \geq 1$ such that $T^j(\mathbf{w}) = \mathbf{u}$ is a characteristic Sturmian word.*

In particular, since there are uncountably many Sturmian sequences on $\{0, 1\}$, Theorem 2 shows that there are uncountably many pairs (ξ, s), where ξ is irrational and $s \in (0, 1 - 1/b)$, such that $s < \{\xi b^n\} < s + 1/b$ for every $n \geq 0$.

At the end of the paper [7] the following problem is posed: prove that, for any real numbers ξ and ν with $\xi > 0$, the numbers $[\xi 2^n + \nu]$ are composite for infinitely many $n \in \mathbf{N}$. Observe that if we have $0 \leq \{\xi 2^{n-1} + (\nu - 1)/2\} < 1/2$, then the number $[\xi 2^n + \nu - 1]$ is even and so $[\xi 2^n + \nu]$ is odd. Thus, since there are uncountably many Sturmian words on the alphabet $\{0, 1\}$, it follows from Theorem 2 that there do exist uncountably many pairs (ξ, ν) for which $[\xi 2^n + \nu]$ is odd for every positive integer n.

3. Proof of Theorem 2

Before giving the proof of Theorem 2, we gather in an auxiliary lemma results from Proposition 2.1.3, Theorem 2.1.5 and Proposition 2.1.22 of Chapter 2 of [10].

Lemma. *Let* $\mathbf{w}$ *be an infinite aperiodic word on* $\{0,1\}$. *Then,* $\mathbf{w}$ *is Sturmian if, and only if, for any finite word* $\mathbf{v}$, *at least one of the words* $0\mathbf{v}0$ *and* $1\mathbf{v}1$ *is not a factor of* $\mathbf{w}$. *Moreover,* $\mathbf{w}$ *is Sturmian characteristic if, and only if, both* $0\mathbf{w}$ *and* $1\mathbf{w}$ *are Sturmian.*

Let us write ξ in the form $g + t_b(\mathbf{x})$, where $g = [\xi]$ is an integer and $t_b(\mathbf{x}) = x_1 b^{-1} + x_2 b^{-2} + x_3 b^{-3} + \ldots = 0.x_1x_2x_3\ldots$ is the b-adic expansion of $\{\xi\} = \xi - g$. As above, $\{\xi b^n\} = 0.x_{n+1}x_{n+2}\ldots$. In particular, since ξ is irrational, this implies that $x_{n+1}b^{-1} < \{\xi b^n\} < x_{n+1}b^{-1} + b^{-1}$. Thus, if there exist x_{i+1} and x_{j+1}, with $i, j \geq 0$, satisfying $x_{j+1} - x_{i+1} \geq 2$, then we get

$$\{\xi b^j\} - \{\xi b^i\} > x_{j+1}b^{-1} - x_{i+1}b^{-1} - b^{-1} \geq 2/b - 1/b = 1/b.$$

Consequently, we can assume without loss of generality that $x_1, x_2, \ldots \in \{k, k+1\}$, where $k = 0, 1, \ldots, b-2$. Thus, we can write ξ in the form $g + k/(b-1) + t_b(\mathbf{w})$, where $\mathbf{w} = w_1w_2\ldots$ is a word on the alphabet $\{0,1\}$ and $t_b(\mathbf{w}) = w_1b^{-1} + w_2b^{-2} + w_3b^{-3} + \ldots = 0.w_1w_2w_3\ldots$. Now, we have

$$\{\xi b^n\} - k/(b-1) = 0.w_{n+1}w_{n+2}\ldots = w_{n+1}b^{-1} + w_{n+2}b^{-2} + \ldots.$$

Since ξ is irrational, the complexity function of the infinite word $\mathbf{w} := w_1w_2\ldots$ is strictly increasing. This implies that, for any $m \geq 1$, there exists (at least) one block $\mathbf{w}_m$ of m letters such that both $0\mathbf{w}_m$ and $1\mathbf{w}_m$ are subblocks of $\mathbf{w}$. In other words, there exist integers $u = u(m)$ and $v = v(m)$ such that $\{\xi b^u\} - k/(b-1) = 0.0\mathbf{w}_m\mathbf{w}'$ and $\{\xi b^v\} - k/(b-1) = 0.1\mathbf{w}_m\mathbf{w}''$. Hence $\{\xi b^v\} - \{\xi b^u\} > b^{-1} - b^{-m}$. By taking m sufficiently large, we conclude that no interval of length strictly smaller than $1/b$ can contain all the $\{\xi b^n\}$ with $n \geq 0$. (Taking $\xi = 0.101010\ldots$ or simply $\xi = 1$ shows that the assumption 'ξ is irrational' is necessary.)

Let us now prove the second statement. Assume that $\mathbf{w}$ is Sturmian. By the lemma, for any finite word $\mathbf{v}$, the words $0\mathbf{v}0$ and $1\mathbf{v}1$ cannot be both factors of $\mathbf{w}$. Consequently, the difference between any two numbers $\{\xi b^j\}$ and $\{\xi b^i\}$ is bounded above in absolute value by $1/b$. The inequality is strict, since $\mathbf{w}$ is aperiodic. Thus, we have shown that, for $\mathbf{w}$ Sturmian, there exists a semi-open interval of length $1/b$ that contains all the $\{\xi b^n\}$, where $n \geq 0$. Furthermore, it follows from [8] that ξ is transcendental.

Assume now that $\mathbf{w}$ is neither Sturmian, nor ultimately periodic. Then, by the lemma, there exists a finite world $\mathbf{u}$ such that both $0\mathbf{u}0$ and $1\mathbf{u}1$ are factors of $\mathbf{w}$. Arguing as above, we see that the difference between corresponding fractional parts is greater than $1/b$. This shows that, for such $\mathbf{w}$, there does not exist a closed interval of length $1/b$ containing all fractional parts $\{\xi b^n\}$, $n \geq 0$, and proves the second part.

Finally, let $\mathbf{w}$ be an infinite Sturmian word. The fact that the numbers $0.w_{n+1}w_{n+2}\ldots$ all belong to a closed interval of length $1/b$ can be expressed in the form

$$t_b(0\mathbf{u}) \leq t_b(T^n\mathbf{w}) \leq t_b(1\mathbf{u}) = t_b(0\mathbf{u}) + b^{-1},$$

where $\mathbf{u}$ is a word on $\{0,1\}$ and where n runs through every non-negative integer. For simplicity (and according to the lexicographical order of words), we can write this inequality in the form

$$0\mathbf{u} \leq T^n\mathbf{w} \leq 1\mathbf{u}, \qquad \text{for any } n \geq 0.$$

Evidently, all $t_b(T^n\mathbf{w})$ belong to an open interval of length $1/b$, unless there is a $h \geq 0$ such that $T^h\mathbf{w} = 0\mathbf{u}$ or $1\mathbf{u}$. Assume that $T^h\mathbf{w} = 0\mathbf{u}$. Then we have

$$0\mathbf{u} < T^n\mathbf{w} < 1\mathbf{u}, \qquad \text{for any } n \geq h+1,$$

that is,

$$0\mathbf{u} < T^n\mathbf{u} < 1\mathbf{u}, \qquad \text{for any } n \geq 0.$$

The case $T^h\mathbf{w} = 1\mathbf{u}$ leads to the same inequalities. These are strict, since $\mathbf{w}$ is aperiodic.

Let us prove now that $\mathbf{u}$ is Sturmian characteristic. In view of the lemma, it is sufficient to show that both $0\mathbf{u}$ and $1\mathbf{u}$ are Sturmian. Observe that $\mathbf{u}$ is aperiodic, since $\mathbf{u} = T^{h+1}(\mathbf{w})$. Assume that $p(0\mathbf{u}, m) \geq m+2$ for some m. The first part of Theorem 2 implies that $0\mathbf{u}$ is a limit point of the sequence $\mathbf{u}, T^1\mathbf{u}, T^2\mathbf{u}, \ldots$, hence, we get that $p(T^n\mathbf{u}, m) \geq m+2$ for some n. This yields

$$m+1 = p(\mathbf{w}, m) \geq p(T^n\mathbf{u}, m) \geq m+2,$$

a contradiction. Consequently, $0\mathbf{u}$ is Sturmian, and so is $1\mathbf{u}$, by a similar argument.

We thus conclude that the numbers $t_b(T^n\mathbf{w})$, $n \geq 0$, all belong to an open interval of length $1/b$, unless there are an integer $h \geq 0$ and a characteristic Sturmian word $\mathbf{u}$ such that $T^h\mathbf{w} = 0\mathbf{u}$ or $1\mathbf{u}$. So $\mathbf{u} = T^j(\mathbf{w})$ with $j = h+1 \geq 1$. The proof of Theorem 2 is completed.

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