Expansions of algebraic numbers

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Abstract. Classical ways to represent a real number are by its continued fraction expansion or by its expansion in some integer base. It is commonly expected that algebraic irrational numbers behave, in many respects, like almost all numbers. For instance, their decimal expansion should contain every finite block of digits from $\{0, \ldots, 9\}$. We are very far away from establishing such a strong assertion. However, there has been some recent progress, and it is now possible to prove that the decimal expansion of an irrational algebraic number cannot be 'too simple', in a suitable sense. The same applies for the continued fraction expansion of an algebraic number of degree at least three (recall that a continued fraction is ultimately periodic if, and only if, it represents a quadratic number). The main tool for the proofs is a deep result from Diophantine approximation, namely the Schmidt Subspace Theorem, a powerful multi-dimensional extension of the Roth Theorem.

2010 Mathematics Subject Classification. Primary 11A63; Secondary 11J70, 11J81, 11J87, 11J04, 11R04, 68R15.

Keywords. Transcendence, digital expansion, continued fraction, algebraic number.

1. Representation of real numbers

The most classical ways to represent real numbers are by means of their continued fraction expansion or their expansion to some integer base, in particular to base two or ten. In this text, we consider only these expansions and deliberately ignore β-expansions, Lüroth expansions, Q-Cantor series, etc., as well as the many variations of the continued fraction algorithm.

The first example of a transcendental number (recall that a real number is *algebraic* if it is root of a nonzero polynomial with integer coefficients and it is *transcendental* otherwise) was given by Liouville [51, 52] in 1844. He showed that if the sequence of partial quotients of an irrational real number grows sufficiently rapidly, then this number is transcendental. He mentioned only at the very end of his note the now classical example of the series (keeping his notation)

$$\frac{1}{a} + \frac{1}{a^{1\cdot 2}} + \frac{1}{a^{1\cdot 2\cdot 3}} + \ldots + \frac{1}{a^{1\cdot 2\cdot 3\cdots m}} + \ldots,$$

where $a \geq 2$ is an integer.

Let b denote an integer at least equal to 2. Any real number ξ has a unique b-ary expansion, that is, it can be uniquely written as

$$\xi = \lfloor \xi \rfloor + \sum_{\ell \geq 1} \frac{a_\ell}{b^\ell} = \lfloor \xi \rfloor + 0 \cdot a_1 a_2 \ldots, \tag{1.1}$$

where $\lfloor\cdot\rfloor$ denotes the integer part function, the *digits* $a_1, a_2, \ldots$ are integers from the set $\{0, 1, \ldots, b-1\}$ and a_ℓ differs from $b-1$ for infinitely many indices ℓ. This notation will be kept throughout this text.

In a seminal paper published in 1909, Émile Borel [21] introduced the notion of *normal number*.

Definition 1.1. *Let $b \geq 2$ be an integer. Let ξ be a real number whose b-ary expansion is given by (1.1). We say that ξ is normal to base b if, for every $k \geq 1$, every finite block of k digits on $\{0, 1, \ldots, b-1\}$ occurs with the same frequency $1/b^k$, that is, if for every $k \geq 1$ and every $d_1, \ldots, d_k \in \{0, 1, \ldots, b-1\}$ we have*

$$\lim_{N \to +\infty} \frac{\#\{\ell : 0 \leq \ell < N, a_{\ell+1} = d_1, \ldots, a_{\ell+k} = d_k\}}{N} = \frac{1}{b^k}.$$

The above definition differs from that given by Borel, but is equivalent to it; see Chapter 4 from [27] for a proof and further equivalent definitions.

We reproduce the fundamental theorem proved by Borel in [21]. Throughout this text, 'almost all' always refers to the Lebesgue measure, unless otherwise specified.

Theorem 1.2. *Almost all real numbers are normal to every integer base $b \geq 2$.*

Despite the fact that normality is a property shared by almost all numbers, we do not know a single explicit example of a number normal to every integer base, let alone of a number normal to base 2 and to base 3. However, Martin [54] gave in 2001 a nice and simple explicit construction of a real number normal to no integer base.

The first explicit example of a real number normal to a given base was given by Champernowne [34] in 1933.

Theorem 1.3. *The real number*

$$0 \cdot 12345678910111213\ldots, \tag{1.2}$$

whose sequence of decimals is the increasing sequence of all positive integers, is normal to base ten.

Further examples also obtained by concatenation of sequences of integers have been given subsequently in [35, 37]. In particular, the real number

$$0 \cdot 235711131719232931\ldots, \tag{1.3}$$

whose sequence of decimals is the increasing sequence of all prime numbers, is normal to base ten. This is due to the fact that the sequence of prime numbers does not increase too rapidly. However, we still do not know whether the real numbers (1.2) and (1.3) are normal to base two.

Constructions of a completely different type were found by Stoneham [66] and Korobov [49]; see Bailey and Crandall [18] for a more general statement which includes the next theorem.

Theorem 1.4. *Let b and c be coprime integers, both at least equal to 2. Let $d \geq 2$ be an integer. Then, the real numbers*

$$\sum_{j \geq 1} \frac{1}{c^j b^{c^j}} \quad \text{and} \quad \sum_{j \geq 1} \frac{1}{c^{d^j} b^{c^{d^j}}}$$

are normal to base b.

Regarding continued fraction expansions, we can as well define a notion of *normal continued fraction expansion* using the Gauss measure (see Section 5) and prove that the continued fraction expansion

$$\alpha = \lfloor \alpha \rfloor + [0; a_1, a_2, \ldots] = \lfloor \alpha \rfloor + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}}$$

of almost every real number α is a normal continued fraction expansion. Here, the positive integers $a_1, a_2, \ldots$ are called the *partial quotients* of α (throughout this text, a_ℓ denotes either a b-ary digit or a partial quotient, but this should be clear from the context; furthermore, we write ξ for a real number when we consider its b-ary expansion and α when we study its continued fraction expansion). In 1981, Adler, Keane, and Smorodinsky [13] have constructed a normal continued fraction in a similar way as Champernowne did for a normal number.

Theorem 1.5. *Let $1/2, 1/3, 2/3, 1/4, 2/4, 3/4, \ldots$ be the infinite sequence obtained in writing the rational numbers in $(0,1)$ with denominator 2, then with denominator 3, denominator 4, etc., ordered with numerators increasing. Let $x_1 x_2 x_3 \ldots$ be the sequence of positive integers constructed by concatenating the partial quotients (we choose the continued fraction expansion which does not end with the digit 1) of this sequence of rational numbers. Then, the real number*

$$[0; x_1, x_2, \ldots] = [0; 2, 3, 1, 2, 4, 2, 1, 3, 5, \ldots]$$

has a normal continued fraction expansion.

All this shows that the b-ary expansion and the continued fraction expansion of a real number taken at random are well understood. But what can be said for a specific number, like $\sqrt[3]{2}, \log 2, \pi$, etc.?

Actually, not much! We focus our attention on algebraic numbers. Clearly, a real number is rational if, and only if, its b-ary expansion is ultimately periodic. Analogously, a real number is quadratic if, and only if, its continued fraction expansion is ultimately periodic; see Section 5. The purpose of the present text is to gather what is known on the b-ary expansion of an irrational algebraic number and on the continued fraction expansion of an algebraic number of degree at least three. It is generally believed that all these expansions are normal, and some numerical computation tend to support this guess, but we are very, very far from proving such a strong assertion. We still do not know whether there is an integer $b \geq 3$ such that at least three different digits occur infinitely often in the b-ary expansion of $\sqrt{2}$. And whether there exist algebraic numbers of degree at least three whose sequence of partial quotients is bounded. Actually, it is widely believed that algebraic numbers should share most of the properties of almost all real numbers. This is indeed the case from the point of view of rational approximation, since Roth's theorem (see Section 6) asserts that algebraic irrational numbers do behave like almost all numbers, in the sense that they cannot be approximated by rational numbers at an order greater than 2.

The present text is organized as follows. Section 2 contains basic results from combinatorics on words. The main results on the complexity of algebraic numbers are stated in Section 3. They are proved by combining combinatorial transcendence criteria given in Section 4 and established in Sections 9 and 10 with a combinatorial lemma proved in Section 8. In Sections 5 and 6 we present various auxiliary results from the theory of continued fractions and from Diophantine approximation, respectively. Section 7 is devoted to a sketch of the proof of Theorem 4.1 and to a short historical discussion. We present in Section 11 another combinatorial transcendence criterion for continued fraction expansions, along with its proof. Section 12 briefly surveys some refined results which complement Theorem 3.1. Finally, in Section 13, we discuss other points of view for measuring the complexity of the b-ary expansion of a number.

A proof of Theorem 4.1 can already be found in the surveys [20, 8] and in the monograph [27]. Here, we provide two different proofs. Historical remarks and discussion on the various results which have ultimately led to Theorem 4.2 are given in [30].

2. Combinatorics on words and complexity

In the sequel, we often identify a real number with the infinite sequence of its b-ary digits or of its partial quotients. It appears to be convenient to use the point of view from combinatorics on words. Throughout, we denote by $\mathcal{A}$ a finite or infinite set. A finite word on the alphabet $\mathcal{A}$ is either the empty word, or a finite string (or block) of elements from $\mathcal{A}$. An infinite word on $\mathcal{A}$ is an infinite sequence of elements from $\mathcal{A}$.

For an infinite word $\mathbf{w} = w_1 w_2 \ldots$ on the alphabet $\mathcal{A}$ and for any positive integer n, we let

$$p(n, \mathbf{w}, \mathcal{A}) := \#\{w_{j+1} \ldots w_{j+n} : j \geq 0\}$$

denote the number of distinct strings (or blocks) of length n occurring in $\mathbf{w}$. Obviously, putting $\#\mathcal{A} = +\infty$ if $\mathcal{A}$ is infinite, we have

$$1 \leq p(n, \mathbf{w}, \mathcal{A}) \leq (\#\mathcal{A})^n,$$

and both inequalities are sharp. Furthermore, the function $n \mapsto p(n, \mathbf{w}, \mathcal{A})$ is non-decreasing.

Definition 2.1. *An infinite word* $\mathbf{w} = w_1 w_2 \ldots$ *is ultimately periodic if there exist positive integers* n_0 *and* T *such that*

$$w_{n+T} = w_n, \quad \text{for every } n \geq n_0.$$

The word $w_{n_0} w_{n_0+1} \ldots w_{n_0+T-1}$ *is a period of* $\mathbf{w}$. *If* n_0 *can be chosen equal to* 1, *then* $\mathbf{w}$ *is (purely) periodic, otherwise,* $w_1 \ldots w_{n_0-1}$ *is a preperiod of* $\mathbf{w}$.

We establish a seminal result from Morse and Hedlund [55, 56].

Theorem 2.2. *Let* $\mathbf{w}$ *be an infinite word over a finite or infinite alphabet* $\mathcal{A}$. *If* $\mathbf{w}$ *is ultimately periodic, then there exists a positive constant* C *such that* $p(n, \mathbf{w}, \mathcal{A}) \leq C$ *for every positive integer* n. *Otherwise, we have*

$$p(n+1, \mathbf{w}, \mathcal{A}) \geq p(n, \mathbf{w}, \mathcal{A}) + 1, \quad \text{for every } n \geq 1,$$

thus,

$$p(n, \mathbf{w}, \mathcal{A}) \geq n+1, \quad \text{for every } n \geq 1.$$

Proof. Throughout the proof, we write $p(\cdot, \mathbf{w})$ instead of $p(\cdot, \mathbf{w}, \mathcal{A})$.

Let $\mathbf{w}$ be an ultimately periodic infinite word, and assume that it has a preperiod of length r and a period of length s. Fix $h = 1, \ldots, s$ and let n be a positive integer. For every $j \geq 1$, the block of length n starting at w_{r+js+h} is the same as the one starting at w_{r+h}. Consequently, there cannot be more than $r+s$ distinct blocks of length n, thus, $p(n, \mathbf{w}) \leq r+s$.

Write $\mathbf{w} = w_1 w_2 \ldots$ and assume that there is a positive integer n_0 such that $p(n_0, \mathbf{w}) = p(n_0+1, \mathbf{w})$. This means that every block of length n_0 extends uniquely to a block of length n_0+1. It implies that $p(n_0, \mathbf{w}) = p(n_0+j, \mathbf{w})$ holds for every positive integer j. By definition of $p(n_0, \mathbf{w})$, two among the words $w_j \ldots w_{n_0+j-1}$, $j = 1, \ldots, p(n_0, \mathbf{w}) + 1$, are the same. Consequently, there are integers k and ℓ with $0 \leq k < \ell \leq p(n_0, \mathbf{w})$ and $w_{k+m} = w_{\ell+m}$ for $m = 1, \ldots, n_0$. Since every block of length n_0 extends uniquely to a block of length n_0+1, this gives $w_{k+m} = w_{\ell+m}$ for every positive integer m. This proves that the word $\mathbf{w}$ is ultimately periodic.

Consequently, if $\mathbf{w}$ is not ultimately periodic, then $p(n+1, \mathbf{w}) \geq p(n, \mathbf{w}) + 1$ holds for every positive integer n. Then, $p(1, \mathbf{w}) \geq 2$ and an immediate induction show that $p(n, \mathbf{w}) \geq n+1$ for every n. The proof of the theorem is complete. □

We complement Theorem 2.2 by pointing out that there exist uncountably many infinite words $\mathbf{w}$ on $\mathcal{A} = \{0, 1\}$ such that

$$p(n, \mathbf{w}, \mathcal{A}) = n+1, \quad \text{for } n \geq 1.$$

These words are called *Sturmian words*; see e.g. [16].

To prove that a real number is normal to some given integer base, or has a normal continued fraction expansion, is in most cases a much too difficult problem. So we are led to consider weaker questions on the sequence of digits (resp. partial quotients), including the following ones:

* Does every digit occur infinitely many times in the b-ary expansion of ξ?

* Are there many non-zero digits in the b-ary expansion of ξ?

* Is the sequence of partial quotients of α bounded from above?

* Does the sequence of partial quotients of α tend to infinity?

We may even consider weaker questions, namely, and this is the point of view we adopt until the very last sections, we wish to bound from below the number of different blocks in the infinite word composed of the digits of ξ (resp. partial quotients of α).

Let $b \geq 2$ be an integer. A natural way to measure the *complexity* of a real number ξ whose b-ary expansion is given by (1.1) is to count the number of distinct blocks of given length in the infinite word $\mathbf{a} = a_1 a_2 a_3 \ldots$ We set $p(n, \xi, b) = p(n, \mathbf{a}, b)$ with $\mathbf{a}$ as above. Clearly, we have

$$p(n, \xi, b) = \#\{a_{j+1} a_{j+2} \ldots a_{j+n} : j \geq 0\} = p(n, \mathbf{a}, \{0, 1, \ldots, b-1\})$$

and

$$1 \leq p(n, \xi, b) \leq b^n,$$

where both inequalities are sharp.

Since the b-ary expansion of a real number is ultimately periodic if, and only if, this number is rational, Theorem 2.2 can be restated as follows.

Theorem 2.3. *Let $b \geq 2$ be an integer. If the real number ξ is irrational, then*

$$p(n, \xi, b) \geq n + 1, \quad \text{for } n \geq 1.$$

Otherwise, the sequence $(p(n, \xi, b))_{n \geq 1}$ is bounded.

Let α be an irrational real number and write

$$\alpha = \lfloor \alpha \rfloor + [0; a_1, a_2, \ldots].$$

Let $\mathbf{a}$ denote the infinite word $a_1 a_2 \ldots$ over the alphabet $\mathbb{Z}_{\geq 1}$. A natural way to measure the intrinsic *complexity* of α is to count the number $p(n, \alpha) := p(n, \mathbf{a}, \mathbb{Z}_{\geq 1})$ of distinct blocks of given length n in the word $\mathbf{a}$.

Since the continued fraction expansion of a real number is ultimately periodic if, and only if, this number is quadratic (see Theorem 5.7), Theorem 2.2 can be restated as follows.

Theorem 2.4. *Let $b \geq 2$ be an integer. If the real number α is irrational and not quadratic, then*

$$p(n, \alpha) \geq n + 1, \quad \text{for } n \geq 1.$$

If the real number α is quadratic, then the sequence $(p(n, \alpha))_{n \geq 1}$ is bounded.

We show in the next section that Theorem 2.3 (resp. 2.4) can be improved when ξ (resp. α) is assumed to be algebraic.

3. Complexity of algebraic numbers

As already mentioned, we focus on the digital expansions and on the continued fraction expansion of algebraic numbers. Until the end of the 20th century, it was only known that the sequence of partial quotients of an algebraic number cannot grow too rapidly; see Section 12. Regarding b-ary expansions, Ferenczi and Mauduit [43] were the first to improve the (trivial) lower bound given by Theorem 2.3 for the complexity function of the b-ary expansion of an irrational algebraic number θ. They showed in 1997 that $p(n, \theta, b)$ strictly exceeds $n + 1$ for every sufficiently large integer n. Actually, as pointed out a few years later by Allouche [14], their approach combined with a combinatorial result of Cassaigne [32] yields a slightly stronger result, namely that

$$\lim_{n\to+\infty} \big(p(n, \theta, b) - n\big) = +\infty, \tag{3.1}$$

for any algebraic irrational number θ.

The estimate (3.1) follows from a good understanding of the combinatorial structure of Sturmian sequences combined with a combinatorial translation of Ridout's theorem 6.6. The transcendence criterion given in Theorem 4.1, established in [10, 3], yields an improvement of (3.1).

Theorem 3.1. *For any irrational algebraic number θ and any integer $b \geq 2$, we have*

$$\lim_{n\to+\infty} \frac{p(n, \theta, b)}{n} = +\infty. \tag{3.2}$$

Although (3.2) considerably strengthens (3.1), it is still very far from what is commonly expected, that is, from confirming that $p(n, \theta, b) = b^n$ holds for every positive n when θ is algebraic irrational.

Regarding continued fraction expansions, it was proved in [15] that

$$\lim_{n\to+\infty} \big(p(n, \theta) - n\big) = +\infty,$$

for any algebraic number θ of degree at least three. This is the continued fraction analogue of (3.1).

Using ideas from [1], the continued fraction analogue of Theorem 3.1 was established in [29].

Theorem 3.2. *For any algebraic number θ of degree at least three, we have*

$$\lim_{n\to+\infty} \frac{p(n, \theta)}{n} = +\infty. \tag{3.3}$$

The main purpose of the present text is to give complete (if one admits Theorem 6.7, whose proof is much too long and involved to be included here) proofs of Theorems 3.1 and 3.2. They are established by combining combinatorial transcendence criteria (Theorems 4.1 and 4.2) and a combinatorial lemma (Lemma 8.1). This is explained in details at the end of Section 8.

4. Combinatorial transcendence criteria

In this section, we state the combinatorial transcendence criteria which, combined with the combinatorial lemma from Section 8, yield Theorems 3.1 and 3.2.

Throughout, the length of a finite word W on the alphabet $\mathcal{A}$, that is, the number of letters composing W, is denoted by $|W|$.

Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition ($\spadesuit$) if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

- (i) For every $n \geq 1$, the word $W_n U_n V_n U_n$ is a prefix of the word $\mathbf{a}$;
- (ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;
- (iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;
- (iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Theorem 4.1. *Let $b \geq 2$ be an integer. Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of elements from $\{0, 1, \dots, b-1\}$. If $\mathbf{a}$ satisfies Condition ($\spadesuit$), then the real number*

$$\xi := \sum_{\ell=1}^{+\infty} \frac{a_\ell}{b^\ell}$$

is transcendental.

Theorem 4.2. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition ($\spadesuit$), then α is transcendental.

The common tool for the proofs of Theorems 4.1 and 4.2 is a powerful theorem from Diophantine approximation, the Subspace Theorem; see Section 6.

Let us comment on Condition ($\spadesuit$) when, for simplicity, the alphabet $\mathcal{A}$ is finite and has $b \geq 2$ elements. Take an arbitrary infinite word $a_1 a_2 \dots$ on $\{0, 1, \dots, b-1\}$. Then, by the *Schubfachprinzip*, for every positive integer m, there exists (at least) one finite word U_m of length m having (at least) two (possibly overlapping) occurrences in the prefix $a_1 a_2 \dots a_{b^m+m}$. If, for simplicity, we suppose that these two occurrences do not overlap, then there exist finite (or empty) words V_m, W_m, X_m such that

$$a_1 a_2 \dots a_{b^m+m} = W_m U_m V_m U_m X_m.$$

This simple argument gives no additional information on the lengths of W_m and V_m, which a priori can be as large as $b^m - m$. In particular, they can be larger than some constant greater than 1 raised to the power the length of U_m.

We demand much more for a sequence $\mathbf{a}$ to satisfy Condition ($\spadesuit$), namely we impose that there exists an integer C such that, for infinitely many m, the lengths of V_m and W_m do not exceed C times the length of U_m. Such a condition occurs quite rarely.

We end this section with a few comments on de Bruijn words.

Definition 4.3. *Let $b \geq 2$ and $n \geq 1$ be integers. A de Bruijn word of order n on an alphabet of cardinality b is a word of length $b^n + n - 1$ in which every block of length n occurs exactly once.*

A recent result of Becher and Heiber [19] shows that we can extend de Bruijn words.

Theorem 4.4. *Every de Bruijn word of order n on an alphabet with at least three letters can be extended to a de Bruijn word of order $n+1$. Every de Bruijn word of order n on an alphabet with two letters can be extended to a de Bruijn word of order $n+2$.*

Theorem 4.4 shows that there exist infinite de Bruijn words obtained as the inductive limit of extended de Bruijn sequences of order n, for each n (when the alphabet has at least three letters; for each even n, otherwise). Let $b \geq 2$ be an integer. By construction, for every $m \geq 1$, the shortest prefix of an infinite de Bruijn word having two occurrences of a same word of length m has at least $b^m + m$ letters if $b \geq 3$ and at least $2^{m-1} + m - 1$ letters if $b = 2$.

5. Continued fractions

In this section, we briefly present classical results on continued fractions which will be used in the proofs of Theorems 4.2 and 11.1. We omit most of the proofs and refer the reader to a text of van der Poorten [58] and to the books of Bugeaud [22], Cassels [33], Hardy and Wright [44], Khintchine [47], Perron [57], Schmidt [64], among many others.

Let $x_0, x_1, \ldots$ be real numbers with $x_1, x_2, \ldots$ positive. A *finite continued fraction* denotes any expression of the form

$$[x_0; x_1, x_2, \ldots, x_n] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ldots + \cfrac{1}{x_n}}}}.$$

We call any expression of the above form or of the form

$$[x_0; x_1, x_2, \ldots] = x_0 + \cfrac{1}{x_1 + \cfrac{1}{x_2 + \cfrac{1}{\ldots}}} = \lim_{n \to +\infty} [x_0; x_1, x_2, \ldots, x_n]$$

a *continued fraction*, provided that the limit exists.

Any rational number r has exactly two different continued fraction expansions. These are $[r]$ and $[r-1;1]$ if r is an integer and, otherwise, one of them reads $[a_0;a_1,\ldots,a_{n-1},a_n]$ with $a_n \ge 2$, and the other one is $[a_0;a_1,\ldots,a_{n-1},a_n-1,1]$. Any irrational number has a unique expansion in continued fraction.

Theorem 5.1. *Let $\alpha = [a_0;a_1,a_2,\ldots]$ be an irrational number. For $\ell \ge 1$, set $p_\ell/q_\ell := [a_0;a_1,a_2,\ldots,a_\ell]$. Let n be a positive integer. Putting*

$$p_{-1}=1, \quad q_{-1}=0, \quad p_0=a_0, \quad \text{and} \quad q_0=1,$$

we have

$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}, \tag{5.1}$$

and

$$p_{n-1}q_n - p_n q_{n-1} = (-1)^n. \tag{5.2}$$

Furthermore, setting $\alpha_{n+1} = [a_{n+1};a_{n+2},a_{n+3},\ldots]$, we have

$$\alpha = [a_0;a_1,\ldots,a_n,\alpha_{n+1}] = \frac{p_n\alpha_{n+1}+p_{n-1}}{q_n\alpha_{n+1}+q_{n-1}}, \tag{5.3}$$

thus

$$q_n\alpha - p_n = \frac{(-1)^n}{q_n\alpha_{n+1}+q_{n-1}},$$

and

$$\frac{1}{(a_{n+1}+2)q_n^2} < \frac{1}{q_n(q_n+q_{n+1})} < \left|\alpha - \frac{p_n}{q_n}\right| < \frac{1}{q_nq_{n+1}} < \frac{1}{a_{n+1}q_n^2} \le \frac{1}{q_n^2}. \tag{5.4}$$

It follows from (5.3) that any real number whose first partial quotients are $a_0,a_1,\ldots,a_n$ belongs to the interval with endpoints $(p_n+p_{n-1})/(q_n+q_{n-1})$ and p_n/q_n. Consequently, we get from (5.2) an upper bound for the distance between two real numbers having the same first partial quotients.

Corollary 5.2. *Let $\alpha = [a_0;a_1,a_2,\ldots]$ be an irrational number. For $\ell \ge 0$, let q_ℓ be the denominator of the rational number $[a_0;a_1,a_2,\ldots,a_\ell]$. Let n be a positive integer and β be a real number such that the first partial quotients of β are $a_0,a_1,\ldots,a_n$. Then,*

$$|\alpha-\beta| \le \frac{1}{q_n(q_n+q_{n-1})} < \frac{1}{q_n^2}.$$

Under the assumption of Theorem 5.1, the rational number p_ℓ/q_ℓ is called the ℓ-th convergent to α. It follows from (5.1) that the sequence of denominators of convergents grows at least exponentially fast.

Theorem 5.3. *Let $\alpha = [a_0;a_1,a_2,\ldots]$ be an irrational number. For $\ell \ge 0$, let q_ℓ be the denominator of the rational number $[a_0;a_1,a_2,\ldots,a_\ell]$. For any positive integers ℓ, h, we have*

$$q_{\ell+h} \ge q_\ell(\sqrt{2})^{h-1}$$

and

$$q_\ell \le (1+\max\{a_1,\ldots,a_\ell\})^\ell.$$

Proof. The first assertion follows from induction on h, since $q_{n+2} \geq q_{n+1}+q_n \geq 2q_n$ for every $n \geq 0$. The second assertion is an immediate consequence of (5.1). $\square$

The next result is sometimes called the *mirror formula*.

Theorem 5.4. *Let $n \geq 2$ be an integer and $a_1, \ldots, a_n$ be positive integers. For $\ell = 1, \ldots, n$, set $p_\ell/q_\ell = [0; a_1, \ldots, a_\ell]$. Then, we have*

$$\frac{q_{n-1}}{q_n} = [0; a_n, a_{n-1}, \ldots, a_1].$$

Proof. We get from (5.1) that

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}},$$

for $n \geq 1$. The theorem then follows by induction. $\square$

An alternative proof of Theorem 5.4 goes as follows. Observe that, if $a_0 = 0$ and $n \geq 1$, then, by (5.1), we have

$$M_n := \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix}.$$

Taking the transpose, we immediately get that

$$\begin{aligned} {}^t M_n &= {}^t\left(\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \right) \\ &= {}^t\begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} {}^t\begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \cdots {}^t\begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} = \begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix}, \end{aligned}$$

which gives Theorem 5.4.

Theorem 5.4 is a particular case of a more general result, which we state below after introducing the notion of *continuant*.

Definition 5.5. *Let $m \geq 1$ and $a_1, \ldots, a_m$ be positive integers. The denominator of the rational number $[0; a_1, \ldots, a_m]$ is called the continuant of $a_1, \ldots, a_m$ and is usually denoted by $K_m(a_1, \ldots, a_m)$.*

Theorem 5.6. *For any positive integers $a_1, \ldots, a_m$ and any integer k with $1 \leq k \leq m-1$, we have*

$$K_m(a_1, \ldots, a_m) = K_m(a_m, \ldots, a_1), \tag{5.5}$$

and

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m) & \\ \leq K_m(a_1, \ldots, a_m) & \\ \leq 2\, K_k(a_1, \ldots, a_k) \cdot K_{m-k}(a_{k+1}, \ldots, a_m). & \end{aligned} \tag{5.6}$$

Proof. The first statement is an immediate consequence of Theorem 5.4. Combining

$$K_m(a_1,\dots,a_m) = a_m K_{m-1}(a_1,\dots,a_{m-1}) + K_{m-2}(a_1,\dots,a_{m-2})$$

with (5.5), we get

$$K_m(a_1,\dots,a_m) = a_1 K_{m-1}(a_2,\dots,a_m) + K_{m-2}(a_3,\dots,a_m),$$

which implies (5.6) for $k=1$. Let k be in $\{1,2,\dots,m-2\}$ such that

$$\begin{aligned} K_m &:= K_m(a_1,\dots,a_m) \\ &= K_k(a_1,\dots,a_k)\cdot K_{m-k}(a_{k+1},\dots,a_m) \\ &\quad + K_{k-1}(a_1,\dots,a_{k-1})\cdot K_{m-k-1}(a_{k+2},\dots,a_m), \end{aligned} \tag{5.7}$$

where we have set $K_0 = 1$. We then have

$$\begin{aligned} K_m &= K_k(a_1,\dots,a_k)\cdot \big(a_{k+1}K_{m-k-1}(a_{k+2},\dots,a_m) + K_{m-k-2}(a_{k+3},\dots,a_m)\big) \\ &\quad + K_{k-1}(a_1,\dots,a_{k-1})\cdot K_{m-k-1}(a_{k+2},\dots,a_m) \\ &= \big(a_{k+1}K_k(a_1,\dots,a_k) + K_{k-1}(a_1,\dots,a_{k-1})\big)\cdot K_{m-k-1}(a_{k+2},\dots,a_m) \\ &\quad + K_k(a_1,\dots,a_k)\cdot K_{m-k-2}(a_{k+3},\dots,a_m), \end{aligned}$$

giving (5.7) for the index $k+1$. This shows that (5.7) and, a fortiori, (5.6) hold for $k=1,\dots,m-1$. □

The 'only if' part of the next theorem is due to Euler [39], and the 'if' part was established by Lagrange [50] in 1770.

Theorem 5.7. *The real irrational number $\alpha = [a_0; a_1, a_2, \dots]$ has a periodic continued fraction expansion (that is, there exist integers $r \ge 0$ and $s \ge 1$ such that $a_{n+s} = a_n$ for all integers $n \ge r+1$) if, and only if, α is a quadratic irrationality.*

We display an elementary result on ultimately periodic continued fraction expansions.

Lemma 5.8. *Let θ be a quadratic real number with ultimately periodic continued fraction expansion*

$$\theta = [0; a_1,\dots,a_r,\overline{a_{r+1},\dots,a_{r+s}}],$$

and denote by $(p_\ell/q_\ell)_{\ell\ge1}$ the sequence of its convergents. Then, θ is a root of the polynomial

$$\begin{aligned} (q_{r-1}q_{r+s} - q_r q_{r+s-1})X^2 - (&q_{r-1}p_{r+s} - q_r p_{r+s-1} \\ &+ p_{r-1}q_{r+s} - p_r q_{r+s-1})X + (p_{r-1}p_{r+s} - p_r p_{r+s-1}). \end{aligned} \tag{5.8}$$

Proof. It follows from (5.3) that

$$\theta = [0; a_1,\dots,a_r,\theta_{r+1}] = \frac{p_r\theta' + p_{r-1}}{q_r\theta' + q_{r-1}} = \frac{p_{r+s}\theta' + p_{r+s-1}}{q_{r+s}\theta' + q_{r+s-1}},$$

where $\theta' = [a_{r+1}; \overline{a_{r+2}, \ldots, a_{r+s}, a_{r+1}}]$. Consequently, we get

$$\theta' = \frac{p_{r-1} - q_{r-1}\theta}{q_r\theta - p_r} = \frac{p_{r+s-1} - q_{r+s-1}\theta}{q_{r+s}\theta - p_{r+s}},$$

from which we obtain

$$(p_{r-1} - q_{r-1}\theta)(q_{r+s}\theta - p_{r+s}) = (p_{r+s-1} - q_{r+s-1}\theta)(q_r\theta - p_r).$$

This shows that θ is a root of (5.8). □

We do not claim that the polynomial in (5.8) is the minimal polynomial of θ over the integers. This is indeed not always true, since its coefficients may have common prime factors.

The sequence of partial quotients of an irrational real number α in $(0,1)$ can be obtained by iterations of the Gauss map $T_{\mathcal{G}}$ defined by $T_{\mathcal{G}}(0) = 0$ and $T_{\mathcal{G}}(x) = \{1/x\}$ for $x \in (0,1)$. Namely, if $[0; a_1, a_2, \ldots]$ denotes the continued fraction expansion of α, then $T_{\mathcal{G}}^n(\alpha) = [0; a_{n+1}, a_{n+2}, \ldots]$ and $a_n = \lfloor 1/T_{\mathcal{G}}^{n-1}(\alpha) \rfloor$ for $n \geq 1$.

In the sequel, it is understood that α is a real number in $(0,1)$, whose partial quotients $a_1(\alpha), a_2(\alpha), \ldots$ and convergents $p_1(\alpha)/q_1(\alpha), p_2(\alpha)/q_2(\alpha), \ldots$ are written $a_1, a_2, \ldots$ and convergents $p_1/q_1, p_2/q_2, \ldots$ when there should be no confusion.

The map $T_{\mathcal{G}}$ possesses an invariant ergodic probability measure, namely the Gauss measure $\mu_{\mathcal{G}}$, which is absolutely continuous with respect to the Lebesgue measure, with density

$$\mu_{\mathcal{G}}(\mathrm{d}x) = \frac{\mathrm{d}x}{(1+x)\log 2}.$$

For every function f in $L^1(\mu_{\mathcal{G}})$ and almost every α in $(0,1)$, we have (Theorem 3.5.1 in [36])

$$\lim_{n\to+\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T_{\mathcal{G}}^k \alpha) = \frac{1}{\log 2} \int_0^1 \frac{f(x)}{1+x}\,\mathrm{d}x. \tag{5.9}$$

For subsequent results in the metric theory of continued fractions, we refer the reader to [47] and to [36].

Definition 5.9. *We say that $[0; a_1, a_2, \ldots]$ is a normal continued fraction, if, for every integer $k \geq 1$ and every positive integers $d_1, \ldots, d_k$, we have*

$$\begin{aligned}\lim_{N\to+\infty} & \frac{\#\{j : 0 \leq j \leq N-k, a_{j+1} = d_1, \ldots, a_{j+k} = d_k\}}{N} \\ &= \int_{r/s}^{r'/s'} \mu_{\mathcal{G}}(\mathrm{d}x) = \mu_{\mathcal{G}}(\Delta_{d_1,\ldots,d_k}),\end{aligned} \tag{5.10}$$

where $r/s, r'/s'$ denote the rational numbers $[0; d_1, \ldots, d_{k-1}, d_k]$ and $[0; d_1, \ldots, d_{k-1}, d_k + 1]$ ordered such that $r/s < r'/s'$, and $\Delta_{d_1,\ldots,d_k} = [r/s, r'/s']$.

Let $d_1, \ldots, d_k$ be positive integers. It follows from Theorem 5.1 that the set $\Delta_{d_1,\ldots,d_k}$ of real numbers α in $(0,1)$ whose first k partial quotients are $d_1, \ldots, d_k$ is an interval of length $1/(q_k(q_k + q_{k-1}))$. Applying (5.9) to the function $f = \mathbf{1}_{\Delta_{d_1,\ldots,d_k}}$, we get that for almost every $\alpha = [0; a_1, a_2, \ldots]$ in $(0,1)$ the limit defined in (5.10) exists and is equal to $\mu_{\mathcal{G}}(\Delta_{d_1,\ldots,d_k})$. Thus, we have established the following statement.

Theorem 5.10. *Almost every α in $(0,1)$ has a normal continued fraction expansion.*

The construction of [13], reproduced in [27], is flexible enough to produce many examples of real numbers with a normal continued fraction expansion.

6. Diophantine approximation

In this section, we survey classical results on approximation to real (algebraic) numbers by rational numbers.

We emphasize one of the results of Theorem 5.1.

Theorem 6.1. *For every real irrational number ξ, there exist infinitely many rational numbers p/q with $q \geq 1$ and*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2}.$$

Theorem 6.1 is often, and wrongly, attributed to Dirichlet, who proved in 1842 a stronger result, namely that, under the assumption of Theorem 6.1 and for every integer $Q \geq 1$, there exist integers p, q with $1 \leq q \leq Q$ and $|\xi - p/q| < 1/(qQ)$. Theorem 6.1 was proved long before 1842.

An easy covering argument shows that, for almost all numbers, the exponent of q in Theorem 6.1 cannot be improved.

Theorem 6.2. *For every $\varepsilon > 0$ and almost all real numbers ξ, there exist only finitely many rational numbers p/q with $q \geq 1$ and*

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^{2+\varepsilon}}. \tag{6.1}$$

Proof. Without loss of generality, we may assume that ξ is in $(0,1)$. If there are infinitely many rational numbers p/q with $q \geq 1$ satisfying (6.1), then ξ belongs to the limsup set

$$\bigcap_{Q \geq 1} \bigcup_{q \geq Q} \bigcup_{p=0}^{q} \left(\frac{p}{q} - \frac{1}{q^{2+\varepsilon}}, \frac{p}{q} + \frac{1}{q^{2+\varepsilon}} \right) \bigcap (0,1).$$

The Lebesgue measure of the latter set is, for every $Q \geq 1$, at most equal to

$$\sum_{q \geq Q} q \, \frac{2}{q^{2+\varepsilon}},$$

which is the tail of a convergent series and thus tends to 0 as Q tends to infinity. This proves the theorem. □

The case of algebraic numbers is of special interest and has a long history. First, we define the (naïve) height of an algebraic number.

Definition 6.3. *Let θ be an irrational, real algebraic number of degree d and let $a_d X^d + \cdots + a_1 X + a_0$ denotes its minimal polynomial over $\mathbb{Z}$ (that is, the integer polynomial of lowest positive degree, with coprime coefficients and positive leading coefficient, which vanishes at θ). Then, the height $H(\theta)$ of θ is defined by*

$$H(\theta) := \max\{|a_0|, |a_1|, \ldots, |a_d|\}.$$

We begin by a result of Liouville [51, 52] proved in 1844, and alluded to in Section 1.

Theorem 6.4. *Let θ be an irrational, real algebraic number of degree d and height at most H. Then,*

$$\left|\theta - \frac{p}{q}\right| \geq \frac{1}{d^2 H (1 + |\theta|)^{d-1} q^d} \tag{6.2}$$

for all rational numbers p/q with $q \geq 1$.

Proof. Inequality (6.2) is true when $|\theta - p/q| \geq 1$. Let p/q be a rational number satisfying $|\theta - p/q| < 1$. Denoting by $P(X)$ the minimal defining polynomial of θ over $\mathbb{Z}$, we have $P(p/q) \neq 0$ and $|q^d P(p/q)| \geq 1$. By Rolle's Theorem, there exists a real number t lying between θ and p/q such that

$$|P(p/q)| = |P(\theta) - P(p/q)| = |\theta - p/q| \times |P'(t)|.$$

Since $|t - \theta| \leq 1$ and

$$|P'(t)| \leq d^2 H (1 + |\theta|)^{d-1},$$

the combination of these inequalities gives the theorem. □

Thue [67] established in 1909 the first significant improvement on Liouville's result. There were subsequent progress by Siegel, Dyson and Gelfond, until Roth [61] proved in 1955 that, as far as approximation by rational numbers is concerned, the irrational, real algebraic numbers do behave like almost all real numbers.

Theorem 6.5. *For every $\varepsilon > 0$ and every irrational real algebraic number θ, there exist at most finitely many rational numbers p/q with $q \geq 1$ and*

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{2+\varepsilon}}. \tag{6.3}$$

For a prime number ℓ and a non-zero rational number x, we set $|x|_\ell := \ell^{-u}$, where $u \in \mathbb{Z}$ is the exponent of ℓ in the prime decomposition of x. Furthermore, we set $|0|_\ell = 0$. The next theorem, proved by Ridout [59], extends Theorem 6.5.

Theorem 6.6. *Let S be a finite set of prime numbers. Let θ be a real algebraic number. Let ε be a positive real number. The inequality*

$$\prod_{\ell \in S} |pq|_\ell \cdot \min\left\{1, \left|\theta - \frac{p}{q}\right|\right\} < \frac{1}{q^{2+\varepsilon}}$$

has only finitely many solutions in non-zero integers p, q.

Theorem 6.5 is ineffective in the sense that their proofs do not allow us to compute explicitly an integer q_0 such that (6.3) has no solution with q greater than q_0. Nevertheless, we are able to bound explicitly the *number* of primitive solutions (that is, of solutions in coprime integers p and q) to inequality (6.3). The first result in this direction was proved in 1955 by Davenport and Roth [38]; see the proof of Theorem 12.1 for a recent estimate.

The Schmidt Subspace Theorem [62, 63, 64] is a powerful multidimensional extension of the Roth Theorem, with many outstanding applications [20, 26, 69]. We quote below a version of it which is suitable for our purpose, but the reader should keep in mind that there are more general formulations.

Theorem 6.7. *Let $m \ge 2$ be an integer. Let S be a finite set of prime numbers. Let $L_{1,\infty}, \ldots, L_{m,\infty}$ be m linearly independent linear forms with real algebraic coefficients. For any prime ℓ in S, let $L_{1,\ell}, \ldots, L_{m,\ell}$ be m linearly independent linear forms with integer coefficients. Let ε be a positive real number. Then, there is an integer T and proper subspaces $S_1, \ldots, S_T$ of $\mathbb{Q}^m$ such that all the solutions $\underline{x} = (x_1, \ldots, x_m)$ in $\mathbb{Z}^m$ to the inequality*

$$\prod_{\ell \in S} \prod_{i=1}^{m} |L_{i,\ell}(\underline{x})|_\ell \cdot \prod_{i=1}^{m} |L_{i,\infty}(\underline{x})| \le (\max\{1, |x_1|, \ldots, |x_m|\})^{-\varepsilon} \tag{6.4}$$

are contained in the union $S_1 \cup \ldots \cup S_T$.

Let us briefly show how Roth's theorem can be deduced from Theorem 6.7. Let θ be a real algebraic number and ε be a positive real number. Consider the two independent linear forms $\theta X - Y$ and X. Theorem 6.7 implies that there are integers $T \ge 1$, $x_1, \ldots, x_T$, $y_1, \ldots, y_T$ with $(x_i, y_i) \ne (0, 0)$ for $i = 1, \ldots, T$, such that, for every integer solution (p, q) to

$$|q| \cdot |q\theta - p| < |q|^{-\varepsilon},$$

there exists an integer k with $1 \le k \le T$ and $x_k p + y_k q = 0$. If θ is irrational, this means that there are only finitely many rational solutions to $|\theta - p/q| < |q|^{-2-\varepsilon}$, which is Roth's theorem.

Note that (like Theorems 6.5 and 6.6) Theorem 6.7 is ineffective in the sense that its proof does not yield an explicit upper bound for the height of the proper rational subspaces containing all the solutions to (6.4). Fortunately, Schmidt [65] was able to give an admissible value for the number T of subspaces; see [42] for a common generalization of Theorems 6.6 and 6.7, usually called the Quantitative Subspace Theorem, and [41] for the current state of the art.

7. Sketch of proof and historical comments

Before proving Theorems 3.1 and 3.2, we wish to highlight the main ideas and explain how weaker results can be deduced from various statements given in Section 6. We focus only on b-ary expansions.

The general idea goes as follows. Let us assume that (3.2) does not hold. Then, the sequence of digits of our real number satisfies a certain combinatorial property. And a suitable transcendence criterion prevents the sequence of digits of an irrational algebraic numbers to fulfill the same combinatorial property.

Let us see how transcendence results listed in Section 6 apply to get combinatorial transcendence criterion. We introduce some more notation. Let W be a finite word. For a positive integer ℓ, we write W^ℓ for the word $W \dots W$ (ℓ times repeated concatenation of the word W) and W^∞ for the infinite word constructed by concatenation of infinitely many copies of W. More generally, for any positive real number x, we denote by W^x the word $W^{\lfloor x \rfloor} W'$, where W' is the prefix of W of length $\lceil (x - \lfloor x \rfloor)|W| \rceil$. Here, $\lceil \cdot \rceil$ denotes the upper integer part function. In particular, we can write

$$aabaaaabaaaa = (aabaa)^{12/5} = (aabaaaabaa)^{6/5} = (aabaa)^{\log 10}.$$

Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. Let $w > 1$ be a real number. We say that $\mathbf{a}$ satisfies Condition $(\spadesuit)_w$ if $\mathbf{a}$ is not ultimately periodic and if there exist two sequences of finite words $(Z_n)_{n \geq 1}$, and $(W_n)_{n \geq 1}$ such that:

- (i) For every $n \geq 1$, the word $W_n Z_n^w$ is a prefix of the word $\mathbf{a}$;
- (ii) The sequence $(|W_n|/|Z_n|)_{n \geq 1}$ is bounded from above;
- (iii) The sequence $(|Z_n|)_{n \geq 1}$ is increasing.

We say that $\mathbf{a}$ satisfies Condition $(\spadesuit)_\infty$ if it satisfies Condition $(\spadesuit)_w$ for every $w > 1$.

Our first result is an application of Theorem 6.4 which gives a combinatorial condition on the sequence b-ary expansion of a real number which ensures that this number is trancendental.

Theorem 7.1. *Let $b \geq 2$ be an integer. Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\{0, 1, \dots, b-1\}$. If $\mathbf{a}$ satisfies Condition $(\spadesuit)_\infty$, then the real number*

$$\xi := \sum_{\ell=1}^{+\infty} \frac{a_\ell}{b^\ell}$$

is transcendental.

Proof. Let $w > 1$ be a real number. By assumption, there exist two sequences of finite words $(Z_n)_{n \geq 1}$, and $(W_n)_{n \geq 1}$ and an integer C such that $|W_n| \leq C|Z_n|$ and $W_n Z_n^w$ is a prefix of $\mathbf{a}$ for $n \geq 1$. Let $n \geq 1$ be an integer. In particular, ξ is very close to the rational number ξ_n whose b-ary expansion is the eventually periodic word $W_n Z_n^\infty$. A rapid calculation shows that there is an integer p_n such that

$$\xi_n = \frac{p_n}{b^{|W_n|}(b^{|Z_n|}-1)}$$

and

$$\begin{aligned} |\xi - \xi_n| = \Bigl| \xi - \frac{p_n}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr| &\le \frac{1}{b^{|W_n|+w|Z_n|}} \\ &\le \Bigl(\frac{1}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr)^{(|W_n|+w|Z_n|)/(|W_n|+|Z_n|)}. \end{aligned}$$

Since $|W_n| \le C|Z_n|$, the quantity $(|W_n| + w|Z_n|)/(|W_n| + |Z_n|)$ is bounded from below by $(C+w)/(C+1)$. Consequently, we get

$$\Bigl| \xi - \frac{p_n}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr| \le \Bigl(\frac{1}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr)^{(C+w)/(C+1)}. \tag{7.1}$$

Let d be the integer part of $(C+w)/(C+1)$. Since (7.1) holds for every $n \ge 1$, it follows from Theorem 6.4 that ξ cannot be algebraic of degree $\le d-1$. Since w can be taken arbitrarily large, one deduces that ξ must be transcendental. □

It is apparent from the proof of Theorem 7.1 that, if we replace the use of Liouville's theorem by that of Roth's (Theorem 6.5) or, even better, by Ridout's Theorem 6.6, then the assumptions of Theorem 7.1 can be substantially weakened. Indeed, Roth's theorem is sufficient to establish the transcendence of ξ as soon as the exponent $(C+w)/(C+1)$ strictly exceeds 2, that is, if $w > 2 + C$. We explain below how Ridout's theorem yields a much better result.

Theorem 7.2. *Let $b \ge 2$ be an integer. Let $\mathbf{a} = (a_\ell)_{\ell \ge 1}$ be a sequence of elements from $\{0, 1, \ldots, b-1\}$. Let $w > 2$ be a real number. If $\mathbf{a}$ satisfies Condition $(\spadesuit)_w$, then the real number*

$$\xi := \sum_{\ell=1}^{+\infty} \frac{a_\ell}{b^\ell}$$

is transcendental.

Proof. We keep the notation of the proof of Theorem 7.1, where it is shown that, for any $n \ge 1$, we have

$$\Bigl| \xi - \frac{p_n}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr| \le \frac{1}{b^{|W_n|+w|Z_n|}},$$

that is,

$$\begin{aligned} b^{-|W_n|} \Bigl| \xi - \frac{p_n}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr| &\le \frac{1}{b^{2|W_n|+w|Z_n|}} \\ &< \Bigl(\frac{1}{b^{|W_n|}(b^{|Z_n|}-1)} \Bigr)^{(2|W_n|+w|Z_n|)/(|W_n|+|Z_n|)}. \end{aligned}$$

Observe that, for $n \geq 1$,

$$\frac{2|W_n| + w|Z_n|}{|W_n| + |Z_n|} = 2 + \frac{(w-2)|Z_n|}{|W_n| + |Z_n|} \geq 2 + \frac{w-2}{C+1}.$$

Consequently, taking for S the set of prime divisors of b and setting $\varepsilon := (w-2)/(C+1)$, we have proved that there are infinitely many rational numbers p/q such that

$$\prod_{\ell \in S} |pq|_\ell \cdot \min\left\{1, \left|\xi - \frac{p}{q}\right|\right\} < \frac{1}{q^{2+\varepsilon}}.$$

By Theorem 6.6, this shows that ξ is transcendental, since $w > 2$. □

We postpone to Section 9 the proof that the conclusion of Theorem 7.2 remains true under the weaker assumption $w > 1$ (the reader can easily check that this corresponds exactly to Theorem 4.1).

8. A combinatorial lemma

The purpose of this section is to establish a combinatorial lemma which allows us to deduce Theorems 3.1 and 3.2 from Theorems 4.1 and 4.2.

Lemma 8.1. *Let* $\mathbf{w} = w_1 w_2 \ldots$ *be an infinite word over a finite or an infinite alphabet* $\mathcal{A}$ *such that*

$$\liminf_{n \to +\infty} \frac{p(n, \mathbf{w}, \mathcal{A})}{n} < +\infty.$$

Then, the word $\mathbf{w}$ *satisfies Condition* ($\spadesuit$) *defined in Section 4.*

Proof. By assumption, there exist an integer $C \geq 2$ and an infinite set $\mathcal{N}$ of positive integers such that

$$p(n, \mathbf{w}, \mathcal{A}) \leq Cn, \quad \text{for every } n \text{ in } \mathcal{N}. \tag{8.1}$$

This implies in particular that $\mathbf{w}$ is written over a finite alphabet.

Let n be in $\mathcal{N}$. By (8.1) and the *Schubfachprinzip*, there exists (at least) one block X_n of length n having (at least) two occurrences in the prefix of length $(C+1)n$ of $\mathbf{w}$. Thus, there are words W_n, W'_n, B_n and B'_n such that $|W_n| < |W'_n|$ and

$$w_1 \ldots w_{(C+1)n} = W_n X_n B_n = W'_n X_n B'_n.$$

If $|W_n X_n| \leq |W'_n|$, then define V_n by the equality $W_n X_n V_n = W'_n$. Observe that

$$w_1 \ldots w_{(C+1)n} = W_n X_n V_n X_n B'_n \tag{8.2}$$

and

$$\frac{|V_n| + |W_n|}{|X_n|} \leq C. \tag{8.3}$$

Set $U_n := X_n$.

If $|W'_n| < |W_n X_n|$, then, recalling that $|W_n| < |W'_n|$, we define X'_n by $W'_n = W_n X'_n$. Since $X_n B_n = X'_n X_n B'_n$ and $|X'_n| < |X_n|$, the word X'_n is a prefix strict of X_n and X_n is the concatenation of at least two copies of X'_n and a (possibly empty) prefix of X'_n. Let t_n be the largest positive integer such that X_n begins with $2t_n$ copies of X'_n. Observe that

$$2t_n|X'_n| + 2|X'_n| \geq |X'_n X_n|,$$

thus

$$n = |X_n| \leq (2t_n + 1)|X'_n| \leq 3t_n|X'_n|.$$

Consequently, $W_n({X'_n}^{t_n})^2$ is a prefix of $\mathbf{w}$ such that

$$|{X'_n}^{t_n}| \geq n/3$$

and

$$\frac{|W_n|}{|{X'_n}^{t_n}|} \leq \frac{3}{n} \cdot \big((C+1)n - 2|{X'_n}^{t_n}|\big) \leq 3C + 1. \tag{8.4}$$

Set $U_n := {X'_n}^{t_n}$ and let V_n be the empty word.

It then follows from (8.2), (8.3), and (8.4) that, for every n in the infinite set $\mathcal{N}$,

$$W_n U_n V_n U_n \quad \text{is a prefix of } \mathbf{w}$$

with

$$|W_n| + |V_n| \leq (3C+1)\,|U_n|.$$

This shows that $\mathbf{w}$ satisfies Condition (♠). □

We are now in position to deduce Theorems 3.1 and 3.2 from Theorems 4.1 and 4.2.

Proof of Theorem 3.1. Let $b \geq 2$ be an integer and ξ be an irrational real number. Assume that $p(n, \xi, b)$ does not tend to infinity with n. It then follows from Lemma 8.1 that the infinite word composed of the digits of ξ written in base b satisfies Condition (♠). Consequently, Theorem 4.1 asserts that ξ cannot be algebraic. By contraposition, we get the theorem.

□

Proof of Theorem 3.2. Let α be a real number not algebraic of degree at most two. Assume that $p(n, \alpha)$ does not tend to infinity with n. It then follows from Lemma 8.1 that the infinite word composed of the partial quotients of ξ satisfies Condition (♠). Furthermore, $p(1, \alpha)$ is finite, thus the sequence of partial quotients of α is bounded, say by M. It then follows from Theorem 5.3 that $q_\ell \leq (M+1)^\ell$ for $\ell \geq 1$, hence the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded. Consequently, all the hypotheses of Theorem 4.2 are satisfied, and one concludes that α cannot be algebraic of degree at least three. By contraposition, we get the theorem.

□

9. Proof of Theorem 4.1

We present two proofs of Theorem 4.1, which was originally established in [10].

Throughout this section, we set $|U_n| = u_n$, $|V_n| = v_n$ and $|W_n| = w_n$, for $n \geq 1$. We assume that ξ is algebraic and we derive a contradiction by a suitable application of Theorem 6.7.

First proof.

Let $n \geq 1$ be an integer. We observe that the real number ξ is quite close to the rational number ξ_n whose b-ary expansion is the infinite word $W_n(U_nV_n)^\infty$. Indeed, there exists an integer p_n such that

$$\xi_n = \frac{p_n}{b^{w_n}(b^{u_n+v_n} - 1)}$$

and

$$|\xi - \xi_n| \leq \frac{1}{b^{w_n+v_n+2u_n}},$$

since ξ and ξ_n have the same first $w_n + v_n + 2u_n$ digits in their b-ary expansion. Consequently, we have

$$|b^{w_n+u_n+v_n}\xi - b^{w_n}\xi - p_n| = |b^{w_n}(b^{u_n+v_n} - 1)\xi - p_n| \leq b^{-u_n}.$$

Consider the three linearly independent linear forms with real algebraic coefficients:

$$\begin{aligned}
L_{1,\infty}(X_1, X_2, X_3) =& X_1,\\
L_{2,\infty}(X_1, X_2, X_3) =& X_2,\\
L_{3,\infty}(X_1, X_2, X_3) =& \xi X_1 - \xi X_2 - X_3.
\end{aligned}$$

Evaluating them on the integer points $\mathbf{x}_n := (b^{w_n+u_n+v_n}, b^{w_n}, p_n)$, we get that

$$\prod_{1\leq j\leq 3} |L_{j,\infty}(\mathbf{x}_n)| \leq b^{2w_n+v_n}. \tag{9.1}$$

For any prime number ℓ dividing b, we consider the three linearly independent linear forms with integer coefficients:

$$\begin{aligned}
L_{1,\ell}(X_1, X_2, X_3) =& X_1,\\
L_{2,\ell}(X_1, X_2, X_3) =& X_2,\\
L_{3,\ell}(X_1, X_2, X_3) =& X_3.
\end{aligned}$$

We get that

$$\prod_{\ell | b} \prod_{1\leq j\leq 3} |L_{j,\ell}(\mathbf{x}_n)|_\ell \leq b^{-2w_n-u_n-v_n}. \tag{9.2}$$

Since $\mathbf{a}$ satisfies Condition (♠), we have

$$\liminf_{n\to+\infty} \frac{u_n}{w_n + u_n + v_n} > 0.$$

It then follows from (9.1) and (9.2) that there exists $\varepsilon > 0$ such that

$$\begin{aligned}\prod_{1\le j\le 3} |L_{j,\infty}(\mathbf{x}_n)| \cdot \prod_{\ell | b} \prod_{1\le j\le 3} |L_{j,\ell}(\mathbf{x}_n)|_\ell &\le b^{-u_n} \\ &\le \max\{b^{w_n+u_n+v_n}, b^{w_n}, p_n\}^{-\varepsilon},\end{aligned}$$

for every $n \ge 1$.

We then infer from Theorem 6.7 that all the points $\mathbf{x}_n$ lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$z_1 b^{w_n+u_n+v_n} + z_2 b^{w_n} + z_3 p_n = 0, \tag{9.3}$$

for any n in $\mathcal{N}_1$.

Dividing (9.3) by $b^{w_n+u_n+v_n}$, we get

$$z_1 + z_2 b^{-u_n-v_n} + z_3 \frac{p_n}{b^{w_n+u_n+v_n}} = 0. \tag{9.4}$$

Since u_n tends to infinity with n, the sequence $(p_n/b^{w_n+u_n+v_n})_{n\ge 1}$ tends to ξ. Letting n tend to infinity along $\mathcal{N}_1$, we then infer from (9.4) that either ξ is rational, or $z_1 = z_3 = 0$. In the latter case, z_2 must be zero, a contradiction. This shows that ξ cannot be algebraic.

□

Second proof.

Here, we follow an alternative approach presented in [2].

Let p_n and p'_n be the rational integers defined by

$$\sum_{\ell=1}^{w_n+v_n+2u_n} \frac{a_\ell}{b^\ell} = \frac{p_n}{b^{w_n+v_n+2u_n}} \quad \text{and} \quad \sum_{\ell=1}^{w_n+u_n} \frac{a_\ell}{b^\ell} = \frac{p'_n}{b^{w_n+u_n}}.$$

Observe that there exist integers f_n and f'_n such that

$$p_n = a_{w_n+v_n+2u_n} + a_{w_n+v_n+2u_n-1} b + \cdots + a_{w_n+v_n+u_n+1} b^{u_n-1} + f_n b^{u_n} \tag{9.5}$$

and

$$p'_n = a_{w_n+u_n} + a_{w_n+u_n-1} b + \cdots + a_{w_n+1} b^{u_n-1} + f'_n b^{u_n}. \tag{9.6}$$

Since, by assumption,

$$a_{w_n+u_n+v_n+j} = a_{w_n+j}, \quad \text{for } j = 1, \ldots, u_n,$$

it follows from (9.5) and (9.6) that $p_n - p'_n$ is divisible by an integer multiple of b^{u_n}. Thus, for any prime number ℓ dividing b, the ℓ-adic distance between p_n and p'_n is very small and we have

$$|p_n - p'_n|_\ell \le |b|_\ell^{u_n}.$$

Furthermore, it is clear that

$$|b^{w_n+u_n}\xi - p'_n| < 1 \quad \text{and} \quad |b^{w_n+v_n+2u_n}\xi - p_n| < 1.$$

Consider now the four linearly independent linear forms with real algebraic coefficients:

$$\begin{aligned} L_{1,\infty}(X_1, X_2, X_3, X_4) =& X_1, \\ L_{2,\infty}(X_1, X_2, X_3, X_4) =& X_2, \\ L_{3,\infty}(X_1, X_2, X_3, X_4) =& \xi X_1 - X_3, \\ L_{4,\infty}(X_1, X_2, X_3, X_4) =& \xi X_2 - X_4. \end{aligned}$$

Evaluating them on the integer points $\mathbf{x}_n := (b^{w_n+v_n+2u_n}, b^{u_n+w_n}, p_n, p'_n)$, we get that

$$\prod_{1\le j\le 4} |L_{j,\infty}(\mathbf{x}_n)| \le b^{2w_n+v_n+3u_n}. \tag{9.7}$$

For any prime number p dividing b, we consider the four linearly independent linear forms with integer coefficients:

$$\begin{aligned} L_{1,\ell}(X_1, X_2, X_3, X_4) =& X_1, \\ L_{2,\ell}(X_1, X_2, X_3, X_4) =& X_2, \\ L_{3,\ell}(X_1, X_2, X_3, X_4) =& X_3, \\ L_{4,\ell}(X_1, X_2, X_3, X_4) =& X_4 - X_3. \end{aligned}$$

We get that

$$\prod_{\ell | b} \prod_{1\le j\le 4} |L_{j,\ell}(\mathbf{x}_n)|_\ell \le b^{-(2w_n+v_n+3u_n)}\, b^{-u_n}. \tag{9.8}$$

Since $\mathbf{a}$ satisfies Condition (♠), we have

$$\liminf_{n\to+\infty} \frac{u_n}{w_n + 2u_n + v_n} > 0.$$

It then follows from (9.7) and (9.8) that there exists $\varepsilon > 0$ such that

$$\begin{aligned} \prod_{1\le j\le 4} |L_{j,\infty}(\mathbf{x}_n)| \cdot \prod_{\ell | b} \prod_{1\le j\le 4} |L_{j,\ell}(\mathbf{x}_n)|_\ell &\le b^{-u_n} \\ &\le \max\{b^{w_n+v_n+2u_n}, b^{u_n+w_n}, p_n, p'_n\}^{-\varepsilon}, \end{aligned}$$

for every $n \ge 1$.

We then infer from Theorem 6.7 that all the points $\mathbf{x}_n$ lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Thus, there exist a non-zero integer quadruple (z_1, z_2, z_3, z_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$z_1 b^{w_n+v_n+2u_n} + z_2 b^{u_n+w_n} + z_3 p_n + z_4 p'_n = 0, \tag{9.9}$$

for any n in $\mathcal{N}_1$.

Dividing (9.9) by $b^{w_n+v_n+2u_n}$, we get

$$z_1 + z_2 b^{-u_n-v_n} + z_3 \frac{p_n}{b^{w_n+v_n+2u_n}} + z_4 b^{-u_n-v_n} \frac{p'_n}{b^{u_n+w_n}} = 0. \tag{9.10}$$

Recall that u_n tends to infinity with n. Thus, the sequences $(p_n/b^{w_n+v_n+2u_n})_{n\geq 1}$ and $(p'_n/b^{u_n+w_n})_{n\geq 1}$ tend to ξ as n tends to infinity. Letting n tend to infinity along $\mathcal{N}_1$, we infer from (9.10) that either ξ is rational, or $z_1 = z_3 = 0$. In the latter case, we obtain that ξ is rational. This is a contradiction, since the sequence $(a_\ell)_{\ell\geq 1}$ is not ultimately periodic. Consequently, ξ cannot be algebraic.

□

Also, the Schmidt Subspace Theorem was applied similarly as in the first proof of Theorem 4.1 by Troi and Zannier [68] to establish the transcendence of the number $\sum_{m\in\mathcal{S}} 2^{-m}$, where $\mathcal{S}$ denotes the set of integers which can be represented as sums of distinct terms 2^k+1, where $k \geq 1$.

Moreover, in his short paper *Some suggestions for further research* published in 1984, Mahler [53] suggested explicitly to apply the Schmidt Subspace Theorem exactly as in the first proof of Theorem 4.1 given in Section 9 to investigate whether the middle third Cantor set contains irrational algebraic elements or not. More precisely, he wrote: *A possible approach to this question consists in the study of the non-homogeneous linear expressions*

$$|3^{p_r+P_r}X - 3^{p_r}X - N_r|.$$

It may be that a p-adic form of Schmidt's theorem on the rational approximations of algebraic numbers [10] holds for such expressions.

The reference [10] above is Schmidt's book [64].

We end this section by mentioning an application of Theorem 4.1. Adamczewski and Rampersad [12] proved that the binary expansion of an algebraic number contains infinitely many occurrences of 7/3-powers. They also established that the ternary expansion of an algebraic number contains infinitely many occurrences of squares or infinitely many occurrences of one of the blocks 010 or 02120.

10. Proof of Theorem 4.2

We reproduce the proof given in [29].

Throughout, the constants implied in $\ll$ depend only on α. Assume that the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ occurring in the definition of Condition ($\spadesuit$) are fixed. For $n \geq 1$, set $u_n = |U_n|$, $v_n = |V_n|$ and $w_n = |W_n|$. We assume that the real number $\alpha := [0; a_1, a_2, \ldots]$ is algebraic of degree at least three. Set $p_{-1} = q_0 = 1$ and $q_{-1} = p_0 = 0$.

We observe that α admits infinitely many good quadratic approximants obtained by truncating its continued fraction expansion and completing by periodicity. Precisely, for every positive integer n, we define the sequence $(b_k^{(n)})_{k\geq 1}$ by

$$b_h^{(n)} = a_h \quad \text{for } 1 \leq h \leq w_n + u_n + v_n,$$
$$b_{w_n+h+j(u_n+v_n)}^{(n)} = a_{w_n+h} \quad \text{for } 1 \leq h \leq u_n + v_n \text{ and } j \geq 0.$$

The sequence $(b_k^{(n)})_{k\geq 1}$ is ultimately periodic, with preperiod W_n and with period $U_n V_n$. Set

$$\alpha_n = [0; b_1^{(n)}, b_2^{(n)}, \ldots, b_k^{(n)}, \ldots]$$

and note that, since the first $w_n + 2u_n + v_n$ partial quotients of α and of α_n are the same, it follows from Corollary 5.2 that

$$|\alpha - \alpha_n| \leq q_{w_n+2u_n+v_n}^{-2}. \tag{10.1}$$

Furthermore, Lemma 5.8 asserts that α_n is root of the quadratic polynomial

$$\begin{aligned} P_n(X) := {} & (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})X^2 \\ & - (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1} \\ & + p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1})X \\ & + (p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}). \end{aligned}$$

By (5.4), we have

$$\begin{aligned} & |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1})| \\ & \quad \leq \; q_{w_n-1}|q_{w_n+u_n+v_n}\alpha - p_{w_n+u_n+v_n}| + q_{w_n}|q_{w_n+u_n+v_n-1}\alpha - p_{w_n+u_n+v_n-1}| \\ & \quad \leq 2 \; q_{w_n} \, q_{w_n+u_n+v_n}^{-1} \end{aligned} \tag{10.2}$$

and, likewise,

$$\begin{aligned} & |(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1})\alpha - (p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1})| \\ & \quad \leq \; q_{w_n+u_n+v_n}|q_{w_n-1}\alpha - p_{w_n-1}| + q_{w_n+u_n+v_n-1}|q_{w_n}\alpha - p_{w_n}| \\ & \quad \leq 2 \; q_{w_n}^{-1} \, q_{w_n+u_n+v_n}. \end{aligned} \tag{10.3}$$

Using (10.1), (10.2), and (10.3), we then get

$$
\begin{aligned}
|P_n(\alpha)| &= |P_n(\alpha) - P_n(\alpha_n)| \\
&= |(q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1})(\alpha - \alpha_n)(\alpha + \alpha_n) \\
&\quad - (q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1} \\
&\qquad\qquad + p_{w_n-1} q_{w_n+u_n+v_n} - p_{w_n} q_{w_n+u_n+v_n-1})(\alpha - \alpha_n)| \\
&= |(q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1})\alpha \\
&\quad - (q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1}) \\
&\quad + (q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1})\alpha \\
&\quad - (p_{w_n-1} q_{w_n+u_n+v_n} - p_{w_n} q_{w_n+u_n+v_n-1}) \\
&\quad + (q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1})(\alpha_n - \alpha)| \cdot |\alpha - \alpha_n| \\
&\ll |\alpha - \alpha_n| \cdot \bigl(q_{w_n}\, q_{w_n+u_n+v_n}^{-1} + q_{w_n}^{-1}\, q_{w_n+u_n+v_n} + q_{w_n} q_{w_n+u_n+v_n} |\alpha - \alpha_n|\bigr) \\
&\ll |\alpha - \alpha_n| q_{w_n}^{-1}\, q_{w_n+u_n+v_n} \\
&\ll q_{w_n}^{-1}\, q_{w_n+u_n+v_n}\, q_{w_n+2u_n+v_n}^{-2}.
\end{aligned}
\tag{10.4}
$$

We consider the four linearly independent linear forms:

$$
\begin{aligned}
L_1(X_1, X_2, X_3, X_4) &= \alpha^2 X_1 - \alpha(X_2 + X_3) + X_4, \\
L_2(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_2, \\
L_3(X_1, X_2, X_3, X_4) &= \alpha X_1 - X_3, \\
L_4(X_1, X_2, X_3, X_4) &= X_1.
\end{aligned}
$$

Evaluating them on the quadruple

$$
\begin{aligned}
\mathbf{x}_n := &(q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1}, q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1}, \\
&p_{w_n-1} q_{w_n+u_n+v_n} - p_{w_n} q_{w_n+u_n+v_n-1}, p_{w_n-1} p_{w_n+u_n+v_n} - p_{w_n} p_{w_n+u_n+v_n-1}),
\end{aligned}
$$

it follows from (10.2), (10.3), (10.4), and Theorem 5.3 that

$$
\begin{aligned}
\prod_{1 \le j \le 4} |L_j(\mathbf{x}_n)| &\ll q_{w_n+u_n+v_n}^2\, q_{w_n+2u_n+v_n}^{-2} \\
&\ll 2^{-u_n} \\
&\ll (q_{w_n} q_{w_n+u_n+v_n})^{-\delta u_n/(2w_n+u_n+v_n)},
\end{aligned}
$$

if n is sufficiently large, where we have set

$$
M = 1 + \limsup_{\ell \to +\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.
$$

Since $\mathbf{a}$ satisfies Condition (♠), we have

$$
\liminf_{n \to +\infty} \frac{u_n}{2w_n + u_n + v_n} > 0.
$$

Consequently, there exists $\varepsilon > 0$ such that

$$\prod_{1 \le j \le 4} |L_j(\mathbf{x}_n)| \ll (q_{w_n}\, q_{w_n+u_n+v_n})^{-\varepsilon}$$

holds for any sufficiently large integer n.

It then follows from Theorem 6.7 that the points $\mathbf{x}_n$ lie in a finite union of proper linear subspaces of $\mathbb{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set $\mathcal{N}_1$ of distinct positive integers such that

$$\begin{aligned} x_1(q_{w_n-1}q_{w_n+u_n+v_n} &- q_{w_n}q_{w_n+u_n+v_n-1}) \\ &+ x_2(q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}) \\ &+ x_3(p_{w_n-1}q_{w_n+u_n+v_n} - p_{w_n}q_{w_n+u_n+v_n-1}) \\ &+ x_4(p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}) = 0, \end{aligned} \tag{10.5}$$

for any n in $\mathcal{N}_1$.

• First case: we assume that there exist an integer ℓ and infinitely many integers n in $\mathcal{N}_1$ with $w_n = \ell$.

By extracting an infinite subset of $\mathcal{N}_1$ if necessary and by considering the real number $[0; a_{\ell+1}, a_{\ell+2}, \ldots]$ instead of α, we may without loss of generality assume that $w_n = \ell = 0$ for any n in $\mathcal{N}_1$.

Then, recalling that $q_{-1} = p_0 = 0$ and $q_0 = p_{-1} = 1$, we deduce from (10.5) that

$$x_1 q_{u_n+v_n-1} + x_2 p_{u_n+v_n-1} - x_3 q_{u_n+v_n} - x_4 p_{u_n+v_n} = 0, \tag{10.6}$$

for any n in $\mathcal{N}_1$. Observe that $(x_1, x_2) \neq (0, 0)$, since, otherwise, by letting n tend to infinity along $\mathcal{N}_1$ in (10.6), we would get that the real number α is rational. Dividing (10.6) by $q_{u_n+v_n}$, we obtain

$$x_1 \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} + x_2 \frac{p_{u_n+v_n-1}}{q_{u_n+v_n-1}} \cdot \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} - x_3 - x_4 \frac{p_{u_n+v_n}}{q_{u_n+v_n}} = 0. \tag{10.7}$$

By letting n tend to infinity along $\mathcal{N}_1$ in (10.7), we get that

$$\beta := \lim_{\mathcal{N}_1 \ni n \to +\infty} \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} = \frac{x_3 + x_4\alpha}{x_1 + x_2\alpha}.$$

Furthermore, observe that, for any sufficiently large integer n in $\mathcal{N}_1$, we have

$$\begin{aligned} \left| \beta - \frac{q_{u_n+v_n-1}}{q_{u_n+v_n}} \right| &= \left| \frac{x_3 + x_4\alpha}{x_1 + x_2\alpha} - \frac{x_3 + x_4 p_{u_n+v_n}/q_{u_n+v_n}}{x_1 + x_2 p_{u_n+v_n-1}/q_{u_n+v_n-1}} \right| \\ &\ll \frac{1}{q_{u_n+v_n-1}q_{u_n+v_n}}, \end{aligned} \tag{10.8}$$

by (5.4). Since the rational number $q_{u_n+v_n-1}/q_{u_n+v_n}$ is under its reduced form and $u_n + v_n$ tends to infinity when n tends to infinity along $\mathcal{N}_1$, we see that, for every positive real number η and every positive integer N, there exists a reduced rational number a/b such that $b > N$ and $|\beta - a/b| \le \eta/b$. This implies that β is irrational.

Consider now the three linearly independent linear forms

$$L_1'(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \quad L_2'(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \quad L_3'(Y_1, Y_2, Y_3) = Y_2.$$

Evaluating them on the triple $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})$ with $n \in \mathcal{N}_1$, we infer from (5.4) and (10.8) that

$$\prod_{1 \le j \le 3} |L_j'(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})| \ll q_{u_n+v_n}^{-1}.$$

It then follows from Theorem 6.7 that the points $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n})$ with $n \in \mathcal{N}_1$ lie in a finite union of proper linear subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2 \subset \mathcal{N}_1$ such that

$$y_1 q_{u_n+v_n} + y_2 q_{u_n+v_n-1} + y_3 p_{u_n+v_n} = 0, \tag{10.9}$$

for any n in $\mathcal{N}_2$. Dividing (10.9) by $q_{u_n+v_n}$ and letting n tend to infinity along $\mathcal{N}_2$, we get

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{10.10}$$

To obtain another equation relating α and β, we consider the three linearly independent linear forms

$$L_1''(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \quad L_2''(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \quad L_3''(Z_1, Z_2, Z_3) = Z_2.$$

Evaluating them on the triple $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})$ with n in $\mathcal{N}_1$, we infer from (5.4) and (10.8) that

$$\prod_{1 \le j \le 3} |L_j''(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})| \ll q_{u_n+v_n}^{-1}.$$

It then follows from Theorem 6.7 that the points $(q_{u_n+v_n}, q_{u_n+v_n-1}, p_{u_n+v_n-1})$ with $n \in \mathcal{N}_1$ lie in a finite union of proper linear subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (z_1, z_2, z_3) and an infinite set of distinct positive integers $\mathcal{N}_3 \subset \mathcal{N}_2$ such that

$$z_1 q_{u_n+v_n} + z_2 q_{u_n+v_n-1} + z_3 p_{u_n+v_n-1} = 0, \tag{10.11}$$

for any n in $\mathcal{N}_3$. Dividing (10.11) by $q_{u_n+v_n-1}$ and letting n tend to infinity along $\mathcal{N}_3$, we get

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0. \tag{10.12}$$

We infer from (10.10) and (10.12) that

$$(z_3\alpha + z_2)(y_3\alpha + y_1) = y_2 z_1.$$

Since β is irrational, we get from (10.10) and (10.12) that $y_3 z_3 \neq 0$. This shows that α is an algebraic number of degree at most two, which is a contradiction with our assumption that α is algebraic of degree at least three.

• Second case: extracting an infinite subset $\mathcal{N}_4$ of $\mathcal{N}_1$ if necessary, we assume that $(w_n)_{n\in\mathcal{N}_4}$ tends to infinity.

In particular $(p_{w_n}/q_{w_n})_{n\in\mathcal{N}_4}$ and $(p_{w_n+u_n+v_n}/q_{w_n+u_n+v_n})_{n\in\mathcal{N}_4}$ both tend to α as n tends to infinity.

We make the following observation. Let a be a letter and U, V, W be three finite words (V may be empty) such that $\mathbf{a}$ begins with $WUVU$ and a is the last letter of W and of UV. Then, writing $W = W'a$, $V = V'a$ if V is non-empty, and $U = U'a$ if V is empty, we see that $\mathbf{a}$ begins with $W'(aU)V'(aU)$ if V is non-empty and with $W'(aU')(aU')$ if V is empty. Consequently, by iterating this remark if necessary, we can assume that for any n in $\mathcal{N}_4$, the last letter of the word U_nV_n differs from the last letter of the word W_n. Said differently, we have $a_{w_n} \neq a_{w_n+u_n+v_n}$ for any n in $\mathcal{N}_4$.

Divide (10.5) by $q_{w_n}\, q_{w_n+u_n+v_n-1}$ and write

$$Q_n := (q_{w_n-1}q_{w_n+u_n+v_n})/(q_{w_n}q_{w_n+u_n+v_n-1}).$$

We then get

$$\begin{aligned} &x_1(Q_n-1) + x_2\left(Q_n\frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) + x_3\left(Q_n\frac{p_{w_n-1}}{q_{w_n-1}} - \frac{p_{w_n}}{q_{w_n}}\right) \\ &+ x_4\left(Q_n\frac{p_{w_n-1}}{q_{w_n-1}}\frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n}}{q_{w_n}}\frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) = 0, \end{aligned} \tag{10.13}$$

for any n in $\mathcal{N}_4$. To shorten the notation, for any $\ell \geq 1$, we put $R_\ell := \alpha - p_\ell/q_\ell$ and rewrite (10.13) as

$$\begin{aligned} &x_1(Q_n-1) + x_2\big(Q_n(\alpha - R_{w_n+u_n+v_n}) - (\alpha - R_{w_n+u_n+v_n-1})\big) \\ &+ x_3\big(Q_n(\alpha - R_{w_n-1}) - (\alpha - R_{w_n})\big) \\ &+ x_4\big(Q_n(\alpha - R_{w_n-1})(\alpha - R_{w_n+u_n+v_n}) - (\alpha - R_{w_n})(\alpha - R_{w_n+u_n+v_n-1})\big) = 0. \end{aligned}$$

This yields

$$\begin{aligned} &(Q_n-1)\big(x_1 + (x_2+x_3)\alpha + x_4\alpha^2\big) \\ &= x_2Q_nR_{w_n+u_n+v_n} - x_2R_{w_n+u_n+v_n-1} + x_3Q_nR_{w_n-1} - x_3R_{w_n} \\ &- x_4Q_nR_{w_n-1}R_{w_n+u_n+v_n} + x_4R_{w_n}R_{w_n+u_n+v_n-1} \\ &+ \alpha(x_4Q_nR_{w_n-1} + x_4Q_nR_{w_n+u_n+v_n} - x_4R_{w_n} - x_4R_{w_n+u_n+v_n-1}). \end{aligned} \tag{10.14}$$

Observe that

$$|R_\ell| \leq q_\ell^{-1}q_{\ell+1}^{-1}, \quad \ell \geq 1, \tag{10.15}$$

by (5.4).

We use (10.14), (10.15) and the assumption that $a_{w_n} \neq a_{w_n+u_n+v_n}$ for any n in $\mathcal{N}_4$ to establish the following claim.

Claim. *We have*

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0.$$

Proof of the Claim. If there are arbitrarily large integers n in $\mathcal{N}_4$ such that $Q_n \ge 2$ or $Q_n \le 1/2$, then the claim follows from (10.14) and (10.15).

Assume that $1/2 \le Q_n \le 2$ holds for every large n in $\mathcal{N}_4$. We then derive from (10.14) and (10.15) that

$$|(Q_n - 1)(x_1 + (x_2 + x_3)\alpha + x_4\alpha^2)| \ll |R_{w_n-1}| \ll q_{w_n-1}^{-1} q_{w_n}^{-1}.$$

If $x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 \neq 0$, then we get

$$|Q_n - 1| \ll q_{w_n-1}^{-1} q_{w_n}^{-1}. \tag{10.16}$$

On the other hand, observe that, by Theorem 5.4, the rational number Q_n is the quotient of the two continued fractions $[a_{w_n+u_n+v_n}; a_{w_n+u_n+v_n-1}, \ldots, a_1]$ and $[a_{w_n}; a_{w_n-1}, \ldots, a_1]$. Since $a_{w_n+u_n+v_n} \neq a_{w_n}$, we have either $a_{w_n+u_n+v_n} - a_{w_n} \ge 1$ or $a_{w_n} - a_{w_n+u_n+v_n} \ge 1$. In the former case, we see that

$$Q_n \ge \cfrac{a_{w_n+u_n+v_n}}{a_{w_n} + \cfrac{1}{1 + \cfrac{1}{a_{w_n-2} + 1}}} \ge \cfrac{a_{w_n} + 1}{a_{w_n} + \cfrac{a_{w_n-2} + 1}{a_{w_n-2} + 2}} \ge 1 + \frac{1}{(a_{w_n} + 1)(a_{w_n-2} + 2)}.$$

In the latter case, we have

$$\frac{1}{Q_n} \ge \frac{a_{w_n} + \cfrac{1}{a_{w_n-1} + 1}}{a_{w_n+u_n+v_n} + 1} \ge 1 + \frac{1}{(a_{w_n-1} + 1)(a_{w_n+u_n+v_n} + 1)} \ge 1 + \frac{1}{(a_{w_n-1} + 1)a_{w_n}}.$$

Consequently, in any case, we have

$$|Q_n - 1| \gg a_{w_n}^{-1} \min\{a_{w_n-2}^{-1}, a_{w_n-1}^{-1}\} \gg a_{w_n}^{-1}\, q_{w_n-1}^{-1}.$$

Combined with (10.16), this gives

$$a_{w_n} \gg q_{w_n} \gg a_{w_n} q_{w_n-1},$$

which implies that n is bounded, a contradiction. This proves the Claim.

□

Since α is irrational and not quadratic, we deduce from the Claim that $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, x_2 is non-zero and, by (10.5), we have, for any n in $\mathcal{N}_4$,

$$q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1} = p_{w_n-1} q_{w_n+u_n+v_n} - p_{w_n} q_{w_n+u_n+v_n-1}.$$

Thus, the polynomial $P_n(X)$ can simply be expressed as

$$\begin{aligned} P_n(X) :=& (q_{w_n-1} q_{w_n+u_n+v_n} - q_{w_n} q_{w_n+u_n+v_n-1})X^2 \\ & - 2(q_{w_n-1} p_{w_n+u_n+v_n} - q_{w_n} p_{w_n+u_n+v_n-1})X \\ & + (p_{w_n-1} p_{w_n+u_n+v_n} - p_{w_n} p_{w_n+u_n+v_n-1}). \end{aligned}$$

Consider now the three linearly independent linear forms

$$\begin{aligned} L_1'''(T_1,T_2,T_3) &= \alpha^2 T_1 - 2\alpha T_2 + T_3, \\ L_2'''(T_1,T_2,T_3) &= \alpha T_1 - T_2, \\ L_3'''(T_1,T_2,T_3) &= T_1. \end{aligned}$$

Evaluating them on the triple

$$\begin{aligned} \mathbf{x}'_n := (q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1}, q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}, \\ p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}), \end{aligned}$$

for n in $\mathcal{N}_4$, it follows from (10.2) and (10.4) that

$$\prod_{1\le j\le 3} |L_j'''(\mathbf{x}'_n)| \ll q_{w_n}\, q_{w_n+u_n+v_n}\, q_{w_n+2u_n+v_n}^{-2} \ll (q_{w_n}\, q_{w_n+u_n+v_n})^{-\varepsilon},$$

with the same ε as above, if n is sufficiently large.

We then deduce from Theorem 6.7 that the points $\mathbf{x}'_n$, $n \in \mathcal{N}_4$, lie in a finite union of proper linear subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (t_1, t_2, t_3) and an infinite set of distinct positive integers $\mathcal{N}_5$ included in $\mathcal{N}_4$ such that

$$\begin{aligned} t_1(q_{w_n-1}q_{w_n+u_n+v_n} - q_{w_n}q_{w_n+u_n+v_n-1}) \\ + t_2(q_{w_n-1}p_{w_n+u_n+v_n} - q_{w_n}p_{w_n+u_n+v_n-1}) \\ + t_3(p_{w_n-1}p_{w_n+u_n+v_n} - p_{w_n}p_{w_n+u_n+v_n-1}) = 0, \end{aligned} \tag{10.17}$$

for any n in $\mathcal{N}_5$.

We proceed exactly as above. Divide (10.17) by $q_{w_n}\, q_{w_n+u_n+v_n-1}$ and set

$$Q_n := (q_{w_n-1}q_{w_n+u_n+v_n})/(q_{w_n}q_{w_n+u_n+v_n-1}).$$

We then get

$$\begin{aligned} t_1(Q_n - 1) + t_2\left(Q_n \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) \\ + t_3\left(Q_n \frac{p_{w_n-1}}{q_{w_n-1}} \frac{p_{w_n+u_n+v_n}}{q_{w_n+u_n+v_n}} - \frac{p_{w_n}}{q_{w_n}} \frac{p_{w_n+u_n+v_n-1}}{q_{w_n+u_n+v_n-1}}\right) = 0, \end{aligned} \tag{10.18}$$

for any n in $\mathcal{N}_5$. We argue as after (10.13). Since p_{w_n}/q_{w_n} and $p_{w_n+u_n+v_n}/q_{w_n+u_n+v_n}$ tend to α as n tends to infinity along $\mathcal{N}_5$, we derive from (10.18) that

$$t_1 + t_2\alpha + t_3\alpha^2 = 0,$$

a contradiction since α is irrational and not quadratic. Consequently, α must be transcendental. This concludes the proof of the theorem.

□

11. A transcendence criterion for quasi palindromic continued fractions

In this section, we present another combinatorial transcendence for continued fractions which was established in [29], based on ideas from [5].

For a finite word $W := w_1 \dots w_k$, we write $\overline{W} := w_k \dots w_1$ its mirror image. The finite word W is called a *palindrome* if $W = \overline{W}$.

Let $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ be a sequence of elements from $\mathcal{A}$. We say that $\mathbf{a}$ satisfies Condition ($\clubsuit$) if $\mathbf{a}$ is not ultimately periodic and if there exist three sequences of finite words $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ such that:

- (i) For every $n \geq 1$, the word $W_n U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
- (ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded from above;
- (iii) The sequence $(|W_n|/|U_n|)_{n\geq 1}$ is bounded from above;
- (iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

Theorem 11.1. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathbf{a}$ satisfies Condition ($\clubsuit$), then α is transcendental.

A slight modification of the proof of Theorem 11.1 allows us to remove the assumption on the growth of the sequence $(q_\ell)_{\ell\geq 1}$, provided that a stronger condition than Condition ($\clubsuit$) is satisfied.

Theorem 11.2. *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers and set*

$$\alpha := [0; a_1, a_2, \dots, a_\ell, \dots].$$

Assume that $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ is not eventually periodic. If there are arbitrarily large integers ℓ such that the word $a_1 \dots a_\ell$ is a palindrome, then α is transcendental.

We leave to the reader the proof of Theorem 11.2, established in [5], and establish Theorem 11.1.

Proof. Throughout, the constants implied in $\ll$ are absolute.

Assume that the sequences $(U_n)_{n\geq 1}$, $(V_n)_{n\geq 1}$ and $(W_n)_{n\geq 1}$ are fixed. Set $w_n = |W_n|$, $u_n = |U_n|$ and $v_n = |V_n|$, for $n \geq 1$. Assume that the real number $\alpha := [0; a_1, a_2, \dots]$ is algebraic of degree at least three.

For $n \geq 1$, consider the rational number P_n/Q_n defined by

$$\frac{P_n}{Q_n} := [0; W_n U_n V_n \overline{U_n}\, \overline{W_n}]$$

and denote by P'_n/Q'_n the last convergent to P_n/Q_n which is different from P_n/Q_n. Since the first $w_n+2u_n+v_n$ partial quotients of α and P_n/Q_n coincide, we deduce from Corollary 5.2 that

$$|Q_n\alpha - P_n| < Q_n q_{w_n+2u_n+v_n}^{-2}, \quad |Q'_n\alpha - P'_n| < Q_n q_{w_n+2u_n+v_n}^{-2}, \tag{11.1}$$

Furthermore, it follows from Theorem 5.4 that the first $w_n+2u_n+v_n$ partial quotients of α and of Q'_n/Q_n coincide, thus

$$|Q_n\alpha - Q'_n| < Q_n q_{w_n+u_n}^{-2}, \tag{11.2}$$

again by Corollary 5.2, and, by Theorem 5.6,

$$Q_n \le 2 q_{w_n} q_{w_n+2u_n+v_n} \le 2 q_{w_n+u_n} q_{w_n+2u_n+v_n}. \tag{11.3}$$

Since

$$\begin{aligned} \alpha(Q_n\alpha - P_n) - (Q'_n\alpha - P'_n) &= \alpha Q_n\left(\alpha - \frac{P_n}{Q_n}\right) - Q'_n\left(\alpha - \frac{P'_n}{Q'_n}\right) \\ &= (\alpha Q_n - Q'_n)\left(\alpha - \frac{P_n}{Q_n}\right) + Q'_n\left(\frac{P'_n}{Q'_n} - \frac{P_n}{Q_n}\right), \end{aligned}$$

it follows from (11.1), (11.2) and (11.3) that

$$\begin{aligned} |\alpha^2 Q_n - \alpha Q'_n - \alpha P_n + P'_n| &\ll Q_n q_{w_n+u_n}^{-2} q_{w_n+2u_n+v_n}^{-2} + Q_n^{-1} \\ &\ll Q_n^{-1}. \end{aligned} \tag{11.4}$$

Consider the four linearly independent linear forms with algebraic coefficients

$$\begin{aligned} L_1(X_1,X_2,X_3,X_4) &= \alpha^2 X_1 - \alpha X_2 - \alpha X_3 + X_4, \\ L_2(X_1,X_2,X_3,X_4) &= \alpha X_2 - X_4, \\ L_3(X_1,X_2,X_3,X_4) &= \alpha X_1 - X_2, \\ L_4(X_1,X_2,X_3,X_4) &= X_2. \end{aligned}$$

We deduce from (11.1), (11.2), (11.3) and (11.4) that

$$\prod_{1\le j\le 4} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^2\, q_{w_n+2u_n+v_n}^{-2}\, q_{w_n+u_n}^{-2} \ll q_{w_n}^2\, q_{w_n+u_n}^{-2}.$$

By combining Theorems 5.3 and 5.6 with (11.3), we have

$$q_{w_n}^2\, q_{w_n+u_n}^{-2} \ll 2^{-u_n} \ll Q_n^{-\delta u_n/(2w_n+2u_n+v_n)},$$

if n is sufficiently large, where we have set

$$M = 1 + \limsup_{\ell\to+\infty} q_\ell^{1/\ell} \quad \text{and} \quad \delta = \frac{\log 2}{\log M}.$$

Since **a** satisfies Condition (♣), we have

$$\liminf_{n\to+\infty} \frac{u_n}{2w_n + 2u_n + v_n} > 0.$$

Consequently, there exists $\varepsilon > 0$ such that

$$\prod_{1\le j\le 4} |L_j(Q_n, Q'_n, P_n, P'_n)| \ll Q_n^{-\varepsilon},$$

for every sufficiently large n.

It then follows from Theorem 6.7 that the points (Q_n, Q'_n, P_n, P'_n) lie in a finite number of proper subspaces of $\mathbb{Q}^4$. Thus, there exist a non-zero integer quadruple (x_1, x_2, x_3, x_4) and an infinite set of distinct positive integers $\mathcal{N}_1$ such that

$$x_1 Q_n + x_2 Q'_n + x_3 P_n + x_4 P'_n = 0, \tag{11.5}$$

for any n in $\mathcal{N}_1$. Dividing by Q_n, we obtain

$$x_1 + x_2 \frac{Q'_n}{Q_n} + x_3 \frac{P_n}{Q_n} + x_4 \frac{P'_n}{Q'_n} \cdot \frac{Q'_n}{Q_n} = 0.$$

By letting n tend to infinity along $\mathcal{N}_1$, we infer from (11.2) and (11.3) that

$$x_1 + (x_2 + x_3)\alpha + x_4\alpha^2 = 0.$$

Since $(x_1, x_2, x_3, x_4) \neq (0, 0, 0, 0)$ and since α is irrational and not quadratic, we have $x_1 = x_4 = 0$ and $x_2 = -x_3$. Then, (11.5) implies that

$$Q'_n = P_n.$$

for every n in $\mathcal{N}_1$. Thus, for n in $\mathcal{N}_1$, we have

$$|\alpha^2 Q_n - 2\alpha Q'_n + P'_n| \ll Q_n^{-1}. \tag{11.6}$$

Consider now the three linearly independent linear forms

$$\begin{aligned} L'_1(X_1, X_2, X_3) &= \alpha^2 X_1 - 2\alpha X_2 + X_3, \\ L'_2(X_1, X_2, X_3) &= \alpha X_2 - X_3, \\ L'_3(X_1, X_2, X_3) &= X_1. \end{aligned}$$

Evaluating them on the triple (Q_n, Q'_n, P'_n) for n in $\mathcal{N}_1$, it follows from (11.1), (11.3) and (11.6) that

$$\begin{aligned} \prod_{1\le j\le 3} |L'_j(Q_n, Q'_n, P'_n)| &\ll Q_n q_{w_n+2u_n+v_n}^{-2} \\ &\ll q_{w_n} q_{w_n+2u_n+v_n}^{-1} \ll q_{w_n} q_{w_n+u_n}^{-1} \ll Q_n^{-\varepsilon/2}, \end{aligned}$$

with the same ε as above, if n is sufficiently large.

It then follows from Theorem 6.7 that the points (Q_n, Q'_n, P'_n) lie in a finite number of proper subspaces of $\mathbb{Q}^3$. Thus, there exist a non-zero integer triple (y_1, y_2, y_3) and an infinite set of distinct positive integers $\mathcal{N}_2$ such that

$$y_1 Q_n + y_2 Q'_n + y_3 P'_n = 0, \tag{11.7}$$

for any n in $\mathcal{N}_2$.

Dividing (11.7) by Q_n, we get

$$y_1 + y_2 \frac{P_n}{Q_n} + y_3 \frac{P'_n}{Q'_n} \cdot \frac{P_n}{Q_n} = 0. \tag{11.8}$$

By letting n tend to infinity along $\mathcal{N}_2$, it thus follows from (11.8) that

$$y_1 + y_2 \alpha + y_3 \alpha^2 = 0.$$

Since (y_1, y_2, y_3) is a non-zero triple of integers, we have reached a contradiction. Consequently, the real number α is transcendental. This completes the proof of the theorem. □

12. Complements

We collect in this section several results which complement Theorems 3.1 and 3.2. The common tool for their proofs (which we omit or just sketch) is the Quantitative Subspace Theorem, that is, a theorem which provides an explicit upper bound for the number T of exceptional subspaces in Theorem 6.7.

We have mentioned at the beginning of Section 3 that the sequence of partial quotients of an algebraic irrational number θ cannot grow too rapidly. More precisely, it can be derived from Roth's Theorem 6.5 that

$$\lim_{n \to +\infty} \frac{\log \log q_n}{n} = 0, \tag{12.1}$$

where $(p_\ell/q_\ell)_{\ell \geq 1}$ denotes the sequence of convergents to θ. This is left as an exercise. The use of a quantitative form of Theorem 6.5 allowed Davenport and Roth [38] to improve (12.1). Their result was subsequently strengthen [4, 25] as follows.

Theorem 12.1. *Let θ be an irrational, real algebraic number and let $(p_n/q_n)_{n \geq 1}$ denote the sequence of its convergents. Then, for any $\varepsilon > 0$, there exists a constant c, depending only on θ and ε, such that*

$$\log \log q_n \leq c\, n^{2/3+\varepsilon}.$$

Proof. We briefly sketch the proof of a slightly weaker result. Let d be the degree of θ. By Theorem 6.4, there exists an integer n_0 such that

$$\left| \theta - \frac{p_n}{q_n} \right| > \frac{1}{q_n^{d+1}},$$

for $n \geq n_0$. Combined with Theorem 5.1, this gives

$$q_{n+1} \leq q_n^d, \quad \text{for } n \geq n_0. \tag{12.2}$$

On the other hand, a quantitative form of Theorem 6.5 (see e.g. [40]) asserts that there exists a positive number $\eta_0 < 1/5$, depending only on θ, such that for every η with $0 < \eta < \eta_0$, the inequality

$$\left|\theta - \frac{p}{q}\right| < \frac{1}{q^{2+\eta}},$$

has at most η^{-4} rational solutions p/q with p and q coprime and $q > 16^{1/\eta}$.

Consequently, for every $n \geq 8/\eta$ with at most η^{-4} exceptions, we have

$$q_{n+1} \leq q_n^{1+\eta}. \tag{12.3}$$

Let $N \geq (8/\eta)^2$ be a large integer and set $h := \lceil \sqrt{N} \rceil$. We deduce from (12.2) and (12.3) that

$$\begin{aligned} \frac{\log q_N}{\log q_h} &= \frac{\log q_N}{\log q_{N-1}} \times \frac{\log q_{N-1}}{\log q_{N-2}} \times \ldots \times \frac{\log q_{h+1}}{\log q_h} \\ &\leq (1+\eta)^N \, d^{\eta^{-4}}, \end{aligned}$$

thus

$$\log\log q_N - \log\log q_h \leq N\,\eta + \eta^{-4}(\log d).$$

Observe that it follows from (12.1) that

$$\log\log q_h \leq N^{1/2},$$

when N is sufficiently large. Choosing $\eta = N^{-1/5}$, we obtain the upper bound

$$\log\log q_N \leq \log\log q_h + N^{4/5}(\log 3d) \leq 2N^{4/5}(\log 3d),$$

when N is large enough. A slight refinement yields the theorem. $\square$

We first observe that, if we assume a slightly stronger condition than

$$\liminf_{n \to +\infty} \frac{p(n, \mathbf{w}, \mathcal{A})}{n} < +\infty$$

in Lemma 8.1, namely if we assume that

$$\limsup_{n \to +\infty} \frac{p(n, \mathbf{w}, \mathcal{A})}{n} < +\infty,$$

then the word $\mathbf{w}$ satisfies a much stronger condition than Condition ($\spadesuit$). Indeed, there then exists an integer $C \geq 2$ such that

$$p(n, \mathbf{w}, \mathcal{A}) \leq Cn, \quad \text{for } n \geq 1,$$

instead of the weaker assumption (8.1). In the case of the first proof of Theorem 4.1 given in Section 9, this means that, keeping its notation, one may assume that, up to extracting subsequences, there exists an integer c such that

$$2(u_n + v_n + w_n) \leq u_{n+1} + v_{n+1} + w_{n+1} \leq c(u_n + v_n + w_n), \quad n \geq 1.$$

This observation is crucial for the proofs of Theorems 12.2 to 12.4 below.

Now, we mention a few applications to the complexity of algebraic numbers, beginning with a result from [31]. Recall that the complexity function $n \mapsto p(n, \theta, b)$ has been defined in Section 2.

Theorem 12.2. *Let $b \geq 2$ be an integer and θ an algebraic irrational number. Then, for any real number η such that $\eta < 1/11$, we have*

$$\limsup_{n \to +\infty} \frac{p(n, \theta, b)}{n(\log n)^\eta} = +\infty.$$

The main tools for the proof of Theorem 12.2 are a suitable extension of the Cugiani–Mahler Theorem and a suitable version of the Quantitative Subspace Theorem, which allows us to get an exponent of $\log n$ independent of the base b. Using the recent results of [41] allows us to show that Theorem 12.2 holds for η in a slightly larger interval than $[0, 1/11)$.

As briefly mentioned in Section 6, one of the main features of the theorems of Roth and Schmidt is that they are ineffective, in the sense that we cannot produce an explicit upper bound for the denominators of the solutions to (6.3) or for the height of the subspaces containing the solutions to (6.4). Consequently, Theorems 3.1 and 3.2 are ineffective, as are the weaker results from [14, 43]. It is shown in [24] that, by means of the Quantitative Subspace Theorem, it is possible to derive an explicit form of a much weaker statement.

Theorem 12.3. *Let $b \geq 2$ be an integer. Let θ be a real algebraic irrational number of degree d and height at most H, with $H \geq e^e$. Set*

$$M = \exp\{10^{190}(\log(8d))^2(\log\log(8d))^2\} + 2^{32\log(240\log(4H))}.$$

Then we have

$$p(n, \theta, b) \geq \left(1 + \frac{1}{M}\right)n, \quad \textit{for } n \geq 1.$$

Unfortunately, the present methods do not seem to be powerful enough to get an effective version of Theorem 3.1.

We have shown that, if the b-ary or the continued fraction expansion of a real number is not ultimately periodic and has small complexity, then this number cannot be algebraic, that is, the distance between this number and the set of algebraic numbers is strictly positive. A natural question then arises: is it possible to get transcendence measures for ξ, that is, to bound from below the distance between ξ and any algebraic number? A positive answer was given in [6], where the authors described a general method to obtain transcendence measures by means of the Quantitative Subspace Theorem. In the next statement, proved in [9], we say that an infinite word $\mathbf{w}$ written on an alphabet $\mathcal{A}$ is of sublinear complexity if there exists a constant C such that the complexity function of $\mathbf{w}$ satisfies

$$p(n, \mathbf{w}, \mathcal{A}) \le Cn, \quad \text{for all } n \ge 1.$$

Recall that a Liouville number is an irrational real number γ such that for every real number w, there exists a rational number p/q with $|\gamma - p/q| < 1/q^w$.

Theorem 12.4. *Let ξ be an irrational real number and $b \ge 2$ be an integer. If the b-ary expansion of ξ is of sublinear complexity, then, either ξ is a Liouville number, or there exists a positive number C such that*

$$|\xi - \theta| > H(\theta)^{-(2d)^{C \log\log(3d)}},$$

for every real algebraic number θ of degree d.

In Theorem 12.4, the quantity $H(\theta)$ is the height of θ as introduced in Definition 6.3.

The analogue of Theorem 12.4 for continued fraction expansions has been established in [7, 28]. The analogues of Theorems 12.2 and 12.3 have not been written yet, but there is little doubt that they hold and can be proved by combining the ideas of the proofs of Theorems 3.2, 12.2 and 12.3.

13. Further notions of complexity

For an integer $b \ge 2$, an irrational real number ξ whose b-ary expansion is given by (1.1), and a positive integer n, set

$$\mathcal{NZ}(n, \xi, b) := \#\{\ell : 1 \le \ell \le n, a_\ell \ne 0\},$$

which counts the number of non-zero digits among the first n digits of the b-ary expansion of ξ.

Alternatively, if $1 \le n_1 < n_2 < \ldots$ denote the increasing sequence of the indices ℓ such that $a_\ell \ne 0$, then for every positive integer n we have

$$\mathcal{NZ}(n, \xi, b) := \max\{j : n_j \le n\}.$$

Let $\varepsilon > 0$ be a real number and θ be an algebraic, irrational number. It follows from Ridout's theorem 6.6 that $n_{j+1} \le (1+\varepsilon)n_j$ holds for every sufficiently large j. Consequently, we get that

$$\lim_{n \to +\infty} \frac{\mathcal{NZ}(n, \theta, b)}{\log n} = +\infty.$$

For the base $b = 2$, this was considerably improved by Bailey, Borwein, Crandall, and Pomerance [17] (see also Rivoal [60]), using elementary considerations and ideas from additive number theory. A minor modification of their method allows us to get a similar result for expansions to an arbitrary integer base. The following statement is extracted from [27] (see also [11]).

Theorem 13.1. *Let $b \geq 2$ be an integer. For any irrational real algebraic number θ of degree d and height H and for any integer n exceeding $(20b^d d^3 H)^d$, we have*

$$\mathcal{NZ}(n, \theta, b) \geq \frac{1}{b-1} \left(\frac{n}{2(d+1)a_d} \right)^{1/d},$$

where a_d denotes the leading coefficient of the minimal polynomial of θ over the integers.

The idea behind the proof of Theorem 13.1 is quite simple and was inspired by a paper by Knight [48]. If an irrational real number ξ has few non-zero digits, then its integer powers $\xi^2, \xi^3, \ldots$, and any finite linear combination of them, cannot have too many non-zero digits. In particular, ξ cannot be a root of an integer polynomial of small degree. This is, in general, not at all so simple, since we have to take much care of the carries. However, for some particular families of algebraic numbers, including roots of positive integers, a quite simple proof of Theorem 13.1 can be given. Here, we follow [60] and (this allows some minor simplification) we treat only the case $b = 2$.

For a non-negative integer x, let $B(x)$ denote the number of 1's in the (finite) binary representation of x.

Theorem 13.2. *Let θ be a positive real algebraic number of degree $d \geq 2$. Let $a_d X^d + \cdots + a_1 X + a_0$ denote its minimal polynomial and assume that $a_1, \ldots, a_d$ are all non-negative. Then, there exists a constant c, depending only on θ, such that*

$$\mathcal{NZ}(n, \theta, 2) \geq B(a_d)^{-1/d} n^{1/d} - c,$$

for $n \geq 1$.

Proof. Observe first that, for all positive integers x and y, we have

$$B(x + y) \leq B(x) + B(y)$$

and

$$B(xy) \leq B(x)\, B(y).$$

For simplicity, let us write $\mathcal{NZ}(n, \cdot)$ instead of $\mathcal{NZ}(n, \cdot, 2)$. Let ξ and η be positive irrational numbers (the assumption of positivity is crucial) and n be a sufficiently large integer. We state without proof several elementary assertions. If $\xi + \eta$ is irrational, then we have

$$\mathcal{NZ}(n, \xi + \eta) \leq \mathcal{NZ}(n, \xi) + \mathcal{NZ}(n, \eta) + 1.$$

If $\xi\eta$ is irrational, then we have

$$\mathcal{NZ}(n, \xi\eta) \leq \mathcal{NZ}(n, \xi) \cdot \mathcal{NZ}(n, \eta) + \log_2(\xi + \eta + 1) + 1,$$

where $\log_2$ denotes the logarithm in base 2. If m is an integer, then we have

$$\mathcal{NZ}(n, m\xi) \leq B(m)(\mathcal{NZ}(n, \xi) + 1).$$

Furthermore, for every positive integer A, we have

$$\mathcal{NZ}(n,\xi)\cdot\mathcal{NZ}(n,A/\xi)\ge n-1-\log_2(\xi+A/\xi+1)). \tag{13.1}$$

Let θ be as in the statement of the theorem. The real number $|a_0|/\theta$ is irrational, as are the numbers $a_j\theta^{j-1}$ for $j=2,\dots,d$ provided that $a_j\neq 0$. Since

$$|a_0|\theta^{-1}=a_1+a_2\theta+\cdots+a_d\theta^{d-1}$$

and $\mathcal{NZ}(n,\theta)$ tends to infinity with n, the various inequalities listed above imply that

$$\begin{aligned}\mathcal{NZ}(n,|a_0|\theta^{-1}) &\le d+B(a_1)+\mathcal{NZ}(n,a_2\theta)+\cdots+\mathcal{NZ}(n,a_d\theta^{d-1})\\ &\le d+B(a_1)+B(a_2)(\mathcal{NZ}(n,\theta)+1)+\cdots\\ &\quad+B(a_d)(\mathcal{NZ}(n,\theta^{d-1})+1)\\ &\le B(a_d)\mathcal{NZ}(n,\theta)^{d-1}+c_1\mathcal{NZ}(n,\theta)^{d-2},\end{aligned} \tag{13.2}$$

where c_1, like c_2,c_3,c_4 below, is a suitable positive real number depending only on θ. By (13.1), we get

$$\mathcal{NZ}(n,|a_0|\theta^{-1})\ge\frac{n}{\mathcal{NZ}(n,\theta)}-c_2.$$

Combined with (13.2), we obtain

$$B(a_d)\mathcal{NZ}(n,\theta)^d+c_3\mathcal{NZ}(n,\theta)^{d-1}\ge n$$

and we finally deduce that

$$\mathcal{NZ}(n,\theta)\ge B(a_d)^{-1/d}n^{1/d}-c_4,$$

as asserted. □

We may also ask for a finer measure of complexity than simply counting the number of non-zero digits and consider the number of digit changes.

For an integer $b\ge 2$, an irrational real number ξ whose b-ary expansion is given by (1.1), and a positive integer n, we set

$$\mathcal{NBDC}(n,\xi,b):=\#\{\ell:1\le\ell\le n, a_\ell\neq a_{\ell+1}\},$$

which counts the number of digits followed by a different digit, among the first n digits in the b-ary expansion of ξ. The functions $n\mapsto\mathcal{NBDC}(n,\xi,b)$ have been introduced in [23]. Using this notion for measuring the complexity of a real number, Theorem 13.3 below, proved in [23, 31], shows that algebraic irrational numbers are 'not too simple'.

Theorem 13.3. *Let $b\ge 2$ be an integer. For every irrational, real algebraic number θ, there exist an effectively computable constant $n_0(\theta,b)$, depending only on θ and b and an effectively computable constant c, depending only on the degree of θ, such that*

$$\mathcal{NBDC}(n,\theta,b)\ge c\,(\log n)^{3/2}\,(\log\log n)^{-1/2}, \tag{13.3}$$

for every integer $n\ge n_0(\theta,b)$.

A weaker result than (13.3), namely that

$$\lim_{n \to +\infty} \frac{\mathcal{NBDC}(n, \theta, b)}{\log n} = +\infty, \tag{13.4}$$

follows quite easily from Ridout's Theorem 6.6. The proof of Theorem 13.3 depends on a quantitative version of Ridout's Theorem. We point out that the lower bound in (13.3) does not depend on b.

Further results on the number of non-zero digits and the number of digit changes in the b-ary expansion of algebraic numbers have been obtained by Kaneko [45, 46].

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