On exponents of homogeneous and inhomogeneous Diophantine Approximation

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Abstract – In Diophantine Approximation, inhomogeneous problems are linked with homogeneous ones by means of the so-called Transference Theorems. We revisit this classical topic by introducing new exponents of Diophantine approximation. We prove that the inhomogeneous exponent of approximation to a generic point in $\mathbf{R}^n$ by a system of n linear forms is equal to the inverse of the uniform homogeneous exponent associated to the system of dual linear forms.

1. Introduction and results.

It is a well-known fact that inhomogeneous problems in Diophantine Approximation are connected to homogeneous ones by means of the so-called Transference Theorems. We revisit this classical topic, refering mainly to the book of Cassels [9], in terms of relations between various homogeneous and inhomogeneous exponents of approximation. Besides the usual exponents defined by the existence of infinitely many solutions to some system of Diophantine inequalities, we consider both in homogeneous and inhomogeneous approximation, uniform exponents which we indicate by a 'hat', following here the conventions of [8]. It turns out that the usual (resp. uniform) inhomogeneous exponents are related to the uniform (resp. usual) homogeneous exponents.

Let us begin with a classical example of a uniform homogeneous exponent of approximation. The well-known Dirichlet Theorem asserts that for any irrational real number ξ and *any* real number $Q \geq 1$, there exist integers p and q with $1 \leq q \leq Q$ and

$$|q\xi - p| \leq Q^{-1}. \tag{1}$$

As observed by Khintchine [33], there is no ξ for which the exponent of Q in (1) can be lowered (see [13] or [44] for a very precise result). However, for any $w > 1$, there clearly

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exist irrational real numbers ξ for which, for *arbitrarily large* values of Q, the equation

$$|q\xi - p| \le Q^{-w}$$

has a solution in integers p and q with $1 \le q \le Q$. Obviously, the quality of approximation strongly depends upon whether we are interested in a uniform statement (i.e. a statement valid for any Q, or for any Q sufficiently large) or in a statement valid for some arbitrarily large values of Q.

Our framework is the following. For any (column) vector $\underline{\theta}$ in $\mathbf{R}^n$, we denote by $|\underline{\theta}|$ the maximum of the absolute values of its coordinates and by

$$\|\underline{\theta}\| = \min_{\underline{y} \in \mathbf{Z}^n} |\underline{\theta} - \underline{y}|$$

the maximum of the distances of its coordinates to the rational integers. Let n and m be two positive integers and let A be a real matrix with n rows and m columns. For any n-tuple $\underline{\theta}$ of real numbers, we denote by $w(A, \underline{\theta})$ the supremum of the real numbers w for which, *for arbitrarily large real numbers* X, the inequalities

$$\|A\underline{x} - \underline{\theta}\| \le X^{-w} \quad \text{and} \quad |\underline{x}| \le X \tag{2}$$

have a solution $\underline{x}$ in $\mathbf{Z}^m$. According to our conventions, let $\hat{w}(A, \underline{\theta})$ be the supremum of the real numbers w for which, *for all sufficiently large positive real numbers* X, the inequalities (2) have an integer solution $\underline{x}$ in $\mathbf{Z}^m$. The lower bounds

$$w(A, \underline{\theta}) \ge \hat{w}(A, \underline{\theta}) \ge 0$$

are then obvious. We define furthermore two homogeneous exponents $w(A)$ and $\hat{w}(A)$ as in (2) with $\underline{\theta} = {}^t(0, \ldots, 0)$, requiring moreover that the integer solution $\underline{x}$ should be non-zero. The uniform exponent $\hat{w}(A)$ was first introduced and studied by Jarník. We shall review in Section 2 some known results concerning $w(A)$ and $\hat{w}(A)$.

The transposed matrix of any matrix A is denoted by ${}^tA$. Furthermore, $1/+\infty$ is understood to be 0. We can now state our main result.

Theorem. *For any n-tuple $\underline{\theta}$ of real numbers, we have the lower bounds*

$$w(A, \underline{\theta}) \ge \frac{1}{\hat{w}({}^tA)} \quad \text{and} \quad \hat{w}(A, \underline{\theta}) \ge \frac{1}{w({}^tA)}, \tag{3}$$

with equality in (3) for almost all $\underline{\theta}$ with respect to the Lebesgue measure on $\mathbf{R}^n$.

Notice that the metrical statement of our Theorem is of same spirit as Satz V b from [33]. It extends and reformulates the results contained in [25, 26, 27]. The existence of a generic behaviour for the problem of inhomogeneous approximation is seemingly a new observation in this full generality.

Let us first examine the simplest case of a 1×1 matrix $A = (\xi)$, where ξ is an irrational real number. We deduce from the observation following (1) that the uniform exponent $\hat{w}({}^tA)$ is equal to 1, as well as the generic inhomogeneous exponent $\inf_{\theta \in \mathbf{R}} w(A, \theta)$. In fact, a more precise result holds in this particular case. Namely, Minkowski has proved that for any real number θ, the system of inequalities

$$|q| \leq Q \quad \text{and} \quad \|q\xi - \theta\| \leq \frac{1}{4} Q^{-1}$$

has an integer solution q for infinitely many integers Q. Moreover, our Theorem shows that -1 is the best possible exponent, regardless of the irrational number ξ. Besides, Cassels has observed that there does not exist any inhomogeneous analogue of the Dirichlet box principle, even if we weaken the property of approximation. In Theorem III of Chapter 3 from [9], he constructed a Liouville number ξ and a real number θ such that, for any $\epsilon > 0$, we have the lower bound

$$\min_{|q| \leq Q} \|q\xi - \theta\| \geq Q^{-\epsilon}$$

for infinitely many integers Q. The result of Cassels follows immediately from our Theorem, since $w((\xi)) = +\infty$ for any Liouville number ξ. The uniform exponent of inhomogeneous approximation $\hat{w}((\xi), \theta)$ therefore vanishes for almost all θ. We postpone to Section 7 further results on that exponent of approximation.

We could refine the first statement of the Theorem (and of Proposition 1 below) by taking into account whether or not there exists a positive constant c such that, *for arbitrarily large real numbers* X, the inequalities

$$\|A\underline{x} - \underline{\theta}\| \leq c\, X^{-w(A, \underline{\theta})} \quad \text{and} \quad |\underline{x}| \leq X$$

have a solution $\underline{x}$ in $\mathbf{Z}^m$. We take into consideration this remark in the statement of the Corollary below.

The next result should be compared with Theorem X of Chapter 5 from [9].

Proposition 1. *Let A be a matrix in $\mathcal{M}_{n,m}(\mathbf{R})$. For any exponent $w > 1/\hat{w}({}^tA)$, there exists a real n-tuple $\underline{\theta}$ such that the lower bound*

$$\|A\underline{x} - \underline{\theta}\| \geq |\underline{x}|^{-w}$$

holds for any integer m-tuple $\underline{x}$ whose norm $|\underline{x}|$ is sufficiently large. Moreover, there exists some real n-tuple $\underline{\theta}$ such that

$$\|A\underline{x} - \underline{\theta}\| \geq \frac{1}{72n^2(8m)^{m/n}}\, |\underline{x}|^{-m/n}$$

holds for all non-zero integer m-tuples $\underline{x}$.

Cassels [9], page 85, has proved the second assertion of Proposition 1 without however computing the constant occurring in the right-hand side of the lower bound. Moreover, our first assertion improves Theorem X of [9] whenever $\hat{w}({}^tA) > n/m$. See Section 6 for examples of such matrices A.

If the subgroup $G = {}^tA\mathbf{Z}^n + \mathbf{Z}^m$ of $\mathbf{R}^m$ generated by the n rows of A together with $\mathbf{Z}^m$ has maximal rank $m+n$, then Kronecker's Theorem asserts that the dual subgroup $\Gamma = A\mathbf{Z}^m + \mathbf{Z}^n$ of $\mathbf{R}^n$ generated by the m columns of A and by $\mathbf{Z}^n$ is dense in $\mathbf{R}^n$. In this respect, our Theorem may be viewed as a measure of the density of Γ. In the case where the rank of G is $< m+n$, we clearly have

$$\hat{w}({}^tA) = w({}^tA) = +\infty \quad \text{and} \quad \hat{w}(A, \underline{\theta}) = w(A, \underline{\theta}) = 0$$

for any n-tuples $\underline{\theta}$ located outside a discrete family of parallel hyperplanes in $\mathbf{R}^n$. The assertion of the Theorem is then obvious. In the sequel of the paper, we shall therefore assume that the rank over $\mathbf{Z}$ of the group G is equal to $m+n$. Notice however that the exponent $\hat{w}({}^tA)$ may be infinite, even when G has rank $m+n$. See Theorem XIV (page 94) of [9] for the construction of such a matrix A and the following Theorem XV concerning the density of the associated group Γ. Various results on the possible values of $\hat{w}({}^tA)$ will be given in Section 2.

Let us illustrate our Theorem by the example of the row (resp. column) matrices

$$A = (\xi, \ldots, \xi^n), \quad \text{resp.} \quad A = {}^t(\xi, \ldots, \xi^n),$$

made up with the successive powers of a transcendental real number ξ. Then, the corresponding exponents $\hat{w}(A)$ are uniformly bounded in terms of n (see [8] for references). Roy [39] determined these exponents for $n = 2$ when ξ is a Fibonacci continued fraction, that is, when we have

$$\xi = [0; a, b, a, a, b, a, b, \ldots],$$

where the sequence of partial quotients of ξ is given by the fixed point of the Fibonacci substitution $a \to ab$, $b \to a$. Here, a and b denote distinct positive integers.

Combining these results with our Theorem, we obtain the following statement.

Corollary. *Let n be a positive integer and let ξ be a real transcendental number.*

(i) There exists a positive constant c such that, for any real number θ, there exist infinitely many polynomials $P(X)$ with integer coefficients, degree at most n, and

$$|P(\xi) - \theta| \leq cH(P)^{-\lceil n/2 \rceil}.$$

(ii) There exists a positive constant c such that, for any real n-tuple $\underline{\theta} = (\theta_1, \dots, \theta_n)$, there exist infinitely many integers q with

$$\max_{1 \le j \le n} \|q\xi^j - \theta_j\| \le c\,|q|^{-1/(2n-1)}.$$

(iii) When $n = 2$, the assertions (i) and (ii) remain valid with the exponents

$$(1+\sqrt{5})/2 \simeq 1.618... \quad \text{and} \quad (3-\sqrt{5})/2 \simeq 0.3819...,$$

respectively. If moreover ξ is a Fibonacci continued fraction and if

$$w > (1+\sqrt{5})/2 \quad \text{and} \quad \lambda > (3-\sqrt{5})/2$$

then, for almost all real numbers θ and for almost all pairs of real numbers (θ_1, θ_2), we have the respective lower bounds

$$|P(\xi) - \theta| \ge H(P)^{-w} \quad \text{and} \quad \max\{\|q\xi - \theta_1\|, \|q\xi^2 - \theta_2\|\} \ge |q|^{-\lambda},$$

for any polynomial $P(X)$ with integer coefficients and degree ≤ 2 and with sufficiently large height, and for any integer q with sufficiently large absolute value.

Thus, for a Fibonacci continued fraction ξ, the critical exponents in degree $n = 2$ for the problems of inhomogeneous approximation (i) and (ii) are respectively $w = 1.618...$ and $\lambda = 0.3819...$, instead of the exponents 2 and 1/2, which occur in the generic situation. Notice that $\hat{w}(A)$ and $\hat{w}({}^tA)$ have also been determined for $A = (\xi, \xi^2)$ when ξ is a Sturmian continued fraction, see [8]. This provides further examples of matrices A with $\hat{w}(A)$ and $\hat{w}({}^tA)$ greater than in the generic case.

Our article is organized as follows. Section 2 provides an account of our knowledge about the homogeneous exponents $w(A)$ and $\hat{w}(A)$. Next, we begin the proof of our Theorem. Section 3 is devoted to the definition and to the properties of the sequence of best approximations. A crucial fact is that it increases at least geometrically. Some transference lemma is stated and proved in Section 4. It is used in the next Section, where we establish the Theorem and Proposition 1. The Corollary is then discussed in Section 6. Finally, questions of Hausdorff dimensions, which arise naturally from the Theorem, are briefly treated in Section 7, where further results on the uniform exponents of inhomogeneous approximation are given.

2. Properties of the homogeneous exponents $w(A)$ and $\hat{w}(A)$.

For any real n by m matrix A, the Dirichlet box principle implies that

$$w(A) \geq \hat{w}(A) \geq \frac{m}{n}. \tag{4}$$

Furthermore, we have both equalities in (4) for almost all matrices A, with respect to the Lebesgue measure on $\mathbf{R}^{mn}$, as follows from the Borel–Cantelli Lemma.

The left-hand side inequality of (4) has been improved by Jarník [28,29] as follows.

Proposition 2. *For any $n \geq 2$ and any $n \times 1$ real matrix A with at least one irrational coefficient, we have*

$$w(A) \geq \frac{\hat{w}^2(A)}{1 - \hat{w}(A)} \quad \text{and} \quad \frac{1}{n} \leq \hat{w}(A) \leq 1.$$

For any $n \geq 1$ and any $n \times 2$ real matrix A, we have

$$w(A) \geq \hat{w}(A)(\hat{w}(A) - 1).$$

For any $n \geq 1$, $m \geq 3$ and any $n \times m$ real matrix A with $\hat{w}(A) > (5m^2)^{m-1}$, we have

$$w(A) \geq \big(\hat{w}(A)\big)^{m/(m-1)} - 3\,\hat{w}(A).$$

It is well-known that $w(A)$ and $w({}^tA)$ are linked by a transference principle. Dyson [16] established the lower bound

$$w(A) \geq \frac{m\,w({}^tA) + m - 1}{(n-1)w({}^tA) + n}, \tag{5}$$

thus extending earlier results of Khintchine [32, 33] who delt with the case $\min\{n, m\} = 1$. For a proof, the reader is referred to Gruber & Lekkerkerker [20, Section 45.3], Cassels [9, Chapter V, Theorem IV], or Schmidt [44, Chapter IV, Section 5]. Inequalities (5) have been shown to be best possible for $\min\{n, m\} = 1$ by Jarník [22,23]. For general m and n, Jarník [30] proved that (5) is best possible except, possibly, when $1 < n < m$ and $w(A) < (m-1)/(n-1)$, in which case his method does not give anything.

Furthermore, extending earlier results of Jarník [24], Apfelbeck [1] established that the uniform exponents $\hat{w}(A)$ and $\hat{w}({}^tA)$ are linked by the same relation

$$\hat{w}(A) \geq \frac{m\,\hat{w}({}^tA) + m - 1}{(n-1)\hat{w}({}^tA) + n}. \tag{6}$$

Jarník [24] and Apfelbeck [1] succeeded in improving (6) when either $\hat{w}(A)$ or $\hat{w}({}^tA)$ is large. Next result is proved in [24] and seems to have been completely forgotten since 1938. It considerably improves (6) when $(m, n) = (1, 2)$ or $(2, 1)$.

Proposition 3. *For any 1×2 real matrix A with at least one irrational coefficient, we have*

$$\hat{w}({}^tA) = 1 - \frac{1}{\hat{w}(A)}.$$

It will be convenient to write $w_{n,m}(A)$ and $\hat{w}_{n,m}(A)$, rather than $w(A)$ and $\hat{w}(A)$, when we wish to remind the dimensions $n \times m$ of the matrix A. We define the *spectrum* of the functions $w_{n,m}$ and $\hat{w}_{n,m}$ to be the set of values taken by these two exponents $w(A)$ and $\hat{w}(A)$, when A ranges over the set of $n \times m$ real matrices A for which the associated group $\Gamma = A\mathbf{Z}^m + \mathbf{Z}^n$ has rank $m+n$, and whose rank over $\mathbf{R}$ is equal to $\min\{m,n\}$. The latter condition on the rank of A means that we wish to avoid trivial constructions coming from smaller dimensions. For instance, the equality $w((\xi)) = w({}^t(\xi,\xi))$ holds for any real number ξ.

Except for $m = n = 1$ (in that case, we can use the continued fraction theory), it is in general a difficult problem to construct explicit examples of $n \times m$ matrices A with prescribed values for $w(A)$ and/or for $\hat{w}(A)$. However, the spectrum of the function $w_{n,m}$ has been completely determined, thanks to some fine calculation of Hausdorff dimension due to Dodson.

Proposition 4. *For any positive integers n and m, the spectrum of the function $w_{n,m}$ is equal to $[m/n, +\infty]$.*

Proof. It follows straightforwardly from [15] that, for any real number $\tau \geq m/n$, we have

$$\begin{aligned} \dim\{A \in \mathcal{M}_{n,m}(\mathbf{R}) : w_{n,m}(A) = \tau\} &= \dim\{A \in \mathcal{M}_{n,m}(\mathbf{R}) : w_{n,m}(A) \geq \tau\} \\ &= (m-1)n + \frac{m+n}{\tau+1}, \end{aligned} \tag{7}$$

where dim stands for the Hausdorff dimension. See also [14] for a more precise result, from which we also get that $+\infty$ is in the spectrum of $w_{n,m}$. □

As for the spectra of the exponents $\hat{w}_{n,m}$, much less is known. They are contained in $[1/n, 1]$ if $m = 1$ and in $[m/n, +\infty]$ if $m \geq 2$. In particular, we have $\hat{w}((\xi)) = 1$ for any irrational real number ξ, as was already observed. The situation is completely different in the case $(m,n) \neq (1,1)$.

Proposition 5. *For any positive integers m, n with $m \geq 2$ there are continuum many $n \times m$ real matrices A whose coefficients are algebraically independent and which satisfy $\hat{w}(A) = +\infty$. For any positive integer n, there are continuum many $n \times 1$ real matrices A whose coefficients are algebraically independent and which satisfy $\hat{w}(A) = 1$.*

Proof. For $(m,n) = (2,1)$ or $(1,2)$ and the coefficients of the corresponding matrices are linearly independent, this is due to Khintchine [33]. Further results have been obtained by Chabauty & Lutz [11]. Jarník [31] completed the proof of the theorem, using a quite different approach (see also Lesca [38]). □

We address the following problem, which is likely to be difficult.

Problem 1. *For positive integers m and n, determine the spectrum of the function $\hat{w}_{n,m}$.*

Partial results when $\min\{n, m\} = 1$ have been established by Jarník [29].

Proposition 6. *For any integer $m \geq 2$, the spectrum of $\hat{w}_{1,m}$ contains the interval $[2^{m-1}, +\infty]$. Consequently, the spectrum of $\hat{w}_{1,2}$ is equal to $[2, +\infty]$ and that of $\hat{w}_{2,1}$ is equal to $[1/2, 1]$. For any integer $n \geq 2$, the spectrum of $\hat{w}_{n,1}$ contains the interval $[(u_n - 2 - u_n^{-n+1})/(u_n - 1), 1]$, where u_n is the largest real root of the polynomial $X^{n-1} - X^{n-2} - \sum_{k=0}^{n-2} X^k$.*

Jarník's proof of Proposition 6 is constructive and rests on the continued fraction theory.

Problem 1 may also be attacked by means of the theory of Hausdorff dimension, which has been used by R. C. Baker [3, 4], Yavid [46], and Rynne [41, 42] in the case $n = 1$, $m \geq 2$. In particular, Corollary 1 from [42] asserts that, for any $m \geq 2$, we have

$$\dim\{A \in \mathcal{M}_{1,m}(\mathbf{R}) : \hat{w}(A) = +\infty\} = m - 2.$$

Furthermore, although quite sharp lower and upper bounds are known, the precise value of

$$\dim\{A \in \mathcal{M}_{n,m}(\mathbf{R}) : \hat{w}(A) \geq \tau\},$$

for a given $\tau \geq m/n$, seems to be hard to predict.

Baker [3,4] proved that

$$\frac{2}{\tau} \leq \dim\{A \in \mathcal{M}_{1,2}(\mathbf{R}) : \hat{w}(A) \geq \tau\} \leq \frac{6}{\tau + 1}$$

holds for any given $\tau > 2$. A combination of Propositions 2 and 3 yields the sharper upper bound

$$\dim\{A \in \mathcal{M}_{1,2}(\mathbf{R}) : \hat{w}(A) \geq \tau\} \leq \frac{3\tau}{\tau^2 - \tau + 1}.$$

Indeed, if $\hat{w}(A) \geq \tau$, we have $\hat{w}({}^tA) \geq (\tau - 1)/\tau$ by Proposition 3, and next $w({}^tA) \geq (\tau - 1)^2/\tau$ by the first assertion of Proposition 2. The above upper estimate then follows from (7).

3. Best approximations.

Following the notations of [9], we denote by

$$M_j(\underline{y}) = \sum_{i=1}^{n} \alpha_{i,j} y_i, \quad \underline{y} = {}^t(y_1, \ldots, y_n), \quad (1 \le j \le m)$$

the linear forms determined by the columns of the matrix $A = (\alpha_{i,j})$ and we set

$$M(\underline{y}) = \max_{1 \le j \le m} \|M_j(\underline{y})\| = \|{}^t A \underline{y}\|.$$

Observe that the quantity $M(\underline{y})$ is positive for all non-zero integer n-tuples $\underline{y}$, since we have assumed that the rank over $\mathbf{Z}$ of the group G is equal to $m + n$. Then, we can build inductively a sequence of integer vectors

$$\underline{y}_i = {}^t(y_{i,1}, \ldots, y_{i,n}), \qquad (i \ge 1),$$

called *a sequence of best approximations* (*) related to the linear forms $M_1, \ldots, M_m$ and to the supremum norm, which satisfies the following properties. Set

$$|\underline{y}_i| = Y_i \quad \text{and} \quad M_i = M(\underline{y}_i).$$

Then, we have

$$1 = Y_1 < Y_2 < \cdots \quad \text{and} \quad M_1 > M_2 > \cdots \ ,$$

and $M(\underline{y}) \ge M_i$ for all non-zero integer vectors $\underline{y}$ of norm $|\underline{y}| < Y_{i+1}$. We start the construction with a smallest *minimal point* $\underline{y}_1$ in the sense of [12], verifying $Y_1 = |\underline{y}_1| = 1$ and $M(\underline{y}) \ge M(\underline{y}_1) = M_1$ for any integer point $\underline{y} \in \mathbf{Z}^n$ with norm $|\underline{y}| = 1$. Suppose that $\underline{y}_1, \ldots, \underline{y}_i$ have already been constructed in such a way that $M(\underline{y}) \ge M_i$ for all non-zero integer point $\underline{y}$ of norm $|\underline{y}| \le Y_i$. Let Y be the smallest positive integer $> Y_i$ for which there exists an integer point $\underline{z}$ verifying $|\underline{z}| = Y$ and $M(\underline{z}) < M_i$. The integer Y does exist by the Dirichlet box principle since $M_i > 0$. Among those points $\underline{z}$, we select an element $\underline{y}$ for which $M(\underline{z})$ is minimal. We then set

$$\underline{y}_{i+1} = \underline{y}, \qquad Y_{i+1} = Y \quad \text{and} \quad M_{i+1} = M(\underline{y}).$$

The sequence $(\underline{y}_i)_{i \ge 1}$ thus obtained satisfies clearly the desired properties.

(*) According to [36], a best approximation should be a vector belonging to $\mathbf{Z}^{n+m}$, by analogy with the usual continued fraction process. We forget here the last m coordinates which are unsignificant for our purpose.

Let w be a real number $< \hat{w}({}^tA)$, so that the system of inequalities

$$M(\underline{y}) \le Y^{-w} \quad \text{and} \quad |\underline{y}| \le Y$$

has a non-zero integer solution $\underline{y}$ for any sufficiently large Y. Choosing $Y < Y_{i+1}$ arbitrarily close to Y_{i+1}, we obtain the upper bound

$$M_i \le Y_{i+1}^{-w} \tag{8}$$

for any sufficiently large index i, using the characteristic property of the best approximations.

Suppose now that $w < w({}^tA)$. Then, there exist infinitely many indices i for which

$$M_i \le Y_i^{-w}. \tag{9}$$

The indices i satisfying (9) are obtained by inserting the norm

$$Y_i \le |\underline{y}| < Y_{i+1}$$

of the integer solutions $\underline{y}$ of the inequation $M(\underline{y}) \le |\underline{y}|^{-w}$ in the sequence $(Y_k)_{k\ge 1}$.

Observe furthermore that the Dirichlet box principle (cf. [9], Theorem VI, page 13) ensures that the system of inequations

$$M(\underline{y}) \le Y^{-n/m} \quad \text{and} \quad |\underline{y}| \le Y$$

has a non-zero integer solution $\underline{y}$ for any $Y \ge 1$. Arguing as above, we obtain the upper bound

$$M_i \le Y_{i+1}^{-n/m} \tag{10}$$

for all $i \ge 1$.

Lemma 1. *There exists a positive constant c such that*

$$Y_i \ge c\, 2^{i/(3^{m+n}-1)}$$

for all $i \ge 1$.

Proof. Lagarias [36] has established in a quite general framework that a sequence of best approximations increases at least geometrically. We take again the argumentation used in the proof of Theorem 2.2 from [35]. Let us consider the $3^{m+n}+1$ consecutive vectors

$$\underline{y}_i, \underline{y}_{i+1}, \dots, \underline{y}_{i+3^{m+n}}.$$

By the usual box principle, there exist two indices r and s, with $0 \le r < s \le 3^{m+n}$, such that

$$\underline{y}_{i+r,j} \equiv \underline{y}_{i+s,j} \pmod 3 \quad \text{for all } j = 1, \dots, n,$$

and

$$\langle M_k(\underline{y}_{i+r}) \rangle \equiv \langle M_k(\underline{y}_{i+s}) \rangle \pmod 3 \quad \text{for all } k = 1, \dots, m,$$

where the notation $\langle x \rangle$ stands for the closest integer to the real number x. Setting

$$\underline{z} = \frac{\underline{y}_{i+s} - \underline{y}_{i+r}}{3},$$

we have

$$|\underline{z}| \le \frac{Y_{i+s} + Y_{i+r}}{3} \quad \text{and} \quad M(\underline{z}) \le \frac{M_{i+r} + M_{i+s}}{3} < M_{i+r}.$$

Since $\underline{z}$ is a non-zero integer vector, we get

$$Y_{i+r+1} \le \frac{Y_{i+s} + Y_{i+r}}{3}$$

and

$$Y_{i+3^{m+n}} \ge Y_{i+s} \ge 3Y_{i+r+1} - Y_{i+r} \ge 2Y_{i+r+1} \ge 2Y_{i+1}$$

for any $i \ge 1$. The expected lower bound then follows by induction on i. $\square$

Lemma 2. *For almost all real n-tuples $\underline{\theta} = {}^t(\theta_1, \dots, \theta_n)$, we have the lower bound*

$$\| y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n \| \ge Y_i^{-\delta},$$

for any $\delta > 0$, and any index i which is sufficiently large in terms of δ and of $\theta_1, \dots, \theta_n$.

Proof. We can assume without restriction that the numbers θ_j are located in the interval $[0, 1]$ and that δ is given. For a fixed i, the reverse inequality

$$\| y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n \| < Y_i^{-\delta} \tag{11}$$

defines in the hypercube $[0, 1]^n$ a subset of Euclidean volume $\le 2Y_i^{-\delta}$. By Lemma 1, the series $\sum_{i \ge 1} Y_i^{-\delta}$ converges. It then follows from the Borel–Cantelli Lemma that the set of $\underline{\theta}$ satisfying (11) for infinitely many indices i has Lebesgue measure zero. $\square$

4. A transference lemma.

Let us consider now the linear forms

$$L_i(\underline{x}) = \sum_{j=1}^{m} \alpha_{i,j} x_j, \quad \underline{x} = {}^t(x_1, \dots, x_m), \quad (1 \le i \le n),$$

determined by the rows of the matrix A. The following result relates the problem of the inhomogeneous simultaneous approximation by the linear forms L_i to the problem of homogeneous simultaneous approximation by the linear forms M_j.

Lemma 3. *Set $\kappa = 2^{1-m-n}((m+n)!)^2$. Let X and Y be two positive real numbers. Suppose that we have the lower bound*

$$M(\underline{y}) \geq \kappa X^{-1}$$

for any non-zero integer n-tuple $\underline{y}$ of norm $|\underline{y}| \leq Y$. Then, for all real n-tuples $(\theta_1, \ldots, \theta_n)$, there exists an integer m-tuple $\underline{x}$ with norm $|\underline{x}| \leq X$ such that

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) - \theta_i\| \leq \kappa Y^{-1}.$$

Proof. This is the first assertion of Lemma 4.1 from [45]. For the convenience of the reader, we reproduce the proof. Let X and Z be two positive real numbers. Part B of Theorem XVII in chapter V of [9] asserts that the system of inequations

$$\max_{1 \leq i \leq n} \|L_i(\underline{x}) - \theta_i\| \leq Z \quad \text{and} \quad |\underline{x}| \leq X$$

has an integer solution $\underline{x} \in \mathbf{Z}^m$, provided that the upper bound

$$\|y_1\theta_1 + \cdots + y_n\theta_n\| \leq \kappa^{-1} \max\{XM(\underline{y}), Z|\underline{y}|\} \tag{12}$$

holds for all integer n-tuples $\underline{y}$. Let us apply this result with $Z = \kappa Y^{-1}$. The condition (12) is satisfied when $|\underline{y}| \geq Y$, since then $\kappa^{-1}Z|\underline{y}|$ is ≥ 1 while the left hand side of (12) is $\leq 1/2$. If $\underline{y}$ is non-zero and $|\underline{y}| \leq Y$, our assumption ensures that $M(\underline{y}) \geq \kappa X^{-1}$, and the right hand side of (12) is ≥ 1 in this case too. Finally, (12) obviously holds for $\underline{y} = 0$. $\square$

Up to the value of the numerical constant κ, the above mentioned Theorem XVII of [9] is a consequence of the more general Theorem VI in Chapter XI of [10], when applied to the distance function

$$F(x_1, \ldots, x_{m+n}) = X^{-1}\left(\sum_{j=1}^{m} |x_j|\right) + Z^{-1}\left(\sum_{i=1}^{n} |L_i(x_1, \ldots, x_m) + x_{m+i}|\right)$$

in $\mathbf{R}^{m+n}$. Notice that this last result provides also an explicit construction of the approximating point $\underline{x}$ in terms of successive minima and of duality.

5. Proof of the Theorem and of Proposition 1.

First, we prove that the lower bounds

$$w(A,\underline{\theta}) \geq \frac{1}{\hat{w}({}^tA)} \quad \text{and} \quad \hat{w}(A,\underline{\theta}) \geq \frac{1}{w({}^tA)} \tag{13}$$

hold for all real n-tuples $\underline{\theta} = {}^t(\theta_1, \ldots, \theta_n)$.

Let $w > \hat{w}({}^tA)$ be a real number. By definition of the exponent $\hat{w}({}^tA)$, there exists a real number Y, which may be chosen arbitrarily large, such that

$$M(\underline{y}) \geq Y^{-w} \tag{14}$$

for any non-zero integer n-tuple $\underline{y}$ of norm $|\underline{y}| \leq Y$. We use Lemma 3 with $X = \kappa Y^w$, where $\kappa = 2^{1-m-n}((m+n)!)^2$. Consequently, there exists an integer m-tuple $\underline{x}$ of norm $|\underline{x}| \leq X$ such that

$$\max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| \leq \kappa Y^{-1} = \kappa^{(1+1/w)} X^{-1/w} \leq \kappa^{(1+1/w)} |\underline{x}|^{-1/w}.$$

We deduce that $w(A,\underline{\theta}) \geq 1/w$. The first assertion of (13) then follows by letting w tend to $\hat{w}({}^tA)$.

The second lower bound of (13) is established along the same lines, observing that for $w > w({}^tA)$ and any sufficiently large real number Y, the inequality (14) is satisfied for any non-zero integer n-tuple $\underline{y}$ of norm $|\underline{y}| \leq Y$.

We shall now prove that the inverse upper bounds

$$w(A,\underline{\theta}) \leq \frac{1}{\hat{w}({}^tA)} \quad \text{and} \quad \hat{w}(A,\underline{\theta}) \leq \frac{1}{w({}^tA)} \tag{15}$$

hold for almost all real n-tuples $\underline{\theta} = {}^t(\theta_1, \ldots, \theta_n)$.

The duality formula ${}^t\underline{y}A\underline{x} = {}^t\underline{x}{}^tA\underline{y}$ written in the form

$$y_1\theta_1 + \cdots + y_n\theta_n = \sum_{j=1}^{m} x_j M_j(y_1, \ldots, y_n) - \sum_{i=1}^{n} y_i \left(L_i(x_1, \ldots, x_m) - \theta_i\right)$$

implies the upper bound

$$\|y_1\theta_1 + \ldots + y_n\theta_n\| \leq n|\underline{y}| \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + m|\underline{x}|M(\underline{y}) \tag{16}$$

for all integer vectors $\underline{x} = {}^t(x_1, \ldots, x_m)$ and $\underline{y} = {}^t(y_1, \ldots, y_n)$.

Let

$$\underline{y}_i = {}^t(y_{i,1}, \cdots, y_{i,n}) \quad \text{and} \quad Y_i = |\underline{y}_i| \quad (i \geq 1)$$

be a sequence of best approximations relative to the matrix ${}^tA$. Suppose that for all $\delta > 0$ we have the lower bound

$$\|y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n\| \geq Y_i^{-\delta} \tag{17}$$

for any index i large enough. By Lemma 2, the inequality (17) holds for almost all real n-tuples $\underline{\theta}$. Let us fix now two real numbers δ and w such that

$$0 < \delta < w < \hat{w}({}^tA).$$

Let $\underline{x}$ be an integer m-tuple with sufficiently large norm $|\underline{x}|$, and let k be the index defined by the inequalities

$$Y_k \leq (2m|\underline{x}|)^{1/(w-\delta)} < Y_{k+1},$$

so that

$$Y_{k+1}^w > (2m|\underline{x}|)^{w/(w-\delta)} \geq 2m|\underline{x}|Y_k^\delta.$$

Combining now the inequalities (8), (16) with $\underline{y} = \underline{y}_k$ and (17) for $i = k$, we obtain

$$\begin{aligned} Y_k^{-\delta} &\leq n\,|\underline{y}_k| \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + m\,|\underline{x}|\,M(\underline{y}_k) \\ &\leq n\,Y_k \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + m\,|\underline{x}|\,Y_{k+1}^{-w} \\ &\leq n\,Y_k \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + \frac{Y_k^{-\delta}}{2}, \end{aligned}$$

from which follows the lower bound

$$\begin{aligned} \|A\underline{x} - \underline{\theta}\| = \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| &\geq \frac{1}{2n} Y_k^{-1-\delta} \\ &\geq (2m)^{-(\delta+1)/(w-\delta)}(2n)^{-1}|\underline{x}|^{-(\delta+1)/(w-\delta)}. \end{aligned}$$

We deduce that

$$w(A, \underline{\theta}) \leq \frac{\delta + 1}{w - \delta}.$$

Choosing δ and w arbitrarily close to 0 and to $\hat{w}({}^tA)$ respectively, we obtain the first upper bound of (15).

In order to prove the second upper bound of (15), we take again the preceding argumentation using now the estimate (9) instead of (8). Let us fix two real numbers δ and w satisfying

$$0 < \delta < w < w({}^tA).$$

Let k be an integer ≥ 1 such that $M_k \leq Y_k^{-w}$. Since $w < w({}^tA)$, there exist infinitely many such integers k. Let $\underline{x}$ be an integer m-tuple with norm $|\underline{x}| \leq X_k := Y_k^{w-\delta}/(2m)$. Combining (9), (16) and (17), we obtain

$$\begin{aligned} Y_k^{-\delta} &\leq n\,|\underline{y}_k| \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + m\,|\underline{x}|\,M(\underline{y}_k) \\ &\leq n\,Y_k \max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| + m\,X_k\,Y_k^{-w}, \end{aligned}$$

from which we deduce that

$$\max_{1\leq i\leq n} \|L_i(\underline{x}) - \theta_i\| \geq \frac{1}{2n} Y_k^{-1-\delta} = (2m)^{-(\delta+1)/(w-\delta)}(2n)^{-1} X_k^{-(\delta+1)/(w-\delta)}.$$

Recall now that the above lower bound holds for any integer point with norm $\leq X_k$ and for infinitely many integers $k \geq 1$. Noting that the sequence $(X_i)_{i\geq 1}$ tends to infinity, it follows that

$$\hat{w}(A, \underline{\theta}) \leq \frac{\delta+1}{w-\delta}.$$

Choosing δ and w arbitrarily close to 0 and to $w({}^tA)$ respectively, we obtain the second upper bound of (15).

Furthermore, the preceding arguments enable us to establish Proposition 1. The first assertion follows immediately from the Theorem, since we just have to pick an n-tuple $\underline{\theta}$ out of a set of full Lebesgue measure. The proof of the second assertion needs more work. We begin by extracting some subsequence from the sequence of best approximations $(\underline{y}_i)_{i\geq 1}$.

We claim that there exists an increasing function $\varphi : \mathbf{Z}_{\geq 1} \to \mathbf{Z}_{\geq 1}$ satisfying $\varphi(1) = 1$ and, for any integer $i \geq 2$,

$$Y_{\varphi(i)} \geq (9n)^{1/2} Y_{\varphi(i-1)} \quad \text{and} \quad Y_{\varphi(i-1)+1} \geq (9n)^{-1} Y_{\varphi(i)}. \tag{18}$$

The function φ is constructed in the following way. Let $j > j'$ be two indices such that $Y_j \geq (9n)^{1/2} Y_{j-1}$ and $Y_{j'} \geq (9n)^{1/2} Y_{j'-1}$. Suppose that $j' - 1 = \varphi(h')$, and that the function φ has already been defined for $1 \leq i \leq h'$. We set $j - 1 = \varphi(h)$ for some $h > h'$, which will be specified later. We let $\varphi(h-1)$ be the largest index $t \geq j'$ for which $Y_{\varphi(h)} \geq (9n)^{1/2}\, Y_t$. We let $\varphi(h-2)$ be the largest index $t \geq j'$ for which $Y_{\varphi(h-1)} \geq (9n)^{1/2}\, Y_t$, and so on until it does not exist any index t as above. We have just defined $\varphi(h), \varphi(h-1), \ldots, \varphi(h-h_0)$. Then, we set $h = h_0 + h' + 1$, and we check that the inequalities (18) are satisfied for $i = h'+1, \ldots, h_0 + h' + 1$.

This process does not apply when there are only finitely many indices j such that $Y_j \geq (9n)^{1/2} Y_{j-1}$. In this case, we denote by g the largest of these indices ($g = 1$ if there is none) and we apply the above process to construct the initial values of the function φ up to $g = \varphi(h)$. Next, we define $\varphi(h+1)$ as the smallest index t for which $Y_t \geq (9n)^{1/2}\, Y_{\varphi(h)}$. Then, we observe that $Y_{\varphi(h+1)-1} < (9n)^{1/2}\, Y_{\varphi(h)}$ and

$$Y_{\varphi(h)+1} \geq Y_{\varphi(h)} > (9n)^{-1/2}\, Y_{\varphi(h+1)-1} > (9n)^{-1} Y_{\varphi(h+1)},$$

as required. We continue in this way, defining $\varphi(h+2)$ as the smallest index t for which $Y_t \geq (9n)^{1/2}\, Y_{\varphi(h+1)}$, and so on. The inequalities (18) are then satisfied.

The first inequalities in (18) enable us to satisfy the assumptions of Lemma 2, page 86, from [9] for the sequence of integer vectors $(\underline{y}_{\varphi(i)})_{i\geq 1}$ with $k = 3$. Consequently, there exists a real n-tuple $\underline{\theta}$ such that

$$\| y_{\varphi(i),1}\theta_1 + \ldots + y_{\varphi(i),n}\theta_n \| \geq \frac{1}{4}, \quad \text{for all } i \geq 1. \tag{19}$$

Let $\underline{x}$ be a non-zero integer m-tuple and let k be the index defined by the inequalities

$$Y_{\varphi(k)} \leq 9n(8m)^{m/n}\, |\underline{x}|^{m/n} < Y_{\varphi(k+1)}.$$

Taking into account (10), (19) and (16) applied with $\underline{y} = \underline{y}_{\varphi(k)}$, we have

$$\frac{1}{4} \leq (9n^2)(8m)^{m/n}\, |\underline{x}|^{m/n}\, \| A\underline{x} - \underline{\theta} \| + m|\underline{x}|\, Y_{\varphi(k)+1}^{-n/m}.$$

By construction of the subsequence $(Y_{\varphi(i)})_{i\geq 1}$, we have $Y_{\varphi(k)+1}^{-1} \cdot Y_{\varphi(k+1)} \leq 9n$, so that

$$\frac{1}{4} \leq (9n^2)(8m)^{m/n}\, |\underline{x}|^{m/n}\, \| A\underline{x} - \underline{\theta} \| + m \left(8m(9n)^{n/m} \right)^{-1} (9n)^{n/m},$$

and

$$\| A\underline{x} - \underline{\theta} \| \geq \frac{1}{72n^2(8m)^{m/n}}\, |\underline{x}|^{-m/n},$$

as announced. □

6. The Corollary.

Our Theorem reduces the determination of the measure of generic density

$$\inf_{\underline{\theta}\in \mathbf{R}^n} w(A, \underline{\theta}) = \frac{1}{\hat{w}({}^tA)}$$

of the group $\Gamma = A\mathbf{Z}^m + \mathbf{Z}^n$ to the computation of the exponent $\hat{w}({}^tA)$. Any upper bound of $\hat{w}({}^tA)$ implies a uniform lower bound for the exponents of approximation $w(A, \underline{\theta})$. When

$$A = (\xi, \ldots, \xi^n),$$

for a real transcendental number ξ, there are known upper bounds for $\hat{w}(A)$ and $\hat{w}({}^tA)$, which depend only upon n. Coming back to the specific notations of [8]:

$$\hat{w}(A) = \hat{w}_n(\xi) \quad \text{and} \quad \hat{w}({}^tA) = \hat{\lambda}_n(\xi),$$

we have

$$\hat{\lambda}_n(\xi) \leq \frac{1}{\lceil n/2 \rceil} \quad \text{and} \quad \hat{w}_n(\xi) \leq 2n-1.$$

The first upper bound is the main result of [37], while Theorem 2b of [12] is equivalent to the second one. Actually, the results of [37] and [12] are slightly sharper: the refinement stated below Proposition 1 hold in these cases. Combined with Lemma 3, they yield the assertions (i) and (ii) of the Corollary.

In degree $n = 2$, the exact upper bounds for the functions $\hat{\lambda}_2(\xi)$ and $\hat{w}_2(\xi)$ are known: Roy [39] and Arbour & Roy [2] have proved that

$$\hat{\lambda}_2(\xi) \leq \frac{\sqrt{5}-1}{2} \quad \text{and} \quad \hat{w}_2(\xi) \leq \frac{\sqrt{5}+3}{2},$$

and both equalities hold when ξ is a Fibonacci continued fraction. The assertion (iii) of the Corollary is then the translation of our Theorem in this particular case.

Remarks. (i) The exponents $\hat{\lambda}_2(\xi)$ and $\hat{w}_2(\xi)$ have been computed more generally in [8] for any Sturmian continued fraction ξ of irrational angle φ. It turns out that

$$\hat{\lambda}_2(\xi) > \frac{1}{2} \quad \text{and} \quad \hat{w}_2(\xi) > 2$$

when the partial quotients in the continued fraction expansion of φ are bounded. The associated subgroups Γ

$$\mathbf{Z} + \mathbf{Z}\xi + \mathbf{Z}\xi^2 \quad \text{and} \quad \mathbf{Z}\begin{pmatrix} \xi \\ \xi^2 \end{pmatrix} + \mathbf{Z}^2$$

are then dense in $\mathbf{R}$ and $\mathbf{R}^2$ respectively, and their generic exponents of density are less than 2 and 1/2, respectively.

(ii) For a class of real numbers ξ connected with the Fibonacci continued fraction, Roy [40] has proved that we have a lower bound of the form

$$|P(\xi) - \xi^3| \gg H(P)^{-(1+\sqrt{5})/2}$$

for any quadratic polynomial $P(X)$ with integer coefficients. This means that the number $\theta = \xi^3$ shares the almost sure property stated in the part (iii) of the Corollary. The difficult point in Roy's proof is to verify that a lower bound similar to (19) is valid for $\underline{\theta} = (\xi^3)$.

7. Spectra and Hausdorff dimension.

Let A be a given real matrix in $\mathcal{M}_{n,m}(\mathbf{R})$. As in Section 2, we may ask for the determination of the *spectrum* of the function $w(A,\cdot)$ (resp. $\hat{w}(A,\cdot)$), which is, by definition, the set of values $w(A,\underline{\theta})$ (resp. $\hat{w}(A,\underline{\theta})$) for $\underline{\theta}$ running through $\mathbf{R}^n$ with the additional requirement that $\underline{\theta} \notin A\mathbf{Z}^m + \mathbf{Z}^n$. When $\underline{\theta}$ belongs to $A\mathbf{Z}^m + \mathbf{Z}^n$, we trivially have $w(A,\underline{\theta}) = \hat{w}(A,\underline{\theta}) = +\infty$, and we conventionally exclude those points with peculiar behaviour. To that purpose, the Hausdorff dimension may be equally used, since it enables us to discriminate between sets of Lebesgue measure zero and thus to prove the existence of real numbers having fine properties of Diophantine approximation (see for instance, Chapter 5 of [6], and the books of Falconer [17, 18] for the definition and the properties of Hausdorff dimension).

Let $w > 1/\hat{w}({}^tA)$ be a real number. Put

$$\mathcal{U}_w(A) = \left\{ \underline{\theta} \in \mathbf{R}^n : \|A\underline{x} - \underline{\theta}\| \le \frac{1}{|\underline{x}|^w} \text{ holds for infinitely many } \underline{x} \text{ in } \mathbf{Z}^m \right\}.$$

Our Theorem asserts that $\mathcal{U}_w(A)$ is a null set. Thus we are led to propose the following

Problem 2. *Find the Hausdorff dimension of the set $\mathcal{U}_w(A)$ for any $w > 1/\hat{w}({}^tA)$.*

This question has been solved in the simplest case when $A = (\xi)$, for any given irrational number ξ, by Bugeaud [5] and, independently, by Schmeling & Troubetzkoy [43]. We then have $w((\xi),\underline{\theta}) = 1$ for almost all real numbers θ and the Hausdorff dimension of the set $\mathcal{U}_w((\xi))$ is equal to $1/w$, for any $w \ge 1$. Furthermore, the proof given in [5] can be adapted (it is sufficient to work with suitable dimension functions) to prove that, indeed, the Hausdorff dimension of the set of real numbers θ for which $w((\xi),\theta) = w$ is equal to $1/w$. Since real numbers $\theta \notin \mathbf{Z} + \xi\mathbf{Z}$ with $w((\xi),\theta) = +\infty$ are easy to construct, this shows that the spectrum of the function $w((\xi),\cdot)$ is equal to $[1,+\infty]$.

In higher dimensions, Bugeaud & Chevallier [7] have proved that, for almost all matrices A in $\mathcal{M}_{n,m}(\mathbf{R})$ and for any real number $w \ge m/n$, we have

$$\dim \mathcal{U}_w(A) = \dim \{\underline{\theta} \in \mathbf{R}^n : w(A,\underline{\theta}) \ge w\} = \frac{m}{w}.$$

Besides, Theorem 3 of [7] asserts that, if A is a column matrix, then

$$\dim \mathcal{U}_w(A) = \dim \{\underline{\theta} \in \mathbf{R}^n : w(A,\underline{\theta}) \ge w\} = \frac{1}{w}$$

for any real number $w \ge 1$. In the above examples, the sets $\{\underline{\theta} \in \mathbf{R}^n : w(A,\underline{\theta}) \ge w\}$ and $\mathcal{U}_w(A)$ have the same Hausdorff dimension (although the first one contains the second).

The results of [7] indicate that the situation is much more complicated when the matrix A is not of the form (ξ), and even a conjectural answer to Problem 2 does not clearly arise from the examples of [7]. Nonetheless, it is possible to show that this dimension is strictly less than n, whenever $w > 1/\hat{w}({}^tA)$.

Proposition 7. *Let A be a matrix in $\mathcal{M}_{n,m}(\mathbf{R})$ and let w be a real number $> 1/\hat{w}({}^tA)$. The set*

$$\{\underline{\theta} \in \mathbf{R}^n : w(A, \underline{\theta}) \geq w\}$$

has Lebesgue measure zero, and its Hausdorff dimension is strictly less than n.

Proof. Set

$$\delta = \frac{w\hat{w}({}^tA) - 1}{2 + w + 1/\hat{w}({}^tA)}$$

and notice that the inequality (11) determines in the hypercube $[0,1]^n$ a subset contained in the union of at most $c(n)\, Y_i \cdot Y_i^{(\delta+1)(n-1)}$ hypercubes with edge $Y_i^{-\delta-1}$, where $c(n)$ denotes some suitable constant, depending only upon n. Since, by Lemma 1, the series

$$\sum_{i \geq 1} Y_i^{1+(\delta+1)(n-1)} \cdot Y_i^{-(\delta+1)s}$$

converges for any $s > n - 1 + 1/(\delta + 1)$, the Hausdorff–Cantelli Lemma (cf. for example [6], Chapter 5) ensures us that the Hausdorff dimension of the set

$$\mathcal{V}_\delta = \{\underline{\theta} \in \mathbf{R}^n : \|y_{i,1}\theta_1 + \cdots + y_{i,n}\theta_n\| < Y_i^{-\delta} \quad \text{for infinitely many } i\}$$

is bounded from above by $n-1+1/(\delta+1)$. This is strictly less than n since δ is positive. Let $\underline{\theta}$ be in the complement of $\mathcal{V}_\delta$, and follow again the proof of our Theorem. The inequality (17) is then satisfied for any sufficiently large integer i. Thus, we have the upper bound

$$w(A, \underline{\theta}) \leq \frac{\delta + 1}{\hat{w}({}^tA) - \delta},$$

and, by our choice of δ,

$$w(A, \underline{\theta}) \leq \frac{1}{2}\left(\frac{1}{\hat{w}({}^tA)} + w\right) = w - \frac{1}{2}\left(w - \frac{1}{\hat{w}({}^tA)}\right).$$

Consequently, the set $\mathcal{U}_w(A)$ is contained in $\mathcal{V}_\delta$. This remark concludes the proof of Proposition 7. ∎

Remark. In view of the results of [39, 40, 8], it may be possible to determine the Hausdorff dimensions of the sets $\mathcal{U}_w(A)$ and $\mathcal{U}_w({}^tA)$, when $A = (\xi, \xi^2)$ and ξ is a Sturmian continued fraction. We plan to return to these questions later. Let us simply remark that our Proposition 7 implies that $\dim \mathcal{U}_2(A) < 1$ and $\dim \mathcal{U}_{1/2}({}^tA) < 2$, unlike in the generic situation.

As for the uniform exponent $\hat{w}(A, \underline{\theta})$, few results are known. In dimension one, Morimoto has proved that $\hat{w}((\xi), \theta) = 1$ holds for all real numbers θ not in $\mathbf{Z} + \mathbf{Z}\xi$, whenever the irrational real number ξ has bounded partial quotients. See Chapter VI of the book of Koksma [34] for references and more information. Next Proposition extends Morimoto's result.

Proposition 8. *Let ξ be an irrational real number. For any real number θ not in $\mathbf{Z}+\xi\mathbf{Z}$, we have*

$$\frac{1}{w((\xi))} \leq \hat{w}((\xi),\theta) \leq w((\xi)).$$

Let w denote $+\infty$ or a real number ≥ 1. There exists a real number ξ for which $w((\xi)) = w$ and the spectrum of the function $\hat{w}((\xi),\cdot)$ is equal to the interval $[1/w, w]$.

Proof. The lower bound $\hat{w}((\xi),\theta) \geq 1/w((\xi))$ follows from our Theorem. Arguing as in Section 3, we construct an infinite sequence of integer couples $\underline{x}_i = (x_i, y_i)_{i\geq 1}$ with increasing norms $|\underline{x}_i|$, such that the inequality

$$0 < |x_i\xi - y_i - \theta| \leq |\underline{x}_{i+1}|^{-w}$$

holds for any real number $w < \hat{w}((\xi),\theta)$ and any index i sufficiently large in terms of w. Using the triangle inequality, we find the upper bounds

$$|(x_i - x_{i-1})\xi - (y_i - y_{i-1})| \leq 2|\underline{x}_i|^{-w} \leq 2(|\underline{x}_i - \underline{x}_{i-1}|/2)^{-w},$$

which show that $w((\xi)) \geq w$. The first assertion is thus proved by choosing w arbitrarily close to $\hat{w}((\xi),\theta)$.

For the second one, we may assume that $w > 1$, taking into account Morimoto's result. We provide a constructive proof. If w is finite, let $(w_n)_{n\geq 0}$ be the constant sequence equals to w, otherwise, put $w_n = n$ for any $n \geq 0$. Let ξ be a real number such that the sequence of the denominators $(q_n)_{n\geq 0}$ of its convergents p_n/q_n satisfies the growth condition

$$q_{n+1} \asymp q_n^{w_n},$$

where the symbol $\asymp$ indicates an asymptotical equivalence. By our Theorem, we know that $\hat{w}((\xi),\theta) = 1/w$ for almost all real numbers θ. Let v be a real number with $1/w < v \leq w$. We now construct a real number θ for which $\hat{w}((\xi),\theta) = v$. When $w = +\infty$, our process furnishes moreover some $\theta \in \mathbf{R} \setminus (\mathbf{Z}+\xi\mathbf{Z})$ with $\hat{w}((\xi),\theta) = +\infty$. This will conclude the proof of Proposition 8.

Let $(u_n)_{n\geq 0}$ be a sequence of positive integers with

$$u_n \asymp q_n^{(w_n - v)/(v+1)}.$$

Set

$$\theta = \sum_{k\geq 0} u_k(q_k\xi - p_k).$$

For any $n \geq 0$, set

$$X_n = \sum_{k=0}^{n} u_k q_k \quad \text{and} \quad Y_n = \sum_{k=0}^{n} u_k p_k.$$

Then, we have the estimates

$$X_n \asymp u_n q_n \asymp q_n^{(w_n+1)/(v+1)} \quad \text{and} \quad |X_n \xi - Y_n - \theta| \asymp u_{n+1} q_{n+1}^{-w_n} \asymp X_{n+1}^{-v}, \tag{20}$$

which imply the lower bound

$$\hat{w}(\xi, \theta) \geq v.$$

When $w = +\infty$, we construct a real number θ, not in $\mathbf{Z} + \xi \mathbf{Z}$ and with $\hat{w}(\xi, \theta) = +\infty$, exactly in the same way, by taking $u_n = 1$ for any $n \geq 0$.

Next we prove that for any sufficiently large n and any integers x and y with $|x| \leq X_n/2$, we have

$$|x\xi - y - \theta| \geq \frac{1}{4} X_n^{-v}. \tag{21}$$

It follows from (21) that $\hat{w}(\xi, \theta) \leq v$, and therefore that the equality $\hat{w}(\xi, \theta) = v$ holds.

Suppose on the contrary that there exist integers x and y with $|x| \leq X_n/2$ for which (21) does not hold. Then, we deduce from (20) and the triangle inequality that

$$|(x - X_{n-1})\xi - (y - Y_{n-1})| \leq 2X_n^{-v}.$$

Write now

$$x - X_{n-1} = a q_{n-1} + b q_n \quad \text{and} \quad y - Y_{n-1} = a p_{n-1} + b p_n$$

for some integers a and b. We have

$$b = \pm \begin{vmatrix} x - X_{n-1} & q_{n-1} \\ y - Y_{n-1} & p_{n-1} \end{vmatrix} = \pm \begin{vmatrix} x - X_{n-1} & q_{n-1} \\ (y - Y_{n-1}) - \xi(x - X_{n-1}) & p_{n-1} - \xi q_{n-1} \end{vmatrix},$$

so that

$$|b| \leq 2 q_{n-1} X_n^{-v} + \left(\frac{X_n}{2} + X_{n-1} \right) q_n^{-1} \leq \frac{2}{3} u_n, \tag{22}$$

since

$$q_{n-1} X_n^{-v} \leq q_{n-1} X_{n-1}^{-1} \asymp u_{n-1}^{-1}.$$

Now we use the formula

$$x\xi - y - \theta = a(q_{n-1}\xi - p_{n-1}) - (u_n - b)(q_n \xi - p_n) - \sum_{k \geq n+1} u_k (q_k \xi - p_k).$$

Observe that

$$u_n q_{n+1}^{-1} \asymp X_n^{-v} = o(q_n^{-1})$$

since $v > 1/w$. When $a \neq 0$, we bound from below

$$|x\xi - y - \theta| \geq |q_{n-1}\xi - p_{n-1}| - \mathcal{O}(u_n q_{n+1}^{-1}) \geq \frac{1}{2q_n},$$

which is better than (21). For $a = 0$, using (22), we find the weaker lower bound

$$|x\xi - y - \theta| \geq \left(\frac{1}{3} - o(1)\right) u_n q_{n+1}^{-1} \asymp \frac{1}{3} X_n^{-v}.$$

In any case, we come to a contradiction. □

Remark. The bounds

$$\frac{1}{w({}^tA)} \leq \hat{w}(A, \underline{\theta}) \leq w(A),$$

established in Proposition 8 for $A = (\xi)$, hold in full generality for any real matrix $A \in \mathcal{M}_{n,m}(\mathbf{R})$ and any point $\underline{\theta} \in \mathbf{R}^n$ not in $A\mathbf{Z}^m + \mathbf{Z}^n$. The proof is similar. As a consequence, observe that, whenever $w(A) = m/n$, we get $\hat{w}(A, \underline{\theta}) = m/n$ for all points $\underline{\theta} \in \mathbf{R}^n$ not in $A\mathbf{Z}^m + \mathbf{Z}^n$.

In anology with $\mathcal{U}_w(A)$, we define for any real number $w > 1/w({}^tA)$ the set

$$\hat{\mathcal{U}}_w(A) = \{\underline{\theta} \in \mathbf{R}^n : \hat{w}(A, \underline{\theta}) \geq w\}.$$

Our Theorem asserts that $\hat{\mathcal{U}}_w(A)$ is a null set. Now the uniform version of Problem 2 is the following

Problem 3. *Let A be a matrix in $\mathcal{M}_{n,m}(\mathbf{R})$ with $w(A) > m/n$. For any real number w with $1/w({}^tA) < w \leq w(A)$, find the Hausdorff dimension of the set $\hat{\mathcal{U}}_w(A)$.*

As far as we know, no special case of Problem 3 has been solved, even in dimension one. However, we have

$$\dim \hat{\mathcal{U}}_w(A) < n,$$

for any $w > 1/w({}^tA)$. We omit the details of the argumentation which follows the same lines as the proof of Proposition 7.

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