Sets of exact approximation order by rational numbers

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1. Introduction

Let $\Psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a decreasing function such that $x \mapsto x^2\Psi(x)$ is non-increasing. In 1924, Khintchine [13] (see also his book [14]) used the theory of continued fractions to prove that the set of Ψ-approximable real numbers

$$\mathcal{K}(\Psi) := \left\{ \xi \in \mathbf{R} : \left| \xi - \frac{p}{q} \right| < \Psi(q) \text{ for infinitely many rational numbers } \frac{p}{q} \right\}$$

has Lebesgue measure zero if the sum $\sum_{x\geq 1} x\Psi(x)$ converges and has full Lebesgue measure otherwise. A few years later, this result has been refined by Jarník [11] and, independently, Besicovitch [3], who established that, for any real number $\tau \geq 1$, the Hausdorff dimension of the set $\mathcal{K}(\tau) := \mathcal{K}(x \mapsto x^{-2\tau})$ is equal to $1/\tau$. Moreover, Jarník [12] determined the Hausdorff $\mathcal{H}^f$-measure of the set $\mathcal{K}(\Psi)$ for a general dimension function f (see Theorem J below, some background on the theory of Hausdorff measure is recalled at the beginning of Section 2). His precise results enabled him [12, Satz 5] to show that, if $\Phi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ is a positive continuous function such that $\Phi(x)/x$ tends monotonically to infinity with x, then the sets $\mathcal{K}(\Psi) \setminus \mathcal{K}(\Psi \circ \Phi)$ and $\mathcal{K}(\Psi)$ have same Hausdorff $\mathcal{H}^f$-measure.

In the same paper, Jarník [12, Satz 6] used the theory of continued fractions to show explicitly that if $\Psi(x) = o(x^{-2})$ then there exist real numbers in $\mathcal{K}(\Psi)$ which do not belong to any set $\mathcal{K}(c\Psi)$ with $0 < c < 1$. This means that the set

$$\mathrm{Exact}(\Psi) := \mathcal{K}(\Psi) \setminus \bigcup_{m\geq 2} \mathcal{K}\big((1 - 1/m)\Psi\big)$$

of real numbers approximable at order Ψ and not better is non empty. In other words, $\mathrm{Exact}(\Psi)$ is the set of real numbers ξ such that

$$|\xi - p/q| \leq \Psi(q) \qquad \text{infinitely often}$$

and

$$|\xi - p/q| \geq c\,\Psi(q) \qquad \text{for any } c < 1 \text{ and any } q \geq q_0(c),$$

where $q_0(c)$ denotes a positive real number depending only on c. Jarník's method, however, provides no metric statements for this set.

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Recently, Beresnevich, Dickinson & Velani [1] applied a theorem of Dickinson & Velani [7] to establish very precise metric statements for sets of Ψ-well approximable systems of linear forms which extend earlier results of Jarník [12]. At the end of [1], they introduced the set $\mathrm{Bad}(\Psi)$ consisting of the real numbers ξ such that

$$|\xi - p/q| \leq \Psi(q) \qquad \text{infinitely often}$$

and

$$|\xi - p/q| \geq c(\xi)\,\Psi(q) \qquad \text{for all rationals } p/q,$$

where $c(\xi) > 0$ is a constant dependent only on ξ. Clearly, the inclusions $\mathrm{Exact}(\Psi) \subset \mathrm{Bad}(\Psi) \subset \mathcal{K}(\Psi)$ hold. However, as pointed out in [1], no metric result is known for a general set $\mathrm{Bad}(\Psi)$ and the authors ask for its Hausdorff dimension. The purpose of the present work is to give a complete answer to this question: we show that the sets $\mathrm{Exact}(\Psi)$ and $\mathcal{K}(\Psi)$ have same 'size', inasmuch as they have same Hausdorff dimension. Furthermore, the sets $\mathrm{Bad}(\Psi)$ and $\mathcal{K}(\Psi)$ have same Hausdorff $\mathcal{H}^f$-measure.

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2. Results

Throughout this paper, rational numbers are always written under their lowest form. We use $[x]$ to denote the integer part of the real number x. Further, the Lebesgue measure is denoted by Leb and we often write $|I|$ rather than $\mathrm{Leb}(I)$ for a bounded real interval I. We briefly recall the definition of the Hausdorff dimension of a real set E.

Let $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a *dimension function,* that is, a strictly increasing, continuous function such that $f(x)$ tends to 0 as x tends to the origin. Let δ be a positive real number and set

$$\mathcal{H}^f_\delta(E) := \inf \sum_j f(|U_j|),$$

where the infimum is taken over all enumerable families of intervals U_j of length $|U_j| < \delta$ which cover E. As δ decreases to 0, $\mathcal{H}^f_\delta(E)$ increases to a finite or infinite limit $\mathcal{H}^f(E)$, called the $\mathcal{H}^f$-measure of E. When f is the function $x \mapsto x^s$, with $0 < s \leq 1$, then $\mathcal{H}^f$ is the usual s-dimensional Hausdorff measure, and we write simply $\mathcal{H}^s$ rather than $\mathcal{H}^f$. Then, the Hausdorff dimension of E, denoted by $\dim E$, is the real number s_0 in $[0,1]$ defined by

$$s_0 = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = +\infty\}.$$

For more results on the theory of Hausdorff measure, the reader is directed to the books of Falconer [9] and of Bernik & Dodson [2].

An indication of how a function $g : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ grows near infinity is provided by its *lower order at infinity* $\lambda(g)$, which is defined by

$$\lambda(g) = \liminf_{x \to +\infty} \frac{\log g(x)}{\log x},$$

and arises naturally in estimating the Hausdorff dimension of the sets $\mathcal{K}(\Psi)$, see e.g. Dodson [8] and Dickinson [6]. Clearly, $\lambda(g)$ is non-negative when $\lim_{x\to+\infty} g(x) = +\infty$. Theorem 2 of [8] gives the exact value of the Hausdorff dimension of the set $\mathcal{K}(\Psi)$ in terms of the lower order at infinity of the function $1/\Psi$. Our first result shows that, in terms of Hausdorff dimension, the sets $\mathcal{K}(\Psi)$ and $\mathrm{Exact}(\Psi)$ have same 'size'. It answers the question asked by Beresnevich, Dickinson & Velani at the end of [1].

Theorem 1. *Let $\Psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be such that $x \mapsto x^2\Psi(x)$ is non-increasing. Assume that the sum $\sum_{x\geq 1} x\Psi(x)$ converges and let λ denote the lower order at infinity of the function $1/\Psi$. Then we have*

$$\dim \mathrm{Exact}(\Psi) = \dim \mathcal{K}(\Psi) = \frac{2}{\lambda}.$$

The assumptions on Ψ in Theorem 1 imply that the lower order at infinity of $1/\Psi$ is at least 2. Let us mention that Dodson [8] determined the exact Hausdorff dimension of $\mathcal{K}(\Psi)$ when $x \mapsto x\Psi(x)$ is non-increasing, which is slightly less restrictive than our assumption.

As for most of the metric problems in Diophantine approximation, the upper bound for the Hausdorff dimension follows easily from a covering argument ([8], Lemma 1). The main difficulty rests on establishing the lower bound. To this end, we construct a Cantor set $\mathcal{C}$ with a nice structure and contained in $\mathrm{Exact}(\Psi)$. More precisely, $\mathcal{C}$ is defined as an intersection $\bigcap_{k\geq 0} E_k$ where each E_k is a finite union of disjoint compact intervals. For any $k \geq 0$, we have $E_{k+1} \subset E_k$ and each interval composing E_k contains 'many' intervals of E_{k+1}, which are 'roughly equally spaced'. A classical result (see Proposition 1 in Section 3) shows that, under these assumptions, we get a lower bound for the Hausdorff dimension of $\bigcap_{k\geq 0} E_k$, hence of $\mathcal{C}$. The inductive construction of the sets E_k is explained in Section 4.

As an immediate corollary, we display the following strengthening of the classical theorem of Jarník and Besicovitch.

Corollary 1. *Let $\tau > 1$ be a real number. Then, the set of real numbers ξ satisfying*

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^{2\tau}} \qquad \text{for infinitely many rational numbers } \frac{p}{q}$$

and, for any positive real number $c < 1$,

$$\left|\xi - \frac{p}{q}\right| \geq \frac{c}{q^{2\tau}} \qquad \text{for } q \text{ sufficiently large,}$$

has Hausdorff dimension equal to $1/\tau$.

Theorem 1 gives the Hausdorff dimension of the set $\mathrm{Exact}(\Psi)$ but provides no information regarding its Hausdorff *measure*. Concerning the set $\mathcal{K}(\Psi)$, Jarník ([12], Sätze 3 und 4) established the following very precise statement.

Theorem J. *Let $\Psi : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be such that $x \mapsto x^2\Psi(x)$ is non-increasing. Let $f : \mathbf{R}_{>0} \to \mathbf{R}_{>0}$ be a dimension function such that $f(x)/x$ tends monotonically to infinity when x tends to 0. Assume that the sum $\sum_{x=1}^{+\infty} x\Psi(x)$ converges and that the function $x \mapsto x^2 f(2\Psi(x))$ is non-increasing. Then $\mathcal{H}^f(\mathcal{K}(\Psi)) = +\infty$ if the sum*

$$\sum_{x=1}^{+\infty} x f(2\Psi(x)) \tag{1}$$

diverges, and $\mathcal{H}^f(\mathcal{K}(\Psi)) = 0$ otherwise.

No importance has to be attached to the 2 in (1) since, in view of the assumptions on f and Ψ, the sums $\sum_{x=1}^{+\infty} x f(2\Psi(x))$ and $\sum_{x=1}^{+\infty} x f(\Psi(x))$ have same behaviour.

The result actually proved by Jarník is more general and deals with simultaneous Diophantine rational approximation. Recently, Theorem J has been extended to systems of n linear forms of m variables by Dickinson & Velani [7]. In collaboration with Beresnevich [1], they applied their results to compute the Hausdorff measure of sets with exact 'logarithmic' order of approximation. In view of their results, it would be desirable to determine the Hausdorff measure of $\mathrm{Exact}(\Psi)$. Unfortunately, we are unable to get such a precise result, and we are led to introduce new sets of real numbers, which contain $\mathrm{Exact}(\Psi)$. For a given positive real number $c < 1$, we denote by $\mathrm{Bad}_c(\Psi)$ the set of real numbers ξ satisfying

$$\left|\xi - \frac{p}{q}\right| < \Psi(q) \qquad \text{for infinitely many rational numbers } \frac{p}{q}$$

and

$$\left|\xi - \frac{p}{q}\right| \geq c\,\Psi(q) \qquad \text{for } q \text{ sufficiently large.}$$

Observe that we have

$$\mathrm{Exact}(\Psi) = \bigcap_{m \geq 2} \mathrm{Bad}_{1-1/m}(\Psi)$$

and, for any $c < 1$,

$$\mathrm{Bad}_c(\Psi) \subset \mathrm{Bad}(\Psi),$$

where $\mathrm{Bad}(\Psi)$ is the set defined in the Introduction. Theorem 2 below gives the Hausdorff measure of $\mathrm{Bad}_c(\Psi)$, thus it considerably improves upon Theorem 3 of [1] in the case $m = n = 1$.

Theorem 2. *We keep the assumptions of Theorem J. For any positive real number $c < 1$, we have $\mathcal{H}^f(\mathrm{Bad}_c(\Psi)) = +\infty$ if the sum*

$$\sum_{x=1}^{+\infty} x f\big(\Psi(x)\big)$$

diverges, and $\mathcal{H}^f(\mathrm{Bad}_c(\Psi)) = 0$ otherwise.

It follows from Theorem 2 that for no dimension function f the $\mathcal{H}^f$-measure of $\mathrm{Bad}_c(\Psi)$ is positive and finite. Thus, the set $\mathrm{Bad}_c(\Psi)$ is not an s-set.

The Hausdorff dimension of the set $\mathrm{Bad}_c(\Psi)$ easily follows from Theorem 1. However, we emphasize that this is not a consequence of Theorem 2, which yields that

$$\dim \mathrm{Bad}_c(\Psi) = \inf\{s : \sum_{x=1}^{+\infty} x\Psi(x)^s < +\infty\}.$$

This result does not imply Theorem 1 when the lower and upper orders at infinity of $1/\Psi$ do not coincide.

The proof of Theorem 2 rests essentially on the same ideas as that of Theorem 1, but is more technically involved.

If the function Ψ is such that the sum $\sum_{x\geq 1} x\Psi(x)$ converges, then the set $\mathcal{K}(\Psi)$ has full Lebesgue measure, according to Khintchine's theorem. If $\Psi(x) = o(x^{-2})$, then $\mathrm{Exact}(\Psi)$ is non empty, as proved by Jarník ([12], Satz 6). However, no metric statement is known for $\mathrm{Exact}(\Psi)$, and the methods developed in the present paper are likely to be useless. We plan to return to this question in a further work.

3. Auxiliary lemmas

In this Section, we gather some well-known results in the theory of continued fractions.

Lemma 1. *Every rational number p/q with $0 < p/q < 1$ has a continued fraction expansion of the shape $p/q = [0; a_1, \ldots, a_n]$ with n even and $a_j \geq 1$ for any j with $1 \leq j \leq n$. Further, if $c > b > 0$ are real numbers, then*

$$[0; a_1, \ldots, a_n] < [0; a_1, \ldots, a_n, c] < [0; a_1, \ldots, a_n, b].$$

PROOF. See e.g. Chapter I of [17]. □

Lemma 2. *For any irrational number $\xi = [0; a_1, a_2, \ldots]$ and any integer $n \geq 1$, the n-th convergent $p_n/q_n := [0; a_1, \ldots, a_n]$ of ξ satisfies*

$$\frac{1}{3q_n^2 a_{n+1}} \leq \frac{1}{q_n^2(a_{n+1}+2)} < \left|\xi - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2 a_{n+1}}.$$

PROOF. See e.g. [14], inequalities (34). □

Lemma 3. *Let $n \geq 1$ be an integer and let $\mathbf{k} = (k_1, \ldots, k_n)$ be a n-tuple of positive integers. Denote by $E_{\mathbf{k}}$ the set composed by the real numbers in $[0, 1[$ whose partial quotients satisfy $a_i = k_i$ for all $1 \leq i \leq n$. Let $S \geq 3$ be a real number and denote by E_S the subset of $E_{\mathbf{k}}$ composed by the real numbers in $E_{\mathbf{k}}$ with $a_{n+1} \leq S$. Then, we have*

$$\mathrm{Leb}(E_S) \geq \left(1 - \frac{3}{S}\right) \mathrm{Leb}(E_{\mathbf{k}}).$$

PROOF. It follows from inequalities (35) of Khintchine [14] that

$$\frac{\mathrm{Leb}(E_S)}{\mathrm{Leb}(E_{\mathbf{k}})} \geq 1 - \sum_{s=[S]+1}^{+\infty} \frac{2}{s^2} = 1 - \frac{2}{[S]} \geq 1 - \frac{3}{S},$$

since $S \geq 3$. □

Lemma 4. *Let $\xi := [0; a_1, a_2, \ldots]$ be a real number and u/v be a rational number with $|\xi - u/v| < 1/v^2$. Then, there exists an integer $n \geq 0$ such that*

$$\frac{u}{v} \in \{[0; a_1, \ldots, a_n], [0; a_1, \ldots, a_n, 1], [0; a_1, \ldots, a_n, a_{n+1} - 1]\}.$$

PROOF. This result is due to Grace [10], and its proof can be found in [15], Theorem I.10. □

Lemma 5. *Let $\Psi : \mathbf{Z}_{>0} \to \mathbf{R}_{>0}$ be a non-increasing function and assume that the sum $\sum_{x \geq 1} \Psi(x)$ diverges (resp. converges). Let $c > 0$ and $M > 0$ be real numbers. Then, the sum $\sum_{x \geq 1} M^x \, \Psi(cM^x)$ also diverges (resp. converges).*

PROOF. This follows from the corresponding property of the integral

$$\int_1^{\infty} M^x f(cM^x) dx = \frac{1}{c \log M} \int_{cM}^{\infty} f(y) dy,$$

where f is monoton and continuous on $\{x : x \geq 1\}$, and coincides with Ψ on $\mathbf{Z}_{>0}$. □

When a Cantor set $\mathcal{C}$ has a nice structure, it is often possible to determine its Hausdorff dimension. Let

$$[0, 1] = E_0 \supset E_1 \supset E_2 \supset \ldots$$

be a decreasing sequence of sets such that each E_k is a finite union of disjoint closed intervals. Assume further that each interval of E_k contains at least two intervals of E_{k+1} and that the maximal length of intervals in E_k tends to 0 as k tends to infinity. Then the set

$$\mathcal{C} := \bigcap_{k=0}^{+\infty} E_k \tag{2}$$

is a totally disconnected subset of $[0, 1]$, and, under some extra hypothesis, we have a lower bound for its Hausdorff dimension.

Proposition 1. *Suppose that in the general construction (2), for any integer $k \geq 1$, each interval of E_{k-1} contains at least m_k intervals of E_k which are separated by gaps of at least ε_k, where $0 < \varepsilon_{k+1} < \varepsilon_k$. We then have*

$$\dim E \geq \liminf_{k \to +\infty} \frac{\log(m_1 \ldots m_{k-1})}{-\log(m_k \varepsilon_k)}. \tag{3}$$

PROOF. This is Example 4.6 of Falconer [9]. □

We keep the hypothesis of Proposition 1 and we assume further that the intervals of E_k are of length δ_k and 'evenly spaced' in the sense that $m_k \varepsilon_k \geq c\delta_{k-1}$, where $c > 0$ is a constant. Then (3) becomes $\dim E \geq \liminf_{k\to+\infty} (\log m_k)/(-\log \delta_k)$. This observation appears to be crucial for the proof of Theorem 1.

A classical tool for obtaining a lower bound for the Hausdorff measure of an arbitrary set of real numbers is the *mass distribution principle.*

Proposition 2. *Let μ be a probability measure with support in a bounded set $E \subset \mathbf{R}$. Let f be a dimension function. Assume that there exist $\lambda > 0$ and $\delta > 0$ such that*

$$\mu(J) \leq \lambda f(|J|)$$

for any interval J with length $|J| \leq \delta$. We then have

$$\mathcal{H}^f(E) \geq \frac{1}{\lambda}.$$

PROOF. See e.g. [7]. □

4. Proof of Theorem 1

Let Ψ be as in the statement of the theorem. In view of the result of Dodson [8] and well-known properties of the Hausdorff dimension, it remains for us to prove that $\dim\big(\mathrm{Exact}(\Psi) \cap [0,1]\big) \geq 2/\lambda$. For any positive integer k, set $c_k = 1 - 2^{-k}$. Our strategy is the following. We construct inductively a (rapidly increasing) sequence $(Q_k)_{k\geq 1}$ of positive integers and, for any $k \geq 1$ a set E_k, which is a finite union of disjoint closed intervals contained in $[p/q + c_k\Psi(q), p/q + \Psi(q)]$, where $Q_k/10 \leq q \leq Q_k$. Each component of E_{k+1} is contained in exactly one interval of E_k, and the length of the largest interval composing E_k tends to zero when k goes to infinity. Furthermore, we proceed in such a way that $\mathcal{C} := \cap_{k\geq 1} E_k$ is contained in $\mathrm{Exact}(\Psi)$. In order to facilitate the exposition, we introduce the following notation.

Notation. *Let p/q be a rational number in $]0,1[$ and let $k \geq 1$ be an integer. Then we set*

$$U_k(p/q) := \big[[0; a_1, \ldots, a_n, A_{n+1}], [0; a_1, \ldots, a_n, B_{n+1}]\big],$$

where n is even,

$$[0; a_1, \ldots, a_n] = p/q,$$

$$A_{n+1} = [1/(c_k q^2 \Psi(q))] - 3 \qquad \textit{and} \qquad B_{n+1} = [1/(q^2\Psi(q))] + 1.$$

Lemma 1 shows that the notation $U_k(p/q)$ is consistent. Further, A_{n+1} and B_{n+1} have been chosen such that

$$U_k(p/q) \subset [p/q + c_k\Psi(q), p/q + \Psi(q)]$$

holds, as follows from Lemma 2. Moreover, there exists an integer Q_0 such that we have

$$2^{-k-1}\Psi(q) \le |U_k(p/q)| \le 2^{-k}\Psi(q), \tag{4}$$

for any $q \ge Q_0$.

Let $M \ge 10$ be an integer such that

$$M^2\Psi(M) \le 1/10 \qquad \text{and} \qquad \prod_{j\ge 1}\big(1 - 9\cdot 2^j\, M^2\, \Psi(2^{j/2}M)\big) \ge 1/2. \tag{5}$$

The existence of M follows from Lemma 5 since $x \mapsto x\Psi(x)$ is non-increasing and the sum $\sum_{x\ge 1} x\Psi(x)$ converges. These two assumptions imply that $x^2\Psi(x)$ tends to 0 as x tends to infinity. Let Q_1 be a large integer with

$$Q_1 \ge \max\{Q_0, 100M\} \qquad \text{and} \qquad Q_1^2\Psi(Q_1) \le 1/100. \tag{6}$$

Dirichlet's theorem asserts that, for any ξ in $[0,1]$, there exists a rational number p/q such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{qQ_1} \quad \text{and} \quad 1 \le q \le Q_1. \tag{7}$$

The Lebesgue measure of the set A_1 of ξ in $[0,1]$ for which (7) is satisfied by a rational number p/q with $q \le Q_1/10$ is less than or equal to

$$\sum_{q=1}^{[Q_1/10]+1} \frac{2}{qQ_1}\,(q+2),$$

which, by (6), is less than $1/4$. Let $\{p_j/q_j\}_{1\le j\le t_1}$ be a maximal family of rational numbers in $[0,1]$ with $Q_1/10 \le q_j \le Q_1$ and $|p_j/q_j - p_\ell/q_\ell| \ge 1/Q_1^2$ for $1 \le j \ne \ell \le t_1$. The union of the intervals

$$\left]\frac{p_j}{q_j} - \frac{11}{Q_1^2}, \frac{p_j}{q_j} + \frac{11}{Q_1^2}\right[, \quad 1 \le j \le t,$$

covers $[0,1]\setminus A_1$, thus its Lebesgue measure is at least $3/4$. Consequently, we get

$$\frac{3}{4} \le \frac{22}{Q_1^2}\, t_1,$$

that is, $t_1 \ge (3Q_1^2)/88$. We have proved that, up to a numerical constant, there are as many as one can expect rational numbers in $[0,1]$ of denominator between $Q_1/10$ and Q_1 which are evenly spaced.

We define the set E_1 as the union of the t_1 intervals $U_1(p_j/q_j)$, with $1 \le j \le t_1$. We point out that we do not have put any restriction on the partial quotients of the rational numbers p_j/q_j. No importance is attached to the quality of the approximation of the elements of the Cantor set $\mathcal{C}$ by rational numbers of denominator less than or equal to Q_1.

By construction, each real number in E_1 is Ψ-approximable by a rational number of denominator between $Q_1/10$ and Q_1. We construct a rapidly increasing sequence $(Q_k)_{k\ge 2}$ of integers and sets $E_2, E_3, \ldots$ such that, for any $k \ge 2$, each element of E_k is Ψ-approximable by a rational number of denominator between $Q_j/10$ and Q_j for any $1 \le j \le k$, and is not so well approximable by any other rational number of denominator between Q_1 and Q_k.

The inductive step is done thanks to the following lemma.

Lemma 6. *Let $k \geq 1$ be an integer and let $I := U_k(p/q)$ denote a connected component of E_k. Assume that the equation*

$$\left|\xi - \frac{u}{v}\right| < \Psi(v) \tag{8}$$

for ξ in I has at least k solutions in rational numbers u/v with $v \leq Q_k$. If $k \geq 2$, assume further that the equation

$$\left|\xi - \frac{u}{v}\right| < c_{k-1}\Psi(v)$$

for ξ in I has no solution in rational numbers u/v with $Q_{k-1} < v \leq Q_k$. Then, for any integer Q with $Q \geq 1000|I|^{-1}\log(1/|I|)$, there are $[Q^2|I|/88]$ rational numbers p_j/q_j in I with the following properties. For any j and ℓ with $1 \leq j < \ell \leq [Q^2|I|/88]$, we have $Q/10 \leq q_j \leq Q$ and $|p_j/q_j - p_\ell/q_\ell| \geq Q^{-2}$. Further, the interval

$$J_j := U_{k+1}(p_j/q_j)$$

is contained in I and, for any ξ in J_j, the equation (8) has at least $k+1$ solutions in rational numbers u/v with $v \leq Q$, while the equation

$$\left|\xi - \frac{u}{v}\right| < c_k\Psi(v) \tag{9}$$

has no solution in rational numbers u and v with $Q_k < v \leq Q$.

Proof. Write $p/q = [0; a_1, \ldots, a_n]$ with n even. Denote by $(a_\ell)_{\ell\geq 1}$ the sequence of partial quotients of a real number in I and by $(p_\ell/q_\ell)_{\ell\geq 1}$ the sequence of its convergents. Let J denote the subset of I consisting of the real numbers ξ whose partial quotients satisfy

$$a_{n+j+1} \leq \frac{1}{3 \cdot 2^j\, M^2\, \Psi(2^{j/2}M)}, \qquad \text{for } j \geq 1. \tag{10}$$

Lemma 3 and (5) yield that

$$\mathrm{Leb}(J) \geq |I| \prod_{j\geq 1} \bigl(1 - 9 \cdot 2^j\, M^2\, \Psi(2^{j/2}M)\bigr)$$

and our choice of M implies that

$$\mathrm{Leb}(J) \geq |I|/2. \tag{11}$$

Further, the sequence $(q_{n+j})_{j\geq 1}$ of the denominators of the convergents of ξ satisfies $q_n \geq Q_1/10$, $q_{n+1} \geq Q_1/10$ and $q_{n+j+2} \geq q_{n+j+1} + q_{n+j}$ for any integer $j \geq 0$. Thus, an easy induction shows that we have $q_{n+j} \geq 2^{j/2}Q_1/100$ for any $j \geq 0$ and we infer from (6) that

$$q_{n+j} \geq M\, 2^{j/2}, \qquad \text{for any } j \geq 1.$$

Since the function $x \mapsto x^2 \Psi(x)$ is non-increasing, we have

$$q_{n+j}^2 \, \Psi(q_{n+j}) \le 2^j \, M^2 \, \Psi(2^{j/2} M), \qquad \text{for any } j \ge 1.$$

Thus, if ξ is in J then (10) yields that

$$a_{n+j+1} \le \frac{1}{3 \, q_{n+j}^2 \, \Psi(q_{n+j})} \tag{12}$$

and we infer from (12) and Lemma 2 that

$$\left| \xi - \frac{p_{n+j}}{q_{n+j}} \right| > \Psi(q_{n+j}), \qquad \text{for any } j \ge 1 \text{ with } \xi \ne p_{n+j}/q_{n+j}.$$

Consequently, for ξ in J, we have

$$\left| \xi - \frac{u}{v} \right| \ge \Psi(v) \qquad \text{for any } u/v \text{ with } v > Q_k \text{ and } u/v \ne \xi. \tag{13}$$

Let $Q > 1$ be a real number with

$$Q \ge 1000 |I|^{-1} \log(1/|I|). \tag{14}$$

By Dirichlet's theorem, for any ξ in I, there exists a rational number p/q such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{qQ} \quad \text{and} \quad 1 \le q \le Q. \tag{15}$$

The Lebesgue measure of the set A of $\xi \in I$ for which (15) is satisfied by a rational number p/q with $q \le Q/10$ is less than or equal to

$$\sum_{q=1}^{[Q/10]+1} \frac{2}{qQ} \, (q|I| + 2) \le \frac{|I|}{4}, \tag{16}$$

by (14). Further, for $\xi \in (I \setminus A) \cap J$, there exists a rational number p/q such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{qQ} \qquad \text{and } Q/10 \le q \le Q. \tag{17}$$

Let $\{p_\ell/q_\ell\}_{1 \le \ell \le t_I}$ be a maximal family of rational numbers in $I \cap J$ with $Q/10 \le q_\ell \le Q$ such that $|p_h/q_h - p_\ell/q_\ell| \ge 1/Q^2$ for $1 \le h \ne \ell \le t_I$. The union of the intervals

$$\left] \frac{p_\ell}{q_\ell} - \frac{11}{Q^2}, \frac{p_\ell}{q_\ell} + \frac{11}{Q^2} \right[, \;\; 1 \le \ell \le t_I,$$

covers $(I \setminus A) \cap J$. Indeed, for ξ in $(I \setminus A) \cap J$, let p/q denote a rational number satisfying (17). In particular, we have $|\xi - p/q| < 1/q^2$ and, since ξ is in J, Lemma 4 implies that the partial quotients of p/q satisfy (10). Thus, p/q is in J. If p/q is not one of the p_j/q_j's, then by maximality, there exists ℓ with $1 \le \ell \le t_I$ and $|p/q - p_\ell/q_\ell| < 1/Q^2$, hence, by the triangle inequality, satisfying $|\xi - p_\ell/q_\ell| < 11/Q^2$. By (11) and (16), we have $\mathrm{Leb}((I \setminus A) \cap J) \ge |I|/4$, thus we get

$$t_I \ge \frac{Q^2|I|}{88},$$

and we have proved that there are at least $[Q^2|I|/88]$ evenly spaced rational numbers in $I \cap J$ of denominator between $Q/10$ and Q. Further, (13) asserts that, for any ℓ with $1 \le \ell \le [Q^2|I|/88]$, the equation

$$0 < \left| \frac{p_\ell}{q_\ell} - \frac{u}{v} \right| < \Psi(v)$$

has no solution in rational numbers u/v with $v > Q_k$, since p_ℓ/q_ℓ is in J. By construction, for any ξ in any of the $[Q^2|I|/88]$ intervals $U_{k+1}(p_\ell/q_\ell)$, equation (9) has no solution u/v with $Q_k < v \le Q$ and equation (8) has at least $k+1$ solutions u/v with $v \le Q$. Consequently, these intervals satisfy the conclusion of the lemma. □

Now we use Lemma 6 in order to construct a system of nested intervals. Assume that we have constructed the set E_k as the union of t_k disjoint intervals $U_j := U_k(p_j^{(k)}/q_j^{(k)})$, with $1 \le j \le t_k$, of length at most $2^{-k}\Psi(Q_k/10)$ and at least $2^{-k-1}\Psi(Q_k)$. Let

$$Q_{k+1} \ge 2^{k+11}\, \Psi(Q_k)^{-1} \log\bigl(2^{k+1}\, \Psi(Q_k)^{-1}\bigr)$$

be an integer. By Lemma 6, for any j with $1 \le j \le t_k$, there exist $[2^{-k-8} Q_{k+1}^2 \Psi(Q_k)]$ disjoint intervals

$$V_h^{(j)} := U_{k+1}(p_h^{(k+1)}/q_h^{(k+1)}), \qquad 1 \le h \le [2^{-k-8} Q_{k+1}^2 \Psi(Q_k)],$$

contained in U_j and mutually distant from at least $Q_{k+1}^{-2}/2$. Then, we set

$$E_{k+1} = \bigcup_{j=1}^{t_k} \bigcup_{h=1}^{[2^{-k-8} Q_{k+1}^2 \Psi(Q_k)]} V_h^{(j)}$$

and we define the Cantor set $\mathcal{C}$ by

$$\mathcal{C} = \bigcap_{k \ge 1} E_k.$$

By construction, for any positive integer k and any ξ in E_k, the equation

$$\left| \xi - \frac{u}{v} \right| < \Psi(v) \tag{18}$$

has at least k solutions. Thus, (18) has infinitely many solutions for any ξ in $\mathcal{C}$. Let ℓ be a positive integer. Again by construction, for any integer $k > \ell$ and for any ξ in E_k, the equation

$$\left|\xi - \frac{u}{v}\right| < c_\ell \Psi(v) \tag{19}$$

has no solution with $Q_\ell < v \le Q_k$. Thus, (19) has no solution u/v with $v > Q_\ell$ for any ξ in $\mathcal{C}$. Consequently, we have $\mathcal{C} \subset \mathrm{Exact}(\Psi)$ and the structure of $\mathcal{C}$ allows us to apply Proposition 1 in order to get a lower bound for its Hausdorff dimension.

Let $\varepsilon > 0$ be given. Up to now, we have made no requirement on the sequence $(Q_k)_{k\ge1}$, excepted that it has to tend quite rapidly to infinity. By definition of the lower order of the function $1/\Psi$, there exists a strictly increasing sequence of real numbers $(X_k)_{k\ge1}$ tending to infinity such that

$$-\log \Psi(X_k) \le (\lambda + \varepsilon)\log X_k, \quad \text{for every } k \ge 1. \tag{20}$$

Since Ψ is monoton, we may assume that $(X_k)_{k\ge1}$ is an increasing sequence of distinct integers.

Taking a sequence $(Q_k)_{k\ge1}$ which increases sufficiently rapidly and which is a subsequence of $(X_k)_{k\ge1}$, the definition of $\mathcal{C}$ and Proposition 1 yield that

$$\begin{aligned}\dim \mathcal{C} &\ge \liminf_{k\to+\infty} \frac{(2-\varepsilon)\log Q_k}{-\log(1-c_k) - \log \Psi(Q_k)} \\ &\ge \liminf_{k\to+\infty} \frac{2(1-\varepsilon)\log Q_k}{-\log \Psi(Q_k)},\end{aligned}$$

hence, by (20), we get

$$\dim \mathcal{C} \ge \frac{2(1-\varepsilon)}{\lambda+\varepsilon}.$$

Since ε is positive and arbitrarily small, it follows that $\dim \mathcal{C} \ge 2/\lambda$, as claimed. □

5. Proof of Theorem 2

In Section 5, we have constructed inductively a Cantor set which was sufficiently 'big' in order to yield the exact Hausdorff dimension of the set $\mathrm{Bad}_c(\Psi)$. However, this approach is not precise enough to determine its Hausdorff *measure*. Proposition 3 below allows us to construct a much bigger Cantor set contained in $\mathrm{Bad}_c(\Psi)$. It should be compared with Proposition 1 of [4] and to Proposition 1 of [5].

Proposition 3. *Let $m \ge 2$ be an integer. Let I denote the interval $[0,1]$ or an interval of the form $U_m(p/q)$ satisfying the assumption of Lemma 6. Let f be a positive, strictly increasing function such that the sum $\sum_{x\ge1} xf(\Psi(x))$ diverges and $x \mapsto x^2 f(\Psi(x))$ is non-increasing and tends to 0 as x goes to infinity. Let a and Q' be positive real numbers. Then there exist rational numbers $u_1/v_1, \ldots, u_t/v_t$ in I, of denominators greater than Q', such* *that the intervals $U_m(u_j/v_j)$ for $1 \le j \le t$ satisfy the assumption of Lemma 6. Further, the intervals*

$$[u_j/v_j - af(\Psi(v_j)), u_j/v_j + af(\Psi(v_j))], \qquad 1 \le j \le t,$$

are included in I, pairwise disjoint, and

$$a \sum_{j=1}^{t} f(\Psi(v_j)) \ge 10^{-6}\,|I|. \tag{21}$$

PROOF. We first deal with the case $I \neq [0,1]$. By assumption, there exist an integer $k \ge 1$ and a rational number p/q such that

$$I = U_m(p/q).$$

Let Q be an integer with

$$Q \ge \max\{Q', 2000|I|^{-1}\log(1/|I|)\}. \tag{22}$$

Since the sum $\sum_{x \ge 1} x f(\Psi(x))$ diverges and the function $x \mapsto x^2 f(\Psi(x))$ is monoton, Lemma 5 yields that

$$\sum_{h \ge 1} Q^{2h} f(\Psi(Q^h/10)) = +\infty. \tag{23}$$

Let ℓ be a positive integer such that

$$ax^2 f(\Psi(x)) \le 10^{-3} \quad \text{and} \quad af(\Psi(x)) \ge \Psi(x), \qquad \text{for any } x \ge Q^\ell/10. \tag{24}$$

By (23) and (24), there exists a positive integer N with

$$\frac{1}{8} \le \sum_{h=\ell}^{\ell+N} aQ^{2h} f(\Psi(Q^h/10)) \le \frac{1}{4}. \tag{25}$$

Applying Lemma 6 with the integer Q^ℓ (this is possible by (22)), we get a set $\mathcal{A}(0)$ of $[|I|Q^{2\ell}/100]$ rational numbers u/v with $Q^\ell/10 \le v \le Q^\ell$ such that the intervals

$$[u/v - af(\Psi(v)), u/v + af(\Psi(v))], \qquad u/v \in \mathcal{A}(0),$$

are contained in I (actually, Lemma 6 gives $[|I|Q^{2\ell}/100]$ rational numbers, but for at most two of them, the corresponding interval may not be contained in I) and, by (24), contain $U_m(u/v)$. Further, these intervals are pairwise disjoint. Indeed, on the one hand, we infer from (24) that

$$af(\Psi(v)) \le 10^{-3} v^{-2} \le Q^{-2\ell}/10,$$

while, on the other hand, we have

$$|u/v - u'/v'| \ge Q^{-2\ell} \qquad \text{for any distinct } u/v,\ u'/v' \text{ in } \mathcal{A}(0).$$

We now repeat this process with $Q^{\ell+j}$ for $1 \leq j \leq N$. Since we want to end up with disjoint intervals, we have to discard some rationals at each step. To do this, we proceed by induction, assuming that, for some $0 \leq j \leq N-1$, we have already constructed disjoint sets of rational numbers $\mathcal{A}(0), \dots, \mathcal{A}(j)$ such that the intervals

$$[u/v - af(\Psi(v)), u/v + af(\Psi(v))], \;\; u/v \in \mathcal{A}(0) \cup \ldots \cup \mathcal{A}(j),$$

are pairwise disjoint.

Applying Lemma 6 with the integer $Q^{\ell+j+1}$, there is a set $\tilde{\mathcal{A}}(j+1)$ of $[|I|Q^{2(\ell+j+1)}/100]$ rational numbers with

$$|u/v - u'/v'| \geq Q^{-2(\ell+j+1)} \qquad \text{for any distinct } u/v,\ u'/v' \text{ in } \tilde{\mathcal{A}}(j+1). \tag{26}$$

and such that the intervals

$$[u/v - af(\Psi(v)), u/v + af(\Psi(v))], \qquad u/v \in \tilde{\mathcal{A}}(j+1),$$

are contained in I and pairwise disjoint. Recall that at each step h, with $0 \leq h \leq j$, we have constructed not more than $[|I|Q^{2(\ell+h)}/100]$ intervals of length at most equal to $2af(\Psi(Q^{\ell+h}/10))$. Hence, to be sure that the new intervals have empty intersection with the intervals constructed previously, we have, by (26), to remove at most

$$N_j := \sum_{h=0}^{j} 2 \cdot 10^{-2} |I| Q^{2(\ell+h)} \big(af(\Psi(Q^{\ell+h}/10))\, Q^{2(\ell+j+1)} + 1 \big)$$

rational points in $\tilde{\mathcal{A}}(j+1)$.

By (22), (25) and $j \leq N$, we have

$$\begin{aligned} N_j &\leq 2 \cdot 10^{-2} |I| Q^{2(\ell+j+1)} \left(\sum_{h=0}^{N} aQ^{2(\ell+h)} f(\Psi(Q^{\ell+h}/10)) + \sum_{h\geq 1} Q^{-2h} \right) \\ &\leq |I| Q^{2(\ell+j+1)}/150. \end{aligned}$$

Thus, by (22), there are at least $|I|Q^{2(\ell+j+1)}/500$ points in $\tilde{\mathcal{A}}(j+1)$ such that the corresponding intervals are pairwise disjoint and do not intersect any of the intervals constructed previously. We denote by $\mathcal{A}(j+1)$ the set of these points, and this completes the induction step.

Since the function $x \mapsto x^2 f(\Psi(x))$ is non-increasing, the total length of all the intervals of the form $[u/v - af(\Psi(v)), u/v + af(\Psi(v))]$ centered at a rational u/v in $\mathcal{A}(0) \cup \cdots \cup \mathcal{A}(N)$ is, by (24) and (25), at least

$$\begin{aligned} \frac{2a}{500} \sum_{h=0}^{N} |I|\, Q^{2(\ell+h)} f\big(\Psi(Q^{\ell+h})\big) &\geq \frac{|I|}{25000} \sum_{h=0}^{N} a\, Q^{2(\ell+h+1)} f(\Psi(Q^{\ell+h+1}/10)) \\ &\geq \frac{|I|}{25000} \left(\frac{1}{8} - aQ^{2\ell} f(\Psi(Q^{\ell}/10)) \right) \\ &\geq \frac{|I|}{25000} \left(\frac{1}{8} - \frac{1}{10} \right) \geq 10^{-6}\, |I|. \end{aligned}$$

This implies that

$$a \sum_{u/v \in \mathcal{A}(0) \cup \ldots \cup \mathcal{A}(N)} f(\Psi(v)) \geq 10^{-6}\,|I|,$$

as asserted.

If I is the interval $[0,1]$, we proceed exactly as above, excepted that, instead of applying Lemma 6, we argue as we did below (7). □

Let $c < 1$ be a positive real number. The fact that $\mathcal{H}^f(\mathrm{Bad}_c(\Psi)) = 0$ when the sum $\sum_{x \geq 1} f(\Psi(x))$ converges follows from Theorem J (actually, this is an easy covering argument). Proposition 3 is the key tool for constructing a Cantor set $\tilde{\mathcal{C}}$ contained in $\mathrm{Bad}_c(\Psi)$ and a probability measure with support in $\tilde{\mathcal{C}}$. The main idea goes back to Jarník [12] and Dickinson & Velani [7] have followed a similar (but, in their context, much more complicated) approach. We describe the inductive process and the key steps: slightly more details are given in the proofs of Theorem 1 of [4] and Theorem 3 of [5], which rest in the same idea.

Let λ be a positive real number. We start the inductive process with the compact $\mathbf{K}(0) := [0,1]$. Our aim is to construct a nested sequence of compact sets $\mathbf{K}(\ell)$, which are finite unions of disjoint intervals, and a measure μ with support in $\cap_{\ell \geq 0} \mathbf{K}(\ell)$. Actually, the sets $\mathbf{K}(\ell)$ depend on c and on λ, but we prefer not to indicate this dependence in order to simplify the notation.

Let m be a positive integer with

$$c < 1 - 2^{-m} < 1.$$

Recall that, for any rational number p/q in $]0,1[$, we have

$$U_m(p/q) \subset [p/q + (1-2^{-m})\Psi(q), p/q + \Psi(q)] \subset [p/q + c\Psi(q), p/q + \Psi(q)]. \tag{27}$$

Set $\mu(\mathbf{K}(0)) = 1$. We apply Proposition 3 with $a = 10^{-6}\lambda$ and Q' large enough in order that $x \mapsto f(x)/x$ is strictly decreasing on $]0, \Psi(Q'/10)]$. We get a set $\mathcal{A}(1)$ of rational numbers such that the intervals $U_m(u/v)$ with u/v in $\mathcal{A}(1)$ satisfy the assumptions of Lemma 6. Denote by $\mathbf{K}(1)$ the union of these intervals $U_m(u/v)$. By (21), we have

$$\sum_{u/v \in \mathcal{A}(1)} f(\Psi(v)) \geq 1/\lambda, \tag{28}$$

and we distribute the mass $\mu(\mathbf{K}(0))$ between the connected components of $\mathbf{K}(1)$ by setting

$$\mu\big(U_m(u/v)\big) = \frac{f(|U_m(u/v)|)}{\sum_{u'/v' \in \mathcal{A}(1)} f(|U_m(u'/v')|)}, \qquad \text{for } u/v \in \mathcal{A}(1).$$

It follows from (4), our choice of Q' and (28) that

$$\mu(U_m(u/v)) \leq 4\lambda f(|U_m(u/v)|), \qquad \text{for any } u/v \in \mathcal{A}(1). \tag{29}$$

Assume that for some integer $\ell \geq 1$, we have constructed a set $\mathcal{A}(\ell)$ of rational numbers such that the intervals $U_m(u/v)$ with u/v in $\mathcal{A}(\ell)$ satisfy the assumptions of Lemma 6, and denote by $\mathbf{K}(\ell)$ the union of these intervals $U_m(u/v)$. Take Q' larger than the denominators of any rational number in $\mathcal{A}(\ell)$ and, for $u/v \in \mathcal{A}(\ell)$, apply Proposition 3 with

$$a = 2^{-m-1}\, 10^{-6} |U_m(u/v)| / f(|U_m(u/v)|) \tag{30}$$

to the interval $U_m(u/v)$. We then get a set $\mathcal{A}(\ell+1, U_m(u/v))$ of rational numbers x/y in $U_m(u/v)$ such that the intervals $U_m(x/y)$ with x/y in $\mathcal{A}(\ell+1, U_m(u/v))$ satisfy the assumptions of Lemma 6. We set

$$\mathcal{A}(\ell+1) := \bigcup_{u/v \in \mathcal{A}(\ell)} \mathcal{A}(\ell+1, U_m(u/v))$$

and

$$\mathbf{K}(\ell+1) := \bigcup_{x/y \in \mathcal{A}(\ell+1)} U_m(x/y).$$

For any u/v in $\mathcal{A}(\ell)$ we distribute the mass $\mu(U_m(u/v))$ between the connected components of $\mathbf{K}(\ell+1)$ contained in $U_m(u/v)$ by setting

$$\mu\big(U_m(x/y)\big) = \frac{f(|U_m(x/y)|)}{\sum_{x'/y' \in \mathcal{A}(\ell+1, U_m(x'/y'))} f(|U_m(x'/y')|)}\, \mu\big(U_m(u/v)\big). \tag{31}$$

We deduce from (4), (21) and (49) that

$$\begin{aligned} \sum_{x/y \in \mathcal{A}(\ell+1, U_m(u/v))} f(|U_m(x/y)|) &\geq \sum_{x/y \in \mathcal{A}(\ell+1, U_m(u/v))} 2^{-m-1} f(\Psi(y)) \\ &\geq 2^{-m-1}\, 10^{-6}\, a^{-1}\, |U_m(u/v)| \geq f(|U_m(u/v)|). \end{aligned} \tag{32}$$

Further, it follows from (32) and the definition (31) of μ that

$$\frac{\mu\big(U_m(x/y)\big)}{f(|U_m(x/y)|)} \leq \frac{\mu(U_m(u/v))}{f(|U_m(u/v)|)}. \tag{33}$$

An easy induction based on (33) and (29) yields that for any $\ell \geq 1$ and any connected component $U_m(u/v)$ of $\mathbf{K}(\ell)$ we have

$$\mu(U_m(u/v)) \leq 4\lambda f(|U_m(u/v)|). \tag{34}$$

Then, we define the Cantor set

$$\tilde{\mathcal{C}} := \bigcap_{\ell=0}^{+\infty} \mathbf{K}(\ell),$$

which, by (27) and Lemma 6, satisfies

$$\tilde{\mathcal{C}} \subset \mathrm{Bad}_c(\Psi).$$

The measure μ defined as above has support in $\tilde{\mathcal{C}}$ and, for any Borelian set F contained in $\mathbf{K}(0)$, it satisfies

$$\mu(F) = \inf_{J \in \mathcal{J}} \mu(J),$$

where the infimum is taken over all the coverings $\mathcal{J}$ of F by intervals J belonging to the union of the $\mathbf{K}(\ell)$, $\ell \geq 0$ (see e.g. [9], Proposition 1.7).

Moreover, if J is an interval in $[0,1]$ of length sufficiently small, then we have

$$\mu(J) \leq 2^{m+4}\, 10^6\, \lambda\, f(|J|).$$

Indeed, if J does not intersect $\tilde{\mathcal{C}}$, then $\mu(J) = 0$. If for any positive integer ℓ the interval J intersects exactly one component of $\mathbf{K}(\ell)$, then we have also $\mu(J) = 0$, by (34). Thus, we assume that there exists an integer ℓ such that J intersects exactly one interval from $\mathbf{K}(\ell)$, which, according to the preceeding notation, we denote by $U_m(u/v)$, and at least two intervals from $\mathbf{K}(\ell+1)$, which we denote by $U_m(x_1/y_1), \ldots, U_m(x_s/y_s)$, with $s \geq 2$. By Proposition 3, the s intervals

$$[x_j/y_j - af(\Psi(y_j)), x_j/y_j + af(\Psi(y_j))], \qquad 1 \leq j \leq s,$$

where a is as in (30), are pairwise disjoint and contained in $U_m(u/v)$. Furthermore, we infer from (30) and (34) that

$$\begin{aligned}
|J \cap U_m(u/v)| &\geq \frac{1}{2} \sum_{j=1}^{s} af(\Psi(y_j)) \\
&\geq 2^{-m-2}\, 10^{-6} \frac{|U_m(u/v)|}{f(|U_m(u/v)|)} \sum_{j=1}^{s} f(|U_m(x_j/y_j)|) \\
&\geq 2^{-m-4}\, 10^{-6}\, \lambda^{-1} \frac{|U_m(u/v)|}{f(|U_m(u/v)|)} \sum_{j=1}^{s} \mu(U_m(x_j/y_j)) \\
&\geq 2^{-m-4}\, 10^{-6}\, \lambda^{-1} \frac{|U_m(u/v)|}{f(|U_m(u/v)|)}\, \mu(J \cap U_m(u/v)) \\
&\geq 2^{-m-4}\, 10^{-6}\, \lambda^{-1} \frac{|J \cap U_m(u/v)|}{f(|J \cap U_m(u/v)|)}\, \mu(J \cap U_m(u/v)),
\end{aligned}$$

since $x \mapsto x/f(x)$ is non-decreasing in a neighbourhood of the origin. Thus, we get

$$\mu(J) = \mu(J \cap U_m(u/v)) \leq 2^{m+4} 10^6 \lambda\, f(|J \cap U_m(u/v)|) \leq 2^{m+4} 10^6 \lambda\, f(|J|),$$

as claimed. Then, Proposition 2 applies and yields that

$$\mathcal{H}^f(\tilde{\mathcal{C}}) \geq \frac{2^{-m-4}\, 10^{-6}}{\lambda}. \tag{35}$$

Since λ can be chosen arbitrarily small and $\tilde{\mathcal{C}} \subset \mathrm{Bad}_c(\Psi)$, the theorem follows from (35). □

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